AUTOMORPHISMS OF DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY \{25, 16, 1; 1, 8, 25\} \(^1\)

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Abstract: Makhnev and Samoilenko have found parameters of strongly regular graphs with no more than 1000 vertices, which may be neighborhoods of vertices in antipodal distance-regular graph of diameter 3 and with \(\lambda = \mu\). They proposed the program of investigation vertex-symmetric antipodal distance-regular graphs of diameter 3 with \(\lambda = \mu\), in which neighborhoods of vertices are strongly regular. In this paper we consider neighborhoods of vertices with parameters \((25, 8, 3, 2)\).

Key words: Strongly regular graph, Distance-regular graph.

Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex \(a\) in a graph \(\Gamma\), we denote by \(\Gamma_i(a)\) the subgraph induced by \(\Gamma\) on the set of all vertices, that are at the distance \(i\) from \(a\). The subgraph \([a] = \Gamma_1(a)\) is called the neighborhood of the vertex \(a\). Let \(\Gamma(a) = \Gamma_1(a)\), \(a^\perp = \{a\} \cup \Gamma(a)\). If graph \(\Gamma\) is fixed, then instead of \(\Gamma(a)\) we write \([a]\). For the set of vertices \(X\) of graph \(\Gamma\) through \(X^\perp\) denote \(\cap_{x \in X} x^\perp\).

Let \(\Gamma\) be an antipodal distance-regular graph of diameter 3 and \(\lambda = \mu\), in which neighborhoods of vertices are strongly-regular graphs. Then \(\Gamma\) has intersection array \(\{k, \mu(r - 1), 1, 1, \mu, k\}\), and spectrum \(k^1, \sqrt{k^f}, -1^k, -\sqrt{k^f}\), where \(f = (k + 1)(r - 1)/2\). In the case \(r = 2\) we obtain Taylor’s graph, in which \(k^1 = 2\mu^\prime\). Conversely, for any strongly regular graph with parameters \((v^\prime, 2\mu^\prime, \lambda^\prime, \mu^\prime)\) there exists a Taylor’s graph, in which neighborhoods of vertices are strongly regular with relevant parameters.

In [1] there were chosen strongly-regular graphs with no more than 1000 vertices, which may be neighborhoods of vertices of antipodal distance-regular graph of diameter 3 and \(\lambda = \mu\). There is provided a research program of the study of vertex-symmetric antipodal distance-regular graphs of diameter 3 with \(\lambda = \mu\), in which neighborhoods of vertices are strongly regular with parameters from Proposition 1.

Proposition 1. Let \(\Delta\) be a strongly-regular graph with parameters \((v, k, \lambda, \mu)\). If \((r - 1)k = v - k - 1, v \leq 1000\) and number \((v + 1)(r - 1)\) is even, then either \(r = 2\), or parameters \((v, k, \lambda, \mu, r)\) belong to the following list:

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(1) (16, 5, 0, 2, 3), (25, 8, 3, 2, 3), (49, 12, 5, 2, 4), (64, 21, 8, 6, 3), (81, 16, 7, 2, 5), (81, 20, 1, 6, 4), (85, 14, 3, 2, 6), (99, 14, 1, 2, 7), (100, 33, 8, 12, 3), (121, 20, 9, 2, 6), (121, 30, 11, 6, 4), (121, 40, 15, 12, 3), (126, 25, 8, 4, 5), (133, 44, 15, 14, 3), (169, 24, 11, 2, 7), (169, 42, 5, 12, 4), (169, 56, 15, 20, 3), (176, 25, 0, 4, 7), (196, 39, 14, 6, 5), (196, 65, 24, 20, 3);

(2) (225, 28, 13, 2, 8), (225, 56, 19, 12, 4), (243, 22, 1, 2, 11), (256, 51, 2, 12, 5), (256, 85, 24, 30, 3), (261, 52, 11, 10, 5), (288, 41, 4, 6, 7), (289, 32, 15, 2, 9), (289, 48, 17, 6, 6), (289, 72, 11, 20, 4), (289, 96, 35, 30, 3), (305, 76, 27, 16, 4), (325, 54, 3, 10, 6), (351, 50, 13, 6, 7), (351, 70, 13, 14, 5), (352, 39, 6, 4, 9), (361, 36, 17, 2, 10), (361, 72, 23, 12, 5), (361, 90, 29, 20, 4), (361, 120, 35, 42, 3);

