URAL MATHEMATICAL JOURNAL, Vol. 5, No. 1, 2019, pp. 13–23
DOI: 10.15826/umj.2019.1.002

ON THE CHERNOUS'KO TIME-OPTIMAL PROBLEM FOR THE EQUATION OF HEAT CONDUCTIVITY IN A ROD[1]

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Abstract: The time-optimal problem for the controllable equation of heat conductivity in a rod is considered. By means of the Fourier expansion, the problem reduced to a countable system of one-dimensional control systems with a combined constraint joining control parameters in one relation. In order to improve the time of a suboptimal control constructed by F.L. Chernous'ko, a method of grouping coupled terms of the Fourier expansion of a control function is applied, and a synthesis of the improved suboptimal control is obtained in an explicit form.

Keywords: Heat equation, Time-optimal problem, Pontryagin maximum principle, Suboptimal control, Synthesis of control.

Introduction

It is known that a time-optimal problem occupied a very important place in the foundation and development of optimal control theory. Even for simple non-trivial cases, the problem required working-out new approaches and lead after all to Pontryagin's maximum principle [3, 10, 30]. Despite 70 years of development, the solution of concrete non-trivial examples of time-optimal control still needs considerable effort [2, 4, 19]. The problem becomes even more difficult when a control system is described by a partial differential equation [11, 24, 25, 34], particularly, for the heat conductivity equation [12, 22, 26, 29, 35, 36]. In [1], the correctness of parabolic equations for heat propagation is discussed and for that purpose, a parabolic equation with time delay is considered.

Here, the maximum principle can be formally written out as well, but it loses its effectiveness as compared with a finite-dimensional case or on cases when the time interval is fixed [2, 9, 18, 25, 32, 33]. Therefore, Chernous'ko suggested [13] another approach based on the Fourier expansion that allowed him to reduce the problem to an infinite system of one-dimensional problems whose control parameters are connected by a condition in the min-max form (see below (1.4)) generating a closed convex control set in a Hilbert space. Unfortunately, to deal with such a constraint is quite difficult (about other kinds of constraints see [17]). In order to overcome this complexity, the mentioned constraint was replaced [13] by an infinite system of separated conditions for scalar control parameters that can be interpreted as if one took Hilbert's brick inscribed into the control set. As a result, this approach made it possible to construct a suboptimal control and to give an explicit upper estimation for an optimal time. In [5], a co-Hilbert's brick inscribed into the control set was considered, and an improved suboptimal control function was constructed. In the present

[1]This work was supported by a grant from the Ministry of Innovative Development of the Republic of Uzbekistan (Project No. OT-Ф4-84).

paper, we suggest another way for constructing a suboptimal control function in the case of the heat conductivity equation in a rod.

1. Preliminaries

As it was noted above, Chernous'ko considered the time-optimal problem for an evolutional equation

$$\frac{\partial u(t,x)}{\partial t} = A[u(\cdot,\cdot)](t,x) + v(t,x) \tag{1.1}$$

with the initial and boundary conditions

$$u(0,x) = u^0(x), \quad Mu(t,s) = u^*(t,s), \tag{1.2}$$

where A is a uniformly elliptic differential operator, $t \geq 0$, $x \in D$, D is a regular domain with Lyapunov boundary Γ, $s \in \Gamma$, and M is a boundary operator [13].

The constraint on the control function in problem (1.1), (1.2) is bounded in the norm of the space L_∞; i.e. $|v(t,x)| \leq v_0$ for almost all t and every $x \in \bar{D}$, where v_0 is a given positive number [31]. It is known that, for every control function $v(t,x)$, problem (1.1), (1.2) has a unique solution $u(t,x)$ [14, 21, 28].

If a solution $u(t,x)$ of problem (1.1), (1.2) satisfies the condition $u(T,x) \equiv 0$ at some T, $T \geq 0$, then the corresponding control function $v(t,x)$ is called admissible, and the number T is called the transition time (from the initial state $u_0(\cdot)$ into the equilibrium state $u(t,x) \equiv 0$). Let V be the class of all admissible controls. Then the quantity $T = T[v(\cdot,\cdot)]$ will be a functional on V at every fixed $u^0(x)$ and $u^*(t,s)$.

