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ESTIMATES OF BEST APPROXIMATIONS OF FUNCTIONS WITH LOGARITHMIC SMOOTHNESS IN THE LORENTZ SPACE WITH ANISOTROPIC NORM[1]

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Abstract: In this paper, we consider the anisotropic Lorentz space $L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ of periodic functions of m variables. The Besov space $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}$ of functions with logarithmic smoothness is defined. The aim of the paper is to find an exact order of the best approximation of functions from the class $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}$ by trigonometric polynomials under different relations between the parameters $\bar{p},\bar{\theta}$, and τ.

The paper consists of an introduction and two sections. In the first section, we establish a sufficient condition for a function $f\in L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$ to belong to the space $L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ in the case $1<\theta^2<\theta_j^{(1)}$, $j=1,\ldots,m$, in terms of the best approximation and prove its unimprovability on the class $E^{\lambda}_{\bar{p},\bar{\theta}}=\{f\in L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)\colon E_n(f)_{\bar{p},\bar{\theta}}\le\lambda_n,$ $n=0,1,\ldots\}$, where $E_n(f)_{\bar{p},\bar{\theta}}$ is the best approximation of the function $f\in L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ by trigonometric polynomials of order n in each variable x_j, $j=1,\ldots,m$, and $\lambda=\{\lambda_n\}$ is a sequence of positive numbers $\lambda_n\downarrow 0$ as $n\to+\infty$. In the second section, we establish order-exact estimates for the best approximation of functions from the class $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ in the space $L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$.

Key words: Lorentz space, Nikol'skii–Besov class, Best approximation.

1. Introduction

Let $\bar{x}=(x_1,\ldots,x_m)\in\mathbb{R}^m$, $\mathbb{I}^m=[0,2\pi]^m$, $\bar{p}=(p_1,\ldots,p_m)$, and $\bar{\theta}=(\theta_1,\ldots,\theta_m)$, where $p_j\in(1,\infty)$ and $\theta_j\in[1,\infty)$ for $j=1,2,\ldots,m$. Denote by $L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ the Lorentz space of real-valued functions $f(\bar{x})$ that are 2π-periodic in each variable and

$$\|f\|^*_{\bar{p},\bar{\theta}}=\left\{\int_0^{2\pi}t_m^{\frac{\theta_m}{p_m}-1}\left[\ldots\left[\int_0^{2\pi}\left(f^{*_1,\ldots,*_m}(t_1,\ldots,t_m)\right)^{\theta_1}t_1^{\frac{\theta_1}{p_1}-1}dt_1\right]^{\frac{\theta_2}{\theta_1}}\ldots\right]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\right\}^{1/\theta_m}<+\infty,$$

where $f^{*_1,\ldots,*_m}$ is a nonincreasing rearrangement of the function $|f(x_1,\ldots,x_m)|$ in each of the variables x_j whereas the other variables are fixed (see [8, 18]).

In the case $p_1=\cdots=p_m=\theta_1=\cdots=\theta_m=p$, the Lorentz space $L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ coincides with the Lebesgue space $L_p(\mathbb{I}^m)$ with the norm

$$\|f\|_p=\left[\int_0^{2\pi}\ldots\int_0^{2\pi}|f(x_1,\ldots,x_m)|^p dx_1\ldots dx_m\right]^{1/p},$$

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where $p \in [1, +\infty)$.

Instead of $L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$, we will write $L^*_{p,\theta}(\mathbb{I}^m)$ in the case $p_1 = \cdots = p_m = p$ and $\theta_1 = \cdots = \theta_m = \theta$ and $L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ if $\overline{p} = (p_1, \ldots, p_m)$ and $\theta_1 = \cdots = \theta_m = \theta^{(2)}$.

Given a natural number M, consider the set

$$\square_M = \{\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m \ : |k_j| < M, \ j = 1, \ldots, m\}.$$

Consider the multiple Dirichlet kernel

$$D_{\square_M}(\bar{x}) = \sum_{\bar{k} \in \square_M} e^{i\langle \bar{k}, \bar{x} \rangle}, \quad \bar{x} \in \mathbb{I}^m,$$

and its convolution with a function $f \in L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$:

$$\sigma_s(f, \bar{x}) = \int_{\mathbb{I}^m} f(\bar{y})(D_{\square_{2^s}}(\bar{x} - \bar{y}) - D_{\square_{2^{s-1}}}(\bar{x} - \bar{y}))d\bar{y},$$

where $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of positive integers.

Let $M \in \mathbb{N}_0$, and let $T_M(\bar{x}) = \sum\limits_{\bar{k} \in \square_M} a_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle}$ be a trigonometric polynomial of order M in each variable $x_j, \ j = 1, \ldots, m$. Denote by $\mathfrak{T}_{\square_M}$ the set of all such polynomials.

Let $E_{M,\ldots,M}(f)_{\bar{p},\bar{\theta}} = \inf\limits_{T \in \mathfrak{T}_{\square_M}} ||f - T||^*_{\bar{p},\bar{\theta}}$ be the best approximation of a function $f \in L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ by the set $\mathfrak{T}_{\square_M}$. Sometimes, we will use the notation $E_M(f)_{\bar{p},\bar{\theta}}$ instead of $E_{M,\ldots,M}(f)_{\bar{p},\bar{\theta}}$. For a given class $F \subset L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$, let $E_M(F)_{\bar{p},\bar{\theta}} = \sup\limits_{f \in F} E_M(f)_{\bar{p},\bar{\theta}}$.