(3) (400, 57, 20, 6, 7), (400, 133, 48, 42, 3), (441, 40, 19, 2, 11), (441, 88, 7, 20, 5), (441, 110, 19, 30, 4), (484, 161, 48, 56, 3), (495, 38, 1, 3, 13), (505, 84, 3, 16, 6), (507, 46, 5, 4, 11), (512, 73, 12, 10, 7), (529, 44, 21, 12, 2), (529, 66, 23, 6, 8), (529, 88, 27, 12, 6), (529, 132, 41, 30, 4), (529, 176, 63, 56, 3), (540, 49, 8, 4, 11), (576, 115, 18, 24, 5);

(4) (625, 48, 23, 2, 13), (625, 156, 29, 42, 4), (625, 208, 63, 72, 3), (640, 71, 6, 8, 9), (649, 72, 15, 7, 9), (649, 216, 63, 76, 3), (676, 75, 26, 6, 9), (676, 135, 14, 30, 5), (704, 37, 0, 2, 19), (729, 52, 25, 2, 14), (729, 104, 31, 12, 7), (729, 182, 55, 42, 4), (736, 105, 20, 14, 7), (768, 59, 10, 4, 13), (784, 261, 80, 90, 3);

(5) (837, 76, 15, 6, 11), (841, 56, 27, 2, 15), (841, 84, 29, 6, 10), (841, 140, 39, 20, 6), (841, 168, 47, 30, 5), (841, 210, 41, 56, 4), (841, 280, 99, 90, 3), (847, 94, 21, 9, 9), (848, 121, 24, 16, 7), (901, 60, 3, 4, 15), (961, 60, 29, 2, 16), (961, 120, 35, 12, 8), (961, 160, 9, 30, 6), (961, 192, 23, 42, 5), (961, 240, 71, 56, 4), (961, 320, 99, 100, 3), (1000, 111, 14, 12, 9).

Graphs with local subgraphs having parameters (64, 21, 8, 6, 3), (81, 16, 7, 2, 3), (85, 14, 3, 2) and (99, 14, 1, 2) were investigated in [2], [3], [4] and [5]. In this article we investigate parameters (25, 8, 3, 2, 3), i.e. this graph is locally 5 \times 5-grid. In [6] it is proved that distance-regular locally 5 \times 5-grid of diameter more then 2 is either isomorphic to the Johnson’s graph J(10, 5) or has an intersection array \{25, 16, 1; 1, 8, 25\}.

**Theorem 1.** Let $\Gamma$ be a distance-regular graph with intersection array \{25, 16, 1; 1, 8, 25\}, $G = \text{Aut}(\Gamma)$, $g$ is an element of prime order $p$ in $G$ and $\Omega = \text{Fix}(g)$ contains exactly $s$ vertices in $t$ antipodal classes. Then $\pi(G) \subseteq \{2, 3, 5, 13\}$ and one of the following assertions holds:

1. $\Omega$ is empty graph and $p \in \{2, 3, 13\}$;
2. $p = 5, t = 1, \alpha_2(g) = 0, \alpha_2(g) = 50t + 25$ and $\alpha_2(g) = 50 - 50t$;
3. $p = 3, s = 3, t = 2, 5, 8, \alpha_3(g) = 0, \alpha_4(g) = 30t + 16 - 11t$ and $\alpha_2(g) = 62 - 30t + 8t$;
4. $p = 2$, and either $s = 1, \Omega$ is $t$-clique, $t = 2, 4, 6, \alpha_3(g) = 2t, \alpha_4(g) = 20t - t + 6$ and $\alpha_2(g) = 72 - 20t - 2t$, or $s = 3, t \leq 8$, $t$ is even, $\alpha_3(g) = 0, \alpha_1(g) = 20t - 11t + 6$ and $\alpha_2(g) = 72 - 20t + 8t$.

**Corollary 1.** Let $\Gamma$ be a distance-regular graph with intersection array \{25, 16, 1; 1, 8, 25\} and a group $G = \text{Aut}(\Gamma)$ acts transitively on the set of vertices of $\Gamma$. Then one of the following assertions holds:

1. $\Gamma$ is a Cayley graph, $G$ is the a Frobenius group with the kernel of order 13 and with the complement of order 6;
2. $\Gamma$ is an arc-transitive Matoni’s graph and the socle of $G$ is isomorphic to $L_2(25)$;
3. $G$ is an extension of a group $Q$ of order 2^{12} by the group $T = L_2(3)$, $|Q : Q_{T_F}| = 2, T_{F_F}$ is an extension of group $E_9$ by $SL_2(3)$, $T$ acts irreducibly on $Q$ and for an element $f$ of order 13 in $G$ we have $C_Q(f) = 1$. 