If an admissible control $v_*(t,x)$ satisfies the condition $T_* = T[v_*(\,\cdot\,,\,\cdot\,)] \leq T[v(\,\cdot\,,\,\cdot\,)]$ for all $v(\,\cdot\,,\,\cdot\,) \in V$, then $v_*(\,\cdot\,,\,\cdot\,)$ is called a time-optimal control, and the value T_* is called optimal transition time.

The direct application of the Pontryagin maximum principle to problem (1.1), (1.2) is a very hard task, unlike optimization problems on a finite interval of time (see [4, 8, 15]). For example, in [25], only theorems on the existence of optimal control and the bang-bang principle are given, but no specific example of a solution was considered. In monograph [11], the time-optimal problem when a control parameter participates in boundary conditions was considered [11, Ch. 5, Sect. 1] and, instead of the necessary conditions, the method of the L-momentum of N.N. Krasovskii [19] was applied [11, Sect. 2]. In the recently published article [20], Butkovsky's approach was applied to the case of a fractional-order diffusion equation. It should be noted that the L-momentum method only allows one to simplify to some degree the time-optimal problem and rarely gives an explicit solution. Therefore, the approach suggested by Chernous'ko [13], where the method of expansions on the system of eigenfunctions of the operator A was used, seems to be more effective. That helped to reduce considering problem to the infinite system of one-dimensional control problems:

$$\dot{y}_k = -\lambda_k y_k + v_k, \quad y_k(0) = y_{k0}, \quad k = 0,1,2,\dots\,. \tag{1.3}$$

(About solution of systems of this kind, see [16]).

In terms of system (1.3), the condition $|v(t,s)| \leq v_0$ means that a counting system of the control parameters v_k, $k = 0,1,2,\dots$, should satisfy the combined constraint

$$\max_{x\in\bar{D}} \Big| \sum_{k=0}^{\infty} \varphi_k(x) v_k \Big| \leq v_0. \tag{1.4}$$

where φ_k are eigenfunctions of the problem.

Condition (1.4) defines some closed convex set L in the Hilbert space l_2, which is difficult to deal with. In this connection, it is natural to try to solve the problem of finding a suboptimal control. (It is essential to note that, if a time interval is fixed, then the method of penalty functions is enough effective for the construction of a suboptimal control. It would be interesting to apply this method for the time-interval problem as well [6].) For this purpose, in [13], constraint (1.4) was replaced by a more rigid system of constraints in the form

$$|v_k| \le U_k, \quad k = 0, 1, 2, \dots, \tag{1.5}$$

where $\alpha_k = \max\limits_{x\in\bar{D}} |\varphi_k(x)|$. Wherein, nonnegative numbers U_k should be chosen satisfying the condition $\sum_{k=0}^{\alpha} \alpha_k U_k = v_0$.

Let T_{*k} be an optimal transition time in the problem

$$\dot{y}_k = -\lambda_k y_k + v_k, \quad y_k(0) = y_k^0, \tag{1.6}$$

such that $y_k(T_{*k}) = 0$, $k = 0, 1, 2 \dots$. In [13], it is shown that the numbers U_k can be chosen so that all T_{*k} coincide: $T_{*k} = \hat{T}$ for some $\hat{T}$. Let $\hat{v}_k(t)$ be the sequence of the corresponding optimal controls. Then $T_* \le \hat{T}$ and $\hat{v}_*(t,x) = \sum_{k=0}^{\infty} \varphi_k(x) v_{*k}(t)$ may serve as the sought suboptimal control.

A new problem arises here: is it possible to use a more exact reduction of the constraint than (1.5)? As mentioned above in [5] it was used Hilbert's co-cube instead of (1.5). Here, we are going to follow another approach based on a special grouping of terms of (1.5). Effectiveness of this approach is tightly related to specific properties of eigenfunctions $\varphi_k(\cdot)$, so here it will be demonstrated for the operator $A = \partial^2/\partial x^2$ connected with the process of the heat conductivity in a rod.