Let $\alpha \geq 0$, $\gamma \in (-\infty, +\infty)$, and $0 < \tau < \infty$. Denote by $\mathbb{A}^{(\alpha,\gamma,\tau)}_{\bar{p},\bar{\theta}}$ the space of all functions $f \in L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ such that the quasi-norm (see [9, 20])

$$\|f\|_{\mathbb{A}^{(\alpha,\gamma,\tau)}_{\bar{p},\bar{\theta}}} = \Big[\sum_{n=1}^{\infty} n^{-1}\big(n^{\alpha}(1 + \log n)^{\gamma} E_n(f)_{\bar{p},\bar{\theta}}\big)^{\tau}\Big]^{1/\tau}$$

is finite, where $\log a$ is the logarithm of the number a to the base 2.

If $\tau = \infty$, then

$$\|f\|_{\mathbb{A}^{\alpha,\gamma,\tau}_{\bar{p},\bar{\theta}}} = \sup_{n \geq 1} n^{\alpha}(1 + \log n)^{\gamma} E_n(f)_{\bar{p},\bar{\theta}} < \infty.$$

It is known that $\mathbb{A}^{(\alpha,\gamma,\tau)}_{\bar{p},\bar{\theta}}$ is a quasi-Banach space (see [9, 10, 20]). It is called an approximate space (see [11]).

In the anisotropic Lorentz space, we consider the space $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}$, $1 \leq \tau \leq \infty$, of all functions $f \in L^*_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ representable in the form of series

$$\sum_{n=0}^{\infty} Q_{2^{2^n}}(f, \bar{x}), \quad Q_{2^{2^n}}(f) \in \mathfrak{T}_{\square_{2^{2^n}}} \tag{1.1}$$

and such that

$$\Big[\sum_{n=0}^{\infty} \big(2^{n\alpha} \|Q_{2^{2^n}}(f)\|^*_{\bar{p},\bar{\theta}}\big)^{\tau}\Big]^{1/\tau} < +\infty \tag{1.2}$$

for $1 \leq \tau < \infty$ and

$$\sup_{n \in \mathbb{N}_0} 2^{n\alpha} \|Q_{2^{2^n}}(f)\|^*_{\bar{p},\bar{\theta}} < \infty$$

for $\tau=\infty$. The infimum of expression (1.2) over all representations (1.1) defines a quasi-norm in this space:

$$\|f\|_{B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}}=\inf\Big[\sum_{n=0}^{\infty}\big(2^{n\alpha}\|Q_{2^{2^n}}(f)\|^{*}_{\bar{p},\bar{\theta}}\big)^{\tau}\Big]^{1/\tau}.$$

The space $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}$ is called the Besov space with logarithmic smoothness. In $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}$, we consider the unit ball

$$\mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}=\Big\{f\in L^{*}_{\bar{p},\bar{\theta}}(\mathbb{I}^m):\ \|f\|_{B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}}\le 1\Big\}.$$

It is known that $f\in\mathbb{B}^{(0,\gamma+1/\tau,\tau)}_{\bar{p},\bar{\theta}}$ if and only if $f\in\mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}}$ (see [10]).

The main aim of the present paper is to obtain an exact order of the best approximation of the function classes $\mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ and $\mathbb{B}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ in anisotropic Lorentz spaces.

In the one-dimensional case, sufficient conditions for a function $f\in L_p(I^1)$ to belong to the space $L_q(\mathbb{I}^1)$ for $1\le p<q<\infty$ in terms of the best approximation and the modulus of continuity were established by P.L. Ul'ynov [30]. This study was continued by V.I. Kolyada [15], V.A. Andrienko [5], N. Temirgaliev [27, 28], E.A. Storozhenko [26], M.F. Timan, P. Oswald, L. Leindler, S.V. Lapin, B.V. Simonov, and others (see the references in [16]).

N. Temirgaliev established [28] a necessary and sufficient condition for a univariate function $f\in L_p(\mathbb{I}^1)$ to belong to the Lorentz space $L_{q,\theta}(\mathbb{I}^1)$ in terms of the modulus of continuity for $1\le\theta<p<\infty$. L.A. Sherstneva studied [22] this problem in terms of the best approximation of a function. Such problems in the Lorentz space were investigated in [1, 4, 23].

Problems of estimating various approximative characteristics of function classes are well known and a survey of the results on this topic is given in [12, 29]. In particular, in the Lebesgue space $L_p(\mathbb{I}^m)$, exact estimates of the best approximation of functions of the Besov class $B^r_{p,\bar{\theta}^{(1)}}$ were established by A.S. Romanyuk [21]. In the case $\theta^{(1)}_j=p_j=p,\ j=1,\dots,m$, estimates of approximative characteristics of the class $\mathbb{B}^{0,\alpha}_{\bar{p},\bar{\theta}^{(1)}}$ were obtained by S.A. Stasyuk [24, 25]. In [13], the embedding and characterization problems of the Besov space with logarithmic smoothness in the Lebesgue space $L_p(\mathbb{I}^m)$ were investigated.

Exact estimates of best approximations of functions from the Besov class in the Lorentz space with a mixed norm were obtained in [2, 6, 7].

The present paper consists of the introduction and two sections. In Section 1, we establish a sufficient condition for a function $f\in L^{*}_{\bar{p},\bar{\theta}}(\mathbb{I}^m)$ to belong to the space $L^{*}_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$, $\theta^{(2)}<\theta^{(1)}_j$, $j=1,\dots,m$, and prove its accuracy on the class

$$E^{\lambda}_{\bar{p},\bar{\theta}}=\big\{f\in L^{*}_{\bar{p},\bar{\theta}}(\mathbb{I}^m):\ E_n(f)_{\bar{p},\bar{\theta}}\le\lambda_n,\ n=0,1,\dots\big\},$$

where $\lambda=\{\lambda_n\}$ is a sequence of positive numbers $\lambda_n\downarrow 0$ as $n\to+\infty$.