1. Proof of the Theorem

Note that there is Delsarte boundary (proposition 4.4.6 from [7]) of maximum order of clique in distance-regular graph with intersection array \(\{25, 16, 1; 1, 8, 25\}\) and spectrum \(25^1, 5^{26}, -1^{25}, -5^{26}\) no more than \(1 - k/\theta_d = 1 + 25/5 = 6\). If \(C\) is 6-clique in \(\Gamma\), then each vertex not in \(C\) is adjacent to 0 or to \(b_1/(\theta_d + 1) = 1 - k/\theta_d = 2\) vertices in \(C\).

**Lemma 1.** Let \(\Gamma\) be a distance-regular graph with intersection array \(\{25, 16, 1; 1, 8, 25\}\), \(G = \text{Aut}(\Gamma)\) and \(g \in G\). If \(\psi\) is the monomial representation of a group \(G\) in \(GL(78, \mathbb{C})\), \(\chi_1\) is the character of the representation \(\psi\) on subspace of eigenvectors of dimension 26, corresponding to the eigenvalue 5, \(\chi_2\) is the character of the representation \(\psi\) on subspace of dimension 25, then \(\chi_1(g) = (10\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) - 5\alpha_3(g))/30\). Therefore \(\chi_1(g) = 78 - \alpha_0(g) - \alpha_1(g) - \alpha_3(g)\), we obtain \(\chi_1(g) = (11\alpha_0(g) + 3\alpha_1(g) - 4\alpha_3(g))/30 - 13/5\).

Similarly, \(\chi_2(g) = 25\alpha_0(g) - \alpha_1(g) - \alpha_2(g) + 25\alpha_3(g))/78\). Substituting \(\alpha_1(g) + \alpha_2(g) = 78 - \alpha_0(g) - \alpha_3(g),\) we obtain \(\chi_2(g) = (\alpha_0(g) + \alpha_3(g))/3 - 1\).

The remaining assertions follow from Lemma 1 in [8]. The proof is complete. \(\square\)

Let further in the paper \(\Gamma\) be a distance-regular graph with intersection array \(\{25, 16, 1; 1, 8, 25\}\), \(G = \text{Aut}(\Gamma)\), \(g\) is an element of prime order \(p\) in \(G\) and \(\Omega = \text{Fix}(g)\).

**Lemma 2.** If \(\Omega\) is an empty graph, then either \(p = 13\), \(\alpha_1(g) = 26\) and \(\alpha_2(g) = 52\), or \(p = 3\), \(\alpha_3(g) = 9s + 6, s < 8, \alpha_1(g) = 54 + 12s - 30l\) and \(\alpha_2(g) = 18 - 21s + 30l, l \leq 5\), or \(p = 2\), \(\alpha_3(g) = 0, \alpha_1(g) = 20l + 6\) and \(\alpha_2(g) = 72 - 20l, l \leq 3\).

**Proof.** Let \(\Omega\) be an empty graph and \(\alpha_i(g) = pw_i\) for \(i > 0\). Since \(v = 78\), we have \(p \in \{2, 3, 13\}\).

Let \(p = 13\). Then \(\alpha_3(g) = 0, \alpha_1(g) + \alpha_2(g) = 78\) and \(\chi_1(g) = (2\alpha_1(g) - \alpha_2(g))/30 = (w_1 - 2)/10\). This implies \(\alpha_1(g) = 26\) and \(\alpha_2(g) = 52\).

Let \(p = 3\). Then \(\chi_2(g) = 25 = \alpha_3(g)/3 - 26\) is divided by 3, \(\alpha_3(g) = 9s + 6, s \leq 8\) and \(\alpha_2(g) = 72 - 9s - \alpha_1(g)\). Furthermore, the number \(\chi_1(g) = (2\alpha_1(g) - \alpha_2(g))/45s - 30)/30 = (3w_1 - 12s - 34)/10\) is congruent to 2 modulo 3. This implies \(\alpha_1(g) = 54 + 12s - 30l\) and \(\alpha_2(g) = 18 - 21s + 30l, l \leq 5\). In case \(s = 8\) we have \(\alpha_3(g) = 78\) and \(\Omega\) acts regularly on each antipodal class. By lemma 4 in [9] 3 must divide \(k + 1 = 26\), we have a contradiction.