2. A method of grouping terms of the Fourier expansion

Consider the following concretization of problem (1.1), (1.2):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + v(t,x), \quad |v(t,x)| \le v_0, \quad t \ge 0, \quad 0 \le x \le \pi, \\ u(0,x) = u^0(x), \quad u(t,0) = 0, \quad u(t,\pi) = 0. \end{cases} \tag{2.1}$$

The system of eigenfunctions $\varphi_k(t) = \sin kx$, $k = 1, 2, \dots$, of the operator $\partial^2/\partial x^2$ forms a complete orthogonal basis of the space $L_2[0,\pi]$ [21, 28].

Let $u(t,x) = \sum_{k=1}^{\infty} y_k \sin kx$ and $v(t,x) = \sum_{k=1}^{\infty} v_k \sin kx$ be the Fourier expansions on the basis $\{\sin kx\}$. Then the restriction (1.4) takes the form

$$\max_{0\le x\le\pi} \left| \sum_{k=1}^{\infty} v_k \sin kx \right| \le v_0. \tag{2.2}$$

Let us consider a more rigid restriction

$$\max_{0\le x\le\pi} \sum_{k\in Q} \Big| v_k \sin kx + v_{3k} \sin 3kx \Big| \le v_0 \tag{2.3}$$

instead of (2.2), thereby replacing the optimal control problem with a suboptimal control problem.

System (1.6) takes the form

$$\dot{y}_k = -k^2 y_k + v_k, \quad k \in Z^+. \tag{2.4}$$

Let Q be the set of all positive integers having the form $3^{2p}q$, where $p = 0, 1, 2, \dots$, and q is relatively prime with 3. It is obvious that the set of all positive integers Z^+ is the union of the two disjoint sets Q and $3Q$. Then (2.4) can be rewritten in the form

$$\dot{y}_k = -k^2 y_k + v_k, \quad \dot{y}_{3k} = -9k^2 y_{3k} + v_{3k}, \quad k \in Q. \tag{2.5}$$

After the substitutions

$$y_k = \frac{\mu_k}{k^2} x^1, \quad y_{3k} = \frac{\mu_k}{k^2} x^2, \quad t = \frac{1}{k^2}\tau, \quad v_k = \mu_k w^1, \quad v_{3k} = \mu_k w^2,$$

all systems (2.5) will be reformulated to the following two-dimensional control system:

$$\dot{x}^1 = -x^1 + w^1, \quad \dot{x}^2 = -9x^2 + w^2. \tag{2.6}$$

Now, following the Chernous'ko way, we replace (2.3) by the even more rigid restriction

$$\max_{0 \le x \le \pi} \left| w_k^1 \sin kx + w_k^2 \sin 3kx \right| \le 1, \quad k \in Q, \tag{2.7}$$

that implies (2.3) if $\sum_{k \in Q} \mu_k = v_0$. Thus, we have reduced the infinite dimensional control problem to the two-dimensional problem.

3. Solution of the auxiliary time-optimal problem on the plane

Let P_k denote the set of all pairs (w_k^1, w_k^2) for which (2.7) holds. Setting

$$P = \left\{ w = (w^1, w^2) \in R^2 : \max_{0 \le t \le \pi} \left| w^1 \sin t + w^2 \sin 3t \right| \le 1 \right\},$$

we have $P_k = \mu_k P$. As a result, the considered problem of constructing a suboptimal control reduces to the concrete problem of time-optimal control for the following two-dimensional system:

$$\dot{x}^1 = -x^1 + w^1, \quad \dot{x}^2 = -9x^2 + w^2, \quad (w^1, w^2) \in P. \tag{3.1}$$

Obviously, P is a convex and compact set with non-empty interior (i.e., a convex body). Since P is symmetric with respect to the origin, we may restrict ourselves to considering only the case $w^1 \ge 0$. It is more convenient to set $\sin t = y$. Then, by the formula

$$\sin 3t = 3 \sin t - 4 \sin^3 t,$$

we get

$$P = \left\{ w = (w^1, w^2) \in R^2 : \max_{0 \le y \le 1} \left| (w^1 + 3w^2)\, y - 4w^2\, y^3 \right| \le 1 \right\}.$$

Just this transformation lay on the base of the separation $Z^+ = Q \cup 3Q$.