In the case $p_j=\theta_j=p,\ j=1,\dots,m$, V.I. Kolyada proved [15] a necessary and sufficient condition for the embedding of classes E^{λ}_p in the space $L_q(\mathbb{I}^1)$, $1\le p<q$.

In Section 2, we establish order-exact estimates of the value $E_n(\mathbb{B}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}})_{\bar{q},\bar{\theta}^{(2)}}$ under various relations between coordinates of the parameters $\bar{p},\bar{\theta}^{(1)},\bar{q},\bar{\theta}^{(2)},\tau$ (see Theorems 5 and 6).

The notation $A(y)\asymp B(y)$ means that there exists positive constants C_1 and C_2 such that $C_1A(y)\le B(y)\le C_2A(y)$. If $B(y)\le C_2A(y)$ or $A(y)\ge C_1B(y)$, then we write $B(y)<<A(y)$ and $A(y)>>B(y)$, respectively.

2. Conditions for embedding classes in the Lorentz space

Theorem 1 [19, Theorem 10]. *Let $1 \le p_j < +\infty$ and $1 \le \theta_j < q_j < +\infty$ for $j = 1, \dots, m$, let $\bar{p} = (p_1, \dots, p_m)$ and $\bar{q} = (q_1, \dots, q_m)$, and let $\bar{\theta} = (\theta_1, \dots, \theta_m)$. Then a trigonometric polynomial*

$$T_{\bar{n}}(\bar{x}) = \sum_{k_1=-n_1}^{n_1} \dots \sum_{k_m=-n_m}^{n_m} b_{\bar{k}} e^{i\langle \bar{x}, \bar{k}\rangle}$$

satsfies the following inequality:

$$\|T_{\bar{n}}\|^*_{\bar{p},\bar{\theta}} \le C(p,q,\theta) \prod_{j=1}^{m} (\ln(1+n_j))^{1/\theta_j - 1/q_j} \|T_{\bar{n}}\|^*_{\bar{p},\bar{q}}.$$

Lemma 1. *Let $1 < p_j < \infty$ and $1 < q_2 < q_j^{(1)} < +\infty$ for $j = 1, \dots, m$. Let $\{u_n\}$ be a sequence of non-negative measurable functions on the cube $\mathbb{I}^m = [0, 2\pi]^m$ such that*

(1)

$$\|u_n\|^*_{\bar{p},\bar{q}^{(1)}} \le \varepsilon_n, \quad \varepsilon_{n+1} \le \beta\varepsilon_n, \quad \beta \in (0,1);$$

(2) *there exists a sequence of positive numbers $\{\Delta_n\}$ such that*

$$\|u_n\|^*_{p,\theta} \le C\Delta_n^{\sum_{j=1}^{m}(1/\theta_j - 1/q_j^{(1)})} \varepsilon_n, \quad n = 1, 2, 3, \dots,$$

for any $\theta_j \in (0, q_j^{(1)})$, $j = 1, \dots, m$.

Then the inequality

$$\|f\|^*_{p,q_2} \le C\Big\{\sum_{n=1}^{\infty} \Delta_n^{\sum_{j=1}^{m}(1/q_2 - 1/q_j^{(1)})} \varepsilon_n^{q_2}\Big\}^{1/q_2}$$

holds for every function of the form $f(\bar{x}) = \sum_{n=1}^{\infty} u_n(\bar{x})$.

This lemma is proved by V.I. Kolyada's method (see [15, Proof of Lemma 4]) as in [3].

Remark 1. Lemma 1 was proved by L.A. Sherstneva [22, Lemma 13] in the one-dimensional case and by the author [3] in the multi-dimensional case for $q_1^{(1)} = \dots = q_m^{(1)}$.

Now, let us consider a condition for a function $f \in L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$ to belong to the space $L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$, $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty, j = 1, \dots, m$.

Theorem 2. *Let $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty$ and $1 < p_j < \infty$ for $j = 1, \dots, m$, and let $\bar{\theta}^{(1)} = (\theta_1^{(1)}, \dots, \theta_m^{(1)})$. Assume that $f \in L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$ and*

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E^{\theta^{(2)}}_{n,\dots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} < +\infty. \tag{2.1}$$

Then $f \in L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ and

$$\|f\|^*_{\bar{p},\theta^{(2)}} << \Big\{\|f\|^*_{\bar{p},\theta^{(1)}} + \Big[\sum_{k=2}^{\infty} \frac{(\ln(k+1))^{\theta^{(2)} \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{k} E^{\theta^{(2)}}_{k,\dots,k}(f)_{\bar{p},\bar{\theta}^{(1)}}\Big]^{1/\theta^{(2)}}\Big\}, \tag{2.2}$$

$$E_{n,\dots,n}(f)_{\bar{p},\theta^{(2)}} << \Big\{(\ln(n+1))^{\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})} E_{n,\dots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} + \\ + \Big[\sum_{k=n+1}^{\infty} \frac{(\ln(k+1))^{\theta^{(2)}\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})-1}}{k} E^{\theta^{(2)}}_{k,\dots,k}(f)_{\bar{p},\bar{\theta}^{(1)}}\Big]^{1/\theta^{(2)}}\Big\}. \tag{2.3}$$

P r o o f. Since $E_{n,\dots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} \equiv \varepsilon_n \downarrow 0$ as $n \to +\infty$ for every function $f \in L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$, $1 < p_j, \theta_j^{(1)} < +\infty$, $j = 1,\dots,m$, there exists a numerical sequence $\{n_\nu\}$ such that (see [15, Sect. 2])

$$\varepsilon_{n_{\nu+1}} < \frac{1}{2}\varepsilon_{n_\nu}, \quad \varepsilon_{n_{\nu+1}-1} \geq \frac{1}{2}\varepsilon_{n_\nu}, \quad \nu = 1,2,\dots .$$