Let \(p = 2\). Then \(\alpha_3(g) = 0, \alpha_1(g) + \alpha_2(g) = 78\), the number \(\chi_1(g) = (\alpha_1(g) - 26)/10\) is even, \(\alpha_1(g) = 20l + 6\) and \(\alpha_2(g) = 72 - 20l, l \leq 3\). \(\square\)

In Lemmas 3–6 it is assumed that there are \(t\) antipodal classes intersecting the \(\Omega\) on \(s\) vertices. Then \(p\) divides \(26 - t\) and \(3 - s\). Let \(F\) be an antipodal class, containing the vertex \(a \in \Omega\), \(F \cap \Omega = \{a, a_2, \ldots, a_s\}, b \in \Omega(a)\). By \(F(x)\) we denote an antipodal class containing vertex \(x\).
Lemma 3. The following assertions hold:

1. If \( t = 1 \), then \( p = 5 \), \( \alpha_3(g) = 0 \), \( \alpha_1(g) = 50l + 25 \) and \( \alpha_2(g) = 50 - 50l \);
2. If \( p \) more than \( 3 \), then \( p = 5 \) and \( t = 1 \);
3. If \( s = 1 \), then \( p = 2 \), \( t = 2, 4, 6 \), \( \alpha_3(g) = 2t \), \( \alpha_1(g) = 20l - t + 6 \) and \( \alpha_2(g) = 72 - 20l - 2t \).

Proof. If \( s = 3 \), then each vertex from \( \Gamma - \Omega \) is adjacent to \( t \) vertices in \( \Omega \), so \( t \leq 8 \).

Let \( t = 1 \). As \( p \) divides \( 26 - t \), then \( p = 5 \), \( s = 3 \), \( \alpha_2(g) = 75 - \alpha_1(g) \), the number \( \chi_1(g) = (\alpha_1(g) - 15)/10 \) is congruent to 1 modulo 5. This implies \( \alpha_1(g) = 50l + 25 \).

Let \( p > 3 \), \( \alpha_1(g) = pw_1 \). Then \( s = 3 \), \( |\Omega| = 3t \), \( \Omega \) is a regular graph by degree \( t - 1 \) and \( p \) divides \( 26 - t \).

If \( p > 7 \), then \( \Omega \) is a distance-regular graph with intersection array \( \{t - 1, 16, 1; 1, 8, t - 1\} \), we come to a contradiction.

Let \( p = 7 \). As \( p \) divides \( 26 - t \), then \( p = 5 \), \( s = 3 \), \( \alpha_2(g) = 75 - \alpha_1(g) \), the number \( \chi_1(g) = (\alpha_1(g) - 15)/10 \) is congruent to 1 modulo 5. This implies \( \alpha_1(g) = 50l + 25 \).

Proof. Let \( p = 7 \). As \( p \) divides \( 26 - t \), then \( t = 1 \). If \( t = 6 \), then the subgraph \( \Omega(b) \) contains 2 vertices in \( a^1 \) and a vertex from \( [a_2] \) and from \( [a_3] \), so \( \Omega \) is a distance-regular graph with intersection array \( \{4, 1, 1; 1, 1, 4\} \), it is a contradiction with the fact that \( r = 3 \).

Let \( s = 1 \). Then \( p = 2 \), \( t \leq 6 \), \( \alpha_3(g) = 2t \), \( \alpha_2(g) = 78 - \alpha_1(g) - 3t \), and \( \chi_1(g) = (\alpha_1(g) + t - 26)/10 \)

Lemma 4. If \( p = 3 \), then \( s = 3 \), \( t = 2, 5, 8 \), \( \alpha_3(g) = 0 \), \( \alpha_1(g) = 30l + 16 - 11t \) and \( \alpha_2(g) = 62 - 30l + 8t \).

Proof. Let \( p = 3 \). Then \( s = 3 \), \( t = 2, 5, 8 \), \( \alpha_2(g) = 78 - \alpha_1(g) - 3t \), and the number \( \chi_1(g) = (11t + \alpha_1(g) - 26)/10 \) is congruent to 2 modulo 3. This implies that \( \alpha_1(g) = 30l + 16 - 11t \).

In the case \( t = 2 \) graph \( \Omega \) is a union of 3 isolated edges.