After elementary calculations, we find that the part of the boundary of the set P lying in the half-plane $w^1 \ge 0$ is given by the formula

$$w^1 = \begin{cases} w^2 + 1 & \text{if} \quad -1 \le w^2 < 0.125, \\ 3(\sqrt[3]{w^2} - w^2) & \text{if} \quad 0.125 \le w^2 \le 1, \end{cases}$$

while the other part is found by central symmetry (see Fig. 1).

Let us recall that, in the auxiliary problem (3.1), a unique optimal time-control function exists at each initial point (x_0^1, x_0^2) [7, 23, 27]. The existence follows from the property $O \in \operatorname{Int} P$. The

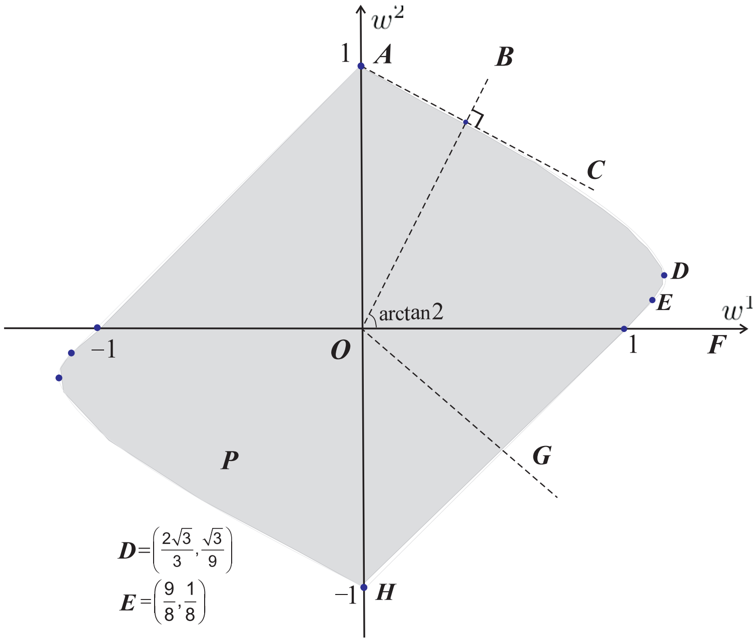

Figure 1. The straight ray AC is tangent at the point A to the curve AD, which is a part of the boundary of P.

uniqueness is a consequence of the following feature of P: the vector (1,1), which is orthogonal to the segment AC, is not an eigenvector of the matrix of system (3.1). Therefore, the optimal control problem (2.5) coincides with the extremal controls of Pontryagin's maximum principle [23, 30].

To calculate the latter, we prefer to use the "backward motion" principle. Let $T(x_0^1, x_0^2)$ be a transition time for the initial point (x_0^1, x_0^2) in the system (3.1). If we set $\tau = T(x_0^1, x_0^2) - t$, then extremals of Pontryagin's maximum principle are defined by the system

$$\begin{cases} \dfrac{dx^1}{d\tau} = x^1 - \bar{w}^1, \quad \dfrac{dx^2}{d\tau} = 9x^2 - \bar{w}^2, \quad \dfrac{d\psi_1}{d\tau} = -\psi_1, \quad \dfrac{d\psi_2}{d\tau} = -9\psi_2. \\ x(0) = y(0) = 0, \quad \psi_1(0) = \cos s, \quad \psi_2(0) = \sin s, \quad -\pi \le s \le \pi. \end{cases} \tag{3.2}$$