Let $T_n(f,\bar{x})$ be a trigonometric polynomial of the best approximation of a function $f \in L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$, $1 < p_j, \theta_j^{(1)} < +\infty$, $j = 1,\dots,m$. Consider the series

$$T_{n_1}(f,\bar{x}) + \sum_{\nu=1}^{\infty}(T_{n_{\nu+1}}(f,\bar{x}) - T_{n_\nu}(f,\bar{x})). \tag{2.4}$$

Let us prove the convergence of this series in the norm of the space $L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$. Suppose that

$$u_\nu(\bar{x}) = |T_{n_{\nu+1}}(f,\bar{x}) - T_{n_\nu}(f,\bar{x})|, \quad \nu = 0,1,\dots .$$

Then

$$\|u_\nu\|^*_{\bar{p},\bar{\theta}^{(1)}} \leq 2\varepsilon_\nu, \quad \nu = 0,1,\dots ,$$

and, by Theorem 1,

$$\|u_\nu\|^*_{\bar{p},\bar{\tau}} << (\ln n_{\nu+1})^{\sum\limits_{j=1}^{m}(1/\tau_j-1/\theta_j^{(1)})}\varepsilon_\nu$$

for any $\tau_j \in (0,\theta_j^{(1)})$, $j = 1,\dots,m$. Hence, by Lemma 1, we obtain

$$\Big\|\sum_{\nu=k+1}^{l}(T_{n_{\nu+1}}(f) - T_{n_\nu}(f))\Big\|^*_{\bar{p},\theta^{(2)}} \leq \Big\|\sum_{\nu=k+1}^{l} u_\nu\Big\|^*_{\bar{p},\theta^{(2)}} << \\ << \Big\{\sum_{\nu=k+1}^{l}(\ln n_{\nu+1})^{\theta^{(2)}\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})}\varepsilon_\nu^{\theta^{(2)}}\Big\}^{1/\theta^{(2)}}. \tag{2.5}$$

Condition (2.1) implies that

$$\sum_{\nu=1}^{\infty}(\ln n_{\nu+1})^{\theta^{(2)}\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})}\varepsilon_{n_\nu}^{\theta^{(2)}} < +\infty. \tag{2.6}$$

It follows from (2.5) and (2.6) that series (2.4) converges to a function $g \in L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ in the norm. It is easy to see that $g(\bar{x}) = f(\bar{x})$ almost everywhere on $\mathbb{I}^m$. Hence, $f \in L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$. Setting $k = 0$ in (2.5), we get

$$\|T_{n_{l+1}}(f)\|^*_{\bar{p},\theta^{(2)}} << \Big[\|f\|^*_{\bar{p},\bar{\theta}^{(1)}} + \sum_{\nu=1}^{l}(\ln n_{\nu+1})^{\theta^{(2)}\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})}\varepsilon_\nu^{\theta^{(2)}}\Big]^{1/\theta^{(2)}} <<$$

$$<< \left\{ \|f\|^*_{\bar{p},\bar{\theta}^{(1)}} + \left[\sum_{n=2}^{\infty} \frac{(\ln(n+1))^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E^{\theta^{(2)}}_{n,\ldots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} \right]^{1/\theta^{(2)}} \right\}.$$

By tending l to $+\infty$ in this inequality, we obtain

$$\|f\|^*_{\bar{p},\theta^{(2)}} << \left\{ \|f\|^*_{\bar{p},\bar{\theta}^{(1)}} + \left[\sum_{n=2}^{\infty} \frac{(\ln(n+1))^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E^{\theta^{(2)}}_{n,\ldots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} \right]^{1/\theta^{(2)}} \right\}.$$

Thus, inequality (2.2) is proved.

Applying inequality (2.2) to the function $f - T_n(f) \in L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$, it is easy to prove inequality (2.3). The proof of Theorem 2 is complete. □

Let us prove that condition (2.1) is exact on the classes $E^{\lambda}_{\bar{p},\bar{\theta}^{(1)}}$.

Theorem 3. *Let $1 < p_j < \infty$ and $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty$ for $j = 1, \ldots, m$. The following condition is necessary and sufficient for the inclusion $E^{\lambda}_{\bar{p},\bar{\theta}^{(1)}} \subset L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$:*

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} \lambda_n^{\theta^{(2)}} < +\infty. \tag{2.7}$$

P r o o f. The sufficiency of condition (2.7) follows from Theorem 2. Let us prove the necessity. Let $E^{\lambda}_{\bar{p},\bar{\theta}^{(1)}} \subset L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$. Assume that condition (2.7) is violated, i.e.,

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} \lambda_n^{\theta^{(2)}} = +\infty. \tag{2.8}$$

We choose a sequence of numbers $\{\nu_k\}$ with the following properties (see [15]):

$$\lambda_{\nu_{k+1}} < \frac{1}{2}\lambda_{\nu_k}, \quad \lambda_{\nu_{k+1}-1} \geq \frac{1}{2}\lambda_{\nu_k}. \tag{2.9}$$

Since the function $(\ln x)^{\beta}/x$ with $\beta \in \mathbb{R}$ decreases to 0 as $x \to +\infty$, we have

$$\sum_{n=\nu_k+1}^{\nu_{k+1}} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} \leq \sum_{n=\nu_k+1}^{\nu_{k+1}} \frac{(\ln(n-\nu_k+1))^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n-\nu_k} <<$$

$$<< (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})}.$$

Thus, (2.8) implies that

$$\sum_{k=1}^{\infty} (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}} = +\infty. \tag{2.10}$$