Lemma 5. If \( p = 2 \), \( s = 3 \), then \( t \) is even, \( t \leq 8 \), \( \alpha_3(g) = 0 \), \( \alpha_1(g) = 20l - 11t + 6 \) and \( \alpha_2(g) = 72 - 20l + 8t \).

Proof. Let \( p = 2 \), \( s = 3 \). Then \( t \) is even, \( t \leq 8 \), \( \alpha_3(g) = 0 \), \( \alpha_2(g) = 78 - 3t - \alpha_1(g) \).

The number \( \chi_1(g) = (11t + \alpha_1(g) - 26)/10 \) is even, so \( \alpha_1(g) = 20l - 11t + 6 \).

Lemmas 2–5 imply the proof of the Theorem.

2. Proof of Corollary

Let the group \( G \) acts transitively on the set of vertices of the graph \( \Gamma \). Then for a vertex \( a \in \Gamma \) subgroup \( H = G_a \) has index 78 in \( G \). By Theorem we have \( \{2, 3, 13\} \subseteq \pi(G) \subseteq \{2, 3, 5, 13\} \).

Lemma 6. Let \( f \) be an element of order 13 in \( G \). Then \( \text{Fix}(f) \) is an empty graph, \( \alpha_1(f) = 26 \) and the following assertions hold:

1. if \( g \) is an element of prime order \( p \neq 13 \) in \( C_G(f) \), then \( p = 2 \), \( \Omega \) is an empty graph, \( \alpha_1(g) = 26 \) and \( |C_G(f)| \) is not divided by 4;
2. either \( |G| = 78 \) or \( F(G) = \Omega_2(G) \);
3. if \( G \) is nonsolvable group, then the socle \( \bar{T} \) of the group \( \bar{G} = G/F(G) \) is isomorphic to \( L_2(25), L_3(3), U_3(4), L_4(3) \) or \( ^2F_4(2)' \).
Lemma 3 in [9] on 12-dimensional module over for the element \( T \) of order 13 of \( G \) we come to a contradiction. So, if \( 1 \leq p \leq 16 \) divided by 4 and an involution \( \alpha \) that \( |\langle \alpha \rangle| = 20 \) is an element of order 3 in \( C_G(g) \). From action \( h \) on \( \{ u \mid d(u, u^h) = 1 \} \) it follows that \( \alpha_1(g) = 20l + 6 \) is divided by 3. In each case \( \alpha_1(g) \) is not divided by 4 and \( |G| = 78 \).

If \( p = 3 \), then \( Q \) fixes some antipodal class. This implies that \( Q \) fixes each antipodal class. By Lemma 3 in [9] \( G \) does not contain subgroups of order 3, which are regular on each antipodal class, we come to a contradiction. So, if \( |G| \neq 78 \) we have \( F(G) = O_2(G) \).

Let \( T \) be the socle of the group \( G = G/F(G) \). Note that 13 divides \( |T| \) and by Theorem 1 in [10] group \( T \) is isomorphic to \( L_2(25), L_3(3), U_3(4), L_4(3), 2F_4(2)' \).

Let us to prove the Corollary. As \( T \) contains a subgroup of index dividing 26, then the group \( T \) is isomorphic to \( L_2(25) \) (and \( T_{1(F)} \) is the extension of a group of order 25 by group of order 12) or \( L_3(3) \) (and \( T_{1(F)} \) is the extension of a group of order 9 by \( SL(2,3) \)).

In the first case \( F(G) \) fixes each antipodal class and \( F(G) = 1 \). This implies that \( \Gamma \) is the arc-transitive Matoušek’s graph.

In the second case for \( Q = F(G) \) we have \( |Q : Q_{1(F)}| = 2 \) and \( T \) acts irreducibly on \( Q \). Further, for the element \( f \) of order 13 of \( G \) by Lemma 6 the number \( |C_Q(f)| \) divides 2. As \( Q \) is either 12-dimensional module over \( F_2 \), or 16-dimensional module over \( F_{16} \), or 26-dimensional module over \( F_2 \), then \( |Q| = 2^{12} \) and \( C_Q(f) = 1 \). The Corollary is proved.

3. Conclusion

We found possible automorphisms of a distance regular graph with intersection array \( \{25, 16, 1; 1, 8, 25\} \). This completes the research program of vertex-symmetric antipodal distance-regular graphs of diameter 3 with \( \lambda = \mu \), in which neighborhoods of vertices are strongly regular with parameters from Proposition 1.

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