Since $\psi_1(\tau, s) = e^{-\tau}\cos s$ and $\psi_2(\tau, s) = e^{-9\tau}\sin s$, an extremal control $\bar{w}(\tau, s)$ should be found by the Pontryagin's maximum principle, i.e., from the equation

$$\bar{w}^1(\tau, s)e^{-\tau}\cos s + \bar{w}^2(\tau, s)e^{-9\tau}\sin s = \max_{w\in P}[w^1 e^{-\tau}\cos s + w^2 e^{-9\tau}\sin s]. \tag{3.3}$$

Equation (3.3) leads to the following construction of the extremal controls.

If $\psi(\tau, s)$ lies in the open angle AOB, then obviously $\bar{w}(\tau, s) = (0, 1)$. Note that, if $s = \pi/2$, then $\psi_1(\tau, s) \equiv 0$. Therefore, $\bar{w}(\tau, \pi/2) = (0, 1)$. Similarly, if $s = 0$, then $\psi_2(\tau, s) \equiv 0$; thus, $\bar{w}(\tau, 0) = (2\sqrt{3}/3,\ \sqrt{3}/9)$.

Consider now the dynamics of $\psi(\tau, s)$. In the case $0 < s < \pi/2$, the vector $\psi(\tau, s)$ lies in the quarter $\psi_1 > 0$, $\psi_2 > 0$ and turns clockwise. Moreover, its direction tends to the axis of abscissas OE as $\tau \to +\infty$. (Similarly, if $-\pi/2 < s < 0$, then $\psi(\tau, s)$ lies in the quarter $\psi_1 > 0$, $\psi_2 < 0$ and turns counterclockwise with the same limit direction.)

Thus, the extremal control has the following structure: if $0 < s \le \arctan 2$ (see Fig. 1), then $\psi(\tau, s)$ lies in the angle BOD for all τ ($\tau \ge 0$), and, hence, $\bar{w}(\tau, s)$ is a point of the arc AD such that its projection to the direction $\psi(\tau, s)$ is maximal (the analytical expression for $\bar{w}(\tau, s)$ is given in Table 1).

Table 1. The analytical expression for $\bar{w}^1(\tau,s)$ and $\bar{w}^2(\tau,s)$.

	$\bar{w}^1(\tau,s)$	$\bar{w}^2(\tau,s)$
$-\dfrac{\pi}{2}<s\le-\dfrac{\pi}{4}$	0 if $0\le\tau\le\tau_{**}$, $3(2-e^{-8\tau}\tan s)M$ if $\tau_{**}\le\tau$	-1 if $0\le\tau\le\tau_{**}$, M if $\tau_{**}\le\tau$
$-\pi/4<s<0$	$3(2-e^{-8\tau}\tan s)M$	M
$s=0$	$2\sqrt{3}/3$	$\sqrt{3}/9$
$0<s\le\arctan 2$	$3(2-e^{-8\tau}\tan s)M$	M
$\arctan 2<s<\dfrac{\pi}{2}$	0 if $0\le\tau\le\tau_{*}$, $3(2-e^{-8\tau}\tan s)M$ if $\tau_{*}\le\tau$	1 if $0\le\tau\le\tau_{*}$, M if $\tau_{*}\le\tau$
$s=\pi/2$	0	1

Table 2. The analytical expression for $x^1(\tau,s)$ and $x^2(\tau,s)$.