Let us consider the function

$$f_0(\bar{x}) = \sum_{k=0}^{\infty} \lambda_{\nu_k} (\ln(\nu_{k+1} - \nu_k + 1))^{-\sum_{j=1}^{m} 1/\theta_j^{(1)}} \tau_k(\bar{x}),$$

where

$$\tau_k(\bar{x}) = \prod_{j=1}^{m} \sum_{n_j=\nu_k+1}^{\nu_{k+1}} (n_j - \nu_k)^{\frac{1}{p_j}-1} \sin n_j x_j.$$

It is known that (see [22])

$$\|\tau_k\|^*_{\bar{p},\bar{\theta}^{(1)}} \asymp (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} 1/\theta_j^{(1)}}, \quad 1 < p_j, \theta_j^{(1)} < +\infty, \quad j = 1, \dots, m. \tag{2.11}$$

Using this relation and (2.9), we can verify that

$$\|f_0\|^*_{\bar{p},\bar{\theta}^{(1)}} \leq \sum_{k=0}^{\infty} \lambda_{\nu_k} (\ln(\nu_{k+1} - \nu_k + 1))^{-\sum\limits_{j=1}^{m} 1/\theta_j^{(1)}} \|\tau_k\|^*_{\bar{p},\bar{\theta}^{(1)}} \leq C \sum_{k=0}^{\infty} \lambda_{\nu_k} < \infty.$$

Hence, $f_0 \in L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$, $1 < p_j, \theta_j^{(1)} < \infty$, $j = 1, \dots, m$.

Let a positive integer n satisfy the inequalities $\nu_l \leq n < \nu_{l+1}$. Then, by the best approximation property and according to relation (2.11) and inequality (2.9), we have

$$E_n(f_0)_{\bar{p},\bar{\theta}^{(1)}} \leq E_{\nu_l}(f_0)_{\bar{p},\bar{\theta}^{(1)}} \leq \sum_{k=l}^{\infty} \lambda_{\nu_k} (\ln(\nu_{k+1} - \nu_k + 1))^{-\sum\limits_{j=1}^{m} 1/\theta_j^{(1)}} \|\tau_k\|^*_{\bar{p},\bar{\theta}^{(1)}} <<$$

$$<< \sum_{k=l}^{\infty} \lambda_{\nu_k} << \lambda_{\nu_l} << 2\lambda_{\nu_{l+1}-1} \leq C_0 \lambda_n.$$

Hence, $f_1 = C_0^{-1} f_0 \in E^{\lambda}_{\bar{p},\bar{\theta}^{(1)}}$.

Let us show that $f_1 \notin L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$, $1 < \theta^{(2)} < \infty$. To this end, we consider the function

$$g_0(\bar{x}) = \sum_{k=0}^{\infty} (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} \xi_k(\bar{x}),$$

where

$$\xi_k(\bar{x}) = \prod_{j=1}^{s} \sum_{n_j=\nu_k+1}^{\nu_{k+1}} (n_j - \nu_k)^{\frac{1}{p'_j}-1} \sin n_j x_j, \quad p'_j = \frac{p_j}{p_j - 1}, \quad j = 1, \dots, m.$$

It is clear that (see (2.11))

$$\|\xi_k\|^*_{\bar{p}',\bar{\theta}} \asymp (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} 1/\theta_j}, \quad 1 < p_j < +\infty, \quad 1 < \theta_j < \infty, \quad j = 1, \dots, m.$$

Further, in view of the orthogonality of the trigonometric system, for any number N, we have

$$B_N \equiv \int\limits_{\mathbb{I}^m} f_1(\bar{x}) \sum_{k=0}^{N} \lambda_{\nu_k}^{\theta^{(2)}-1} (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \xi_k(\bar{x}) d\bar{x} =$$

$$= C \sum_{k=0}^{N} [\ln(\nu_{k+1} - \nu_k + 1)]^{-\theta^{(2)} \sum\limits_{j=1}^{m} 1/\theta_j^{(1)}} \lambda_{\nu_k}^{\theta^{(2)}} \prod_{j=1}^{m} \sum_{n_j=\nu_k+1}^{\nu_{k+1}} \frac{1}{n_j - \nu_k} >> \tag{2.12}$$

$$>> \sum_{k=0}^{N} [\ln(\nu_{k+1} - \nu_k + 1)]^{\theta^{(2)} \sum\limits_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}}.$$

Using the integral Hölder inequality, we obtain

$$B_N << \|f_1\|^*_{\bar{p},\theta^{(2)}} \left\| \sum_{k=0}^{N} (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} \xi_k \right\|^*_{\bar{p}',\theta^{(2)\prime}}, \tag{2.13}$$

where

$$\theta^{(2)\prime} = \frac{\theta^{(2)}}{\theta^{(2)} - 1}.$$

We set $u_k(\bar{x}) = (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} |\xi_k(\bar{x})|$. Then (see (2.11))

$$\|u_k\|^*_{\bar{p}',\frac{\bar{\theta}^{(1)}}{\theta^{(2)}-1}} << \lambda_{\nu_k}^{\theta^{(2)}-1} \equiv \beta_k,$$

$$\|u_k\|^*_{\bar{p}',\bar{\tau}} << [\ln(\nu_{k+1} - \nu_k + 1)]^{\sum\limits_{j=1}^{m} (\frac{1}{\tau_j} - \frac{\theta^{(2)}-1}{\theta_j^{(1)}})} \beta_k, \quad k = 0, 1, \dots .$$

Thus, all the conditions of Lemma 1 hold for the sequence of functions $\{u_k(\bar{x})\}$. Therefore,

$$\begin{aligned} &\left\| \sum_{k=0}^{N} (\ln(\nu_{k+1} - \nu_k + 1))^{\sum\limits_{j=1}^{m} \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} \xi_k \right\|^*_{\bar{p}',\theta^{(2)\prime}} << \\ &<< \left\{ \sum_{k=0}^{N} (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum\limits_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}} \right\}^{1-1/\theta^{(2)}}. \end{aligned} \tag{2.14}$$

Now, it follows from inequalities (2.12), (2.13), and (2.14) that

$$\left\{ \sum_{k=0}^{N} (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum\limits_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}} \right\}^{1/\theta^{(2)}} << \|f_1\|^*_{\bar{p},\theta^{(2)}}.$$

By (2.10), we find that $f_1 \notin L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$, $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty$, $j = 1, \dots, m$. This contradicts the inclusion $E^{\lambda}_{\bar{p},\bar{\theta}^{(1)}} \subset L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$. The proof of Theorem 3 is complete. □

Remark 2. The results of L.A. Sherstneva [22] follow from Theorems 2 and 3 in the case $m = 1$.

3. Estimates of best approximations of functions with logarithmic smoothness

Now, let us prove estimates of the value $E_M(F)_{\bar{p},\bar{\theta}^{(2)}}$ for the classes $F = \mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ and $F = \mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$.

Theorem 4. *Let* $1 < p_j < \infty$ *and* $1 \le \theta^{(2)} < \theta_j^{(1)} < \infty$ *for* $j = 1, \dots, m$, *and let* $1 \le \tau \le \infty$. *If* $\alpha > \sum\limits_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})$, *then* $B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}} \subset L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ *and*

$$\|f\|^*_{\bar{p},\theta^{(2)}} << \|f\|_{B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}}}.$$

P r o o f. Let $f \in B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$. Then, by the definition of the class, this function can be represented in the form of the series

$$\sum_{\nu=0}^{\infty} Q_{2^{2^\nu}}(f,\bar{x}), \quad Q_{2^{2^\nu}}(f,\bar{x}) \in \mathfrak{F}_{\square_{2^{2^n}}},$$

in the sense of convergence in the quasi-norm of the space $L^*_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$ and

$$\Big[\sum_{\nu=0}^{\infty} \big(2^{\nu\alpha}\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}}\big)^{\tau}\Big]^{1/\tau} < +\infty.$$

If $\theta^{(2)} < \tau < \infty$, then, using the Hölder inequality and taking into account that $\alpha > \sum\limits_{j=1}^{m} (1/\theta^{(2)} - 1/\theta^{(1)}_j)$, we obtain

$$\begin{gathered}\Big\{\sum_{\nu=0}^{\infty} 2^{\nu\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta^{(1)}_j)\theta^{(2)}} \big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\theta^{(2)}}\Big\}^{1/\theta^{(2)}} \leq \\ \leq \Big\{\sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha}\big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\tau}\Big\}^{1/\tau}\Big\{\sum_{\nu=0}^{\infty} 2^{\nu\theta^{(2)}\beta'(\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta^{(1)}_j)-\alpha)}\Big\}^{\frac{1}{\theta^{(2)}\beta'}} \leq \\ \leq C\Big\{\sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha}\big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\tau}\Big\}^{1/\tau},\end{gathered} \tag{3.1}$$

where

$$\beta = \frac{\tau}{\theta^{(2)}}, \quad \beta' = \frac{\beta}{\beta - 1}.$$

If $\tau = \infty$, then

$$\begin{gathered}\Big\{\sum_{\nu=0}^{\infty} 2^{\nu\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta^{(1)}_j)\theta^{(2)}} \big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\theta^{(2)}}\Big\}^{1/\theta^{(2)}} \leq \\ \leq \sup_{\nu\in\mathbb{N}_0} 2^{\nu\alpha}\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\Big\{\sum_{\nu=0}^{\infty} 2^{\nu\theta^{(2)}(\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta^{(1)}_j)-\alpha)}\Big\}^{1/\theta^{(2)}}.\end{gathered} \tag{3.2}$$

If $\tau \leq \theta^{(2)}$, then, using the Jensen inequality (see [17, p. 125]), we obtain

$$\Big\{\sum_{\nu=0}^{\infty} 2^{\nu\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta^{(1)}_j)\theta^{(2)}} \big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\theta^{(2)}}\Big\}^{1/\theta^{(2)}} \leq \Big\{\sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha}\big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\tau}\Big\}^{1/\tau}. \tag{3.3}$$

Thus, (3.1)–(3.3) imply that the series

$$\sum_{\nu=0}^{\infty} 2^{\nu\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta^{(1)}_j)\theta^{(2)}} \big(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}}\big)^{\theta^{(2)}} \tag{3.4}$$

is convergent for every function $f \in B^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$.

Taking into account the monotonicity of the best approximation and the properties of the norm, it is easy to verify that

$$\sum_{n=2}^{\infty}\frac{(\ln n)^{\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})-1}}{n}E_{n,\ldots,n}^{\theta^{(2)}}(f)_{\bar p,\bar\theta^{(1)}} << \sum_{\nu=0}^{\infty}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}E_{2^{2^\nu},\ldots,2^{2^\nu}}^{\theta^{(2)}}(f)_{\bar p,\bar\theta^{(1)}} <<$$
$$<< \sum_{\nu=0}^{\infty}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}\Big(\Big\|\sum_{l=\nu}^{\infty}Q_{2^{2^l}}(f)\Big\|_{\bar p,\bar\theta^{(1)}}^{*}\Big)^{\theta^{(2)}} << \tag{3.5}$$
$$<< \sum_{\nu=0}^{\infty}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}\Big(\sum_{l=\nu}^{\infty}\|Q_{2^{2^l}}(f)\|_{\bar p,\bar\theta^{(1)}}^{*}\Big)^{\theta^{(2)}}.$$

Since $\theta^{(2)} < \theta_j^{(1)}$, $j = 1,\ldots,m$, we have

$$\sum_{\nu=0}^{n}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}} << 2^{n\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}, \quad n\in\mathbb{N}_0.$$

Therefore, according to [14, Lemma 2.2], we find from (3.5) that

$$\sum_{n=2}^{\infty}\frac{(\ln n)^{\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}-1}}{n}E_{n,\ldots,n}^{\theta^{(2)}}(f)_{\bar p,\bar\theta^{(1)}} << \sum_{\nu=0}^{\infty}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}\big(\|Q_{2^{2^\nu}}(f)\|_{\bar p,\bar\theta^{(1)}}^{*}\big)^{\theta^{(2)}}. \tag{3.6}$$

Since the series (3.4) converges, it follows from (3.6) that

$$\sum_{n=2}^{\infty}\frac{(\ln n)^{\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}-1}}{n}E_{n,\ldots,n}^{\theta^{(2)}}(f)_{\bar p,\bar\theta^{(1)}} < \infty.$$

Hence, by Theorem 3, we have $f\in L_{\bar p,\theta^{(2)}}^{*}(\mathbb{I}^m)$.

Let us estimate the quasi-norm $\|f\|_{\bar p,\bar\theta^{(1)}}^{*}$. By the quasi-norm property and the Hölder inequality, we obtain

$$\|f\|_{\bar p,\bar\theta^{(1)}}^{*} << \sum_{\nu=0}^{\infty}\|Q_{2^{2^\nu}}(f)\|_{\bar p,\bar\theta^{(1)}}^{*} <<$$
$$<< \Big(\sum_{\nu=0}^{\infty}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}\big(\|Q_{2^{2^\nu}}(f)\|_{\bar p,\bar\theta^{(1)}}^{*}\big)^{\theta^{(2)}}\Big)^{1/\theta^{(2)}}. \tag{3.7}$$

Therefore, according to relations (2.2), (3.6), and (3.7), we have

$$\|f\|_{\bar p,\bar\theta^{(1)}}^{*} << \Big\{\sum_{\nu=0}^{\infty}2^{\nu\sum_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})\theta^{(2)}}\big(\|Q_{2^{2^\nu}}(f)\|_{\bar p,\bar\theta^{(1)}}^{*}\big)^{\theta^{(2)}}\Big\}^{1/\theta^{(2)}}. \tag{3.8}$$

Taking into account (3.1)–(3.3) and (3.8), we obtain

$$\|f\|_{\bar p,\theta^{(2)}}^{*} << \Big\{\sum_{\nu=0}^{\infty}2^{\nu\tau(\gamma+1/\tau)}\big(\|Q_{2^{2^\nu}}(f)\|_{\bar p,\bar\theta^{(1)}}^{*}\big)^{\tau}\Big\}^{1/\tau} \tag{3.9}$$

for every function $f\in B_{\bar p,\bar\theta^{(1)}}^{(0,\alpha,\tau)}$. The proof of Theorem 4 is complete. □

Theorem 5. *Let $1 < p_j < \infty$ and $1 \le \theta^{(2)} < \theta_j^{(1)} < \infty$ for $j = 1, \dots, m$, and let $1 \le \tau \le \infty$. If $\alpha > \sum_{j=1}^{m} (1/\theta^{(2)} - 1/\theta_j^{(1)})$, then*

$$E_M(\mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}})_{\bar{p},\bar{\theta}^{(2)}} \asymp (\log(M+1))^{-(\alpha - \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}))}, \quad M \in \mathbb{N}.$$

P r o o f. Let $f \in \mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$. We have $\alpha > \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)})$; therefore, $f \in L^*_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ by Theorem 4. Take a positive integer l such that $2^{2^l} \le M < 2^{2^{l+1}}$. Then, using the best approximation property and inequality (3.9), we have

$$E_M(f)_{\bar{p},\theta^{(2)}} \le E_{2^{2^l}}(f)_{\bar{p},\theta^{(2)}} << \left\{ \sum_{\nu=l}^{\infty} 2^{\nu \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)}} \left(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}} \right)^{\theta^{(2)}} \right\}^{1/\theta^{(2)}}. \tag{3.10}$$

If $\theta^{(2)} < \tau$, then by the Hölder inequality and in view of the fact that $\alpha > \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)})$, (3.10) implies that (see formula (3.1))

$$\begin{aligned} E_M(f)_{\bar{p},\theta^{(2)}} \le \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha} \left(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}} \right)^{\tau} \right\}^{1/\tau} \times \\ \left\{ \sum_{\nu=l}^{\infty} 2^{\nu\theta^{(2)}\beta'(\sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}) - \alpha)} \right\}^{\frac{1}{\theta^{(2)}\beta'}} << 2^{-l(\alpha - \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}))} \end{aligned} \tag{3.11}$$

for every function $f \in \mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ in the case $\theta^{(2)} < \tau$.

If $\tau \le \theta^{(2)}$, then, arguing as in the proof of formula (3.3), by means of the Jensen inequality, we find from (3.10) that

$$E_M(f)_{\bar{p},\theta^{(2)}} \le \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha} \left(\|Q_{2^{2^\nu}}(f)\|^*_{\bar{p},\bar{\theta}^{(1)}} \right)^{\tau} \right\}^{1/\tau} 2^{-l(\alpha - \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}))}. \tag{3.12}$$

Now, taking into account that $2^{2^l} \le M < 2^{2^{l+1}}$, by formulas (3.11) and (3.12), we obtain

$$E_M(f)_{\bar{p},\theta^{(2)}} << (\log(M+1))^{-(\alpha - \sum_{j=1}^{m}(1/\theta^{(2)} - 1/\theta_j^{(1)}))}$$

for every function $f \in \mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$. Thus, the upper estimates are proved.

Let us prove the lower estimates. Consider the function

$$f_2(\bar{x}) = 2^{-(n+1)(\alpha + \sum_{j=1}^{m} 1/\theta_j^{(1)})} \sum_{s=2^{n+1}+1}^{2^{n+2}} \sum_{\bar{k} \in \square_{2^s} \setminus \square_{2^{s-1}}} \prod_{j=1}^{m} (k_j - 2^{s-1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle},$$

where $\bar{x} \in \mathbb{I}^m$ and $n \in \mathbb{N}_0$. It is well known that

$$\Big\| \sum_{s=2^{n+1}}^{2^{n+2}} \sigma_s(f_2) \Big\|^*_{\bar{p},\bar{\theta}^{(1)}} = 2^{-(n+1)(\alpha + \sum_{j=1}^{m} 1/\theta_j^{(1)})} \Big\| \sum_{s=2^{n+1}+1}^{2^{n+2}} \sum_{\bar{k} \in \square_{2^s} \setminus \square_{2^{s-1}}} \prod_{j=1}^{m} (k_j - 2^{s-1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle} \Big\|^*_{\bar{p},\bar{\theta}^{(1)}} <<$$

$$<< 2^{-(n+1)(\alpha+\sum\limits_{j=1}^{m}1/\theta_j^{(1)})}(\log(2^{2^{n+2}}-2^{2^{n+1}}))^{\sum\limits_{j=1}^{m}1/\theta_j^{(1)}} << 2^{-(n+1)\alpha}.$$

Thus,

$$\Big\{\sum_{\nu=0}^{\infty}2^{\nu\tau\alpha}\Big(\Big\|\sum_{s=2^{\nu}}^{2^{\nu+1}}\sigma_s(f_2)\Big\|^{*}_{\bar{p},\bar{\theta}^{(1)}}\Big)^{\tau}\Big\}^{1/\tau}=2^{(n+1)\alpha}\Big\|\sum_{s=2^{n+1}}^{2^{n+2}}\sigma_s(f_2)\Big\|^{*}_{\bar{p},\bar{\theta}^{(1)}}\le C_1.$$

Hence, $C_1^{-1}f_2\in\mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ for $1<\theta^{(2)}<\infty$ and $1\le\tau<\infty$. Next, by the definition of the best approximation and the estimate

$$\Big\|\sum_{s=2^{n+1}+1}^{2^{n+2}}\sum_{\bar{k}\in\square_{2^s}\setminus\square_{2^{s-1}}}\prod_{j=1}^{m}(k_j-2^{s-1}+1)^{\frac{1}{p_j}-1}e^{i\langle\bar{k},\bar{x}\rangle}\Big\|^{*}_{\bar{p},\theta^{(2)}} >> 2^{n\frac{m}{\theta^{(2)}}},$$

we have

$$E_{2^{2^n}}(f_2)_{\bar{p},\theta^{(2)}}=C_1^{-1}\|f_2\|^{*}_{\bar{p},\theta^{(2)}}=$$

$$=C_1^{-1}2^{-(n+1)(\alpha+\sum\limits_{j=1}^{m}1/\theta_j^{(1)})}\Big\|\sum_{s=2^{n+1}+1}^{2^{n+2}}\sum_{\bar{k}\in\square_{2^s}\setminus\square_{2^{s-1}}}\prod_{j=1}^{m}(k_j-2^{s-1}+1)^{\frac{1}{p_j}-1}e^{i\langle\bar{k},\bar{x}\rangle}\Big\|^{*}_{\bar{p},\theta^{(2)}} >>$$

$$>> 2^{-(n+1)(\alpha-\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)}))}.$$

Taking into account that $2^{2^n}\le M<2^{2^{n+1}}$, we obtain

$$E_M(f_2)_{\bar{p},\theta^{(2)}} >> (\log(M+1))^{(\alpha-\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)}))}$$

for $1\le\theta^{(2)}<\infty$ and $1\le\tau\le\infty$. Thus, the proof of Theorem 5 is compete. □

Theorem 6. *Let $1<p_j<\infty$ and $1\le\theta^{(2)}<\theta_j^{(1)}<\infty$ for $j=1,\dots,m$, and let $1\le\tau\le\infty$. If $\gamma>\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)})-1/\tau$, then*

$$E_M(\mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}})_{\bar{p},\bar{\theta}^{(2)}}\asymp(\log(M+1))^{-(\gamma+1/\tau-\sum\limits_{j=1}^{m}(1/\theta^{(2)}-1/\theta_j^{(1)}))}.$$

P r o o f. Since $\mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ and $\mathbb{B}^{(0,\gamma+1/\tau,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ coincide, the statement of Theorem 6 follows from Theorem 5. □

4. Conclusion

The best approximations of functions of the classes $\mathbb{B}^{(0,\alpha,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ and $\mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}}$ in the space $L^{*}_{\bar{p},\theta^{(2)}}(\mathbb{I}^m)$ have logarithmic order.

Note that estimates of the quantities $E_M(\mathbb{B}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}})_{\bar{p},\bar{\theta}^{(2)}}$ and $E_M(\mathbb{A}^{(0,\gamma,\tau)}_{\bar{p},\bar{\theta}^{(1)}})_{\bar{p},\bar{\theta}^{(2)}}$ are unknown in the case $\theta_j^{(1)}=\theta^{(2)}$, $j=1,\dots,m$.

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