	$x^1(\tau,s)$	$x^2(\tau,s)$
$-\dfrac{\pi}{2}<s\le-\dfrac{\pi}{4}$	0 if $0\le\tau\le\tau_{**}$, $\dfrac{3e^{\tau}}{4\tan s}\displaystyle\int_m^n\dfrac{1-p^2}{p^2q^7}dp$ if $\tau_{**}\le\tau$	$(e^{9\tau}-1)/9$ if $0\le\tau\le\tau_{**}$, $\dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n qdp$ if $\tau_{**}\le\tau$
$-\pi/4<s<0$	$\dfrac{3e^{\tau}}{4\tan s}\displaystyle\int_m^n\dfrac{1-p^2}{p^2q^7}dp$	$\dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n qdp$
$s=0$	$2\sqrt{3}(1-e^{\tau})/3$	$\sqrt{3}(1-e^{9\tau})/81$
$0<s\le\arctan 2$	$\dfrac{3e^{\tau}}{4\tan s}\displaystyle\int_m^n\dfrac{1-p^2}{p^2q^7}dp$	$\dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n qdp$
$\arctan 2<s<\dfrac{\pi}{2}$	0 if $0\le\tau\le\tau_{*}$, $\dfrac{3e^{\tau}}{4\tan s}\displaystyle\int_m^n\dfrac{1-p^2}{p^2q^7}dp$ if $\tau_{*}\le\tau$	$(1-e^{9\tau})/9$ if $0\le\tau\le\tau_{*}$, $\dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n qdp$ if $\tau_{*}\le\tau$
$s=\pi/2$	0	$(1-e^{9\tau})/9$

Further, in the case $\arctan 2<s<\pi/2$, we have $\bar{w}(\tau,s)=(0,1)$ on the interval $[0,\tau_*)$, where $\tau_*=-1/8\cdot\log(2\cot s)$. At the time $\tau=\tau_*$, the vector $\psi(\tau,s)$ becomes orthogonal to the right side tangent to the curve ∂P at the point $(0,1)$ and it occurs "switching" of the extremal control from the value $(0,1)$ to a continuous mode. Namely, $\bar{w}(\tau,s)$ begins sliding along the arc AC (see Table 1) and tends to the point C as $\tau\to+\infty$.

Similarly, if $(-\pi/2<s<-\pi/4)$, then $\bar{w}(\tau,s)=(0,-1)$ at $0\le\tau<\tau_{**}$, where $\tau_{**}=-1/8\cdot\log(-\cot s)$ and $w(\tau,s)$ is a switching time. On the interval $(\tau_{**},+\infty)$, $\bar{w}(\tau,s)$ slides along the arc ED tending to the point D.

The entire synthesis of the extremal control is given in Table 1. Due to the central symmetry, the values of s are considered only on the range $-\pi/2<s\le\pi/2$ and the following notation is used:

$$M=(3-e^{-8\tau}\tan s)^{-3/2},\quad m=(3e^{8\tau}\cot s-1)^{-1/2},\quad n=(3\cot s-1)^{-1/2},$$

$$p=(3-e^{-8\tau}\tan s)^{-1/2},\quad q=((3-p^{-2})\cot s)^{1/8}.$$

Now, extremal trajectories can be easily calculated by (3.2). The corresponding formulas are gathered in Table 2. They are illustrated in Fig. 2.

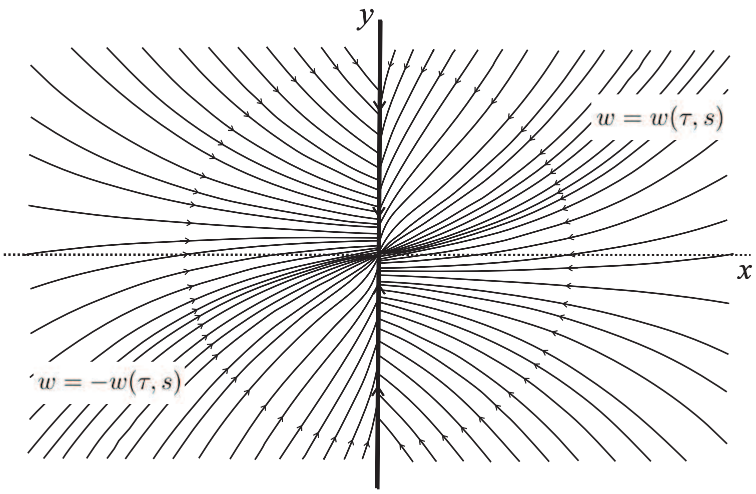

Figure 2. The extremal trajectories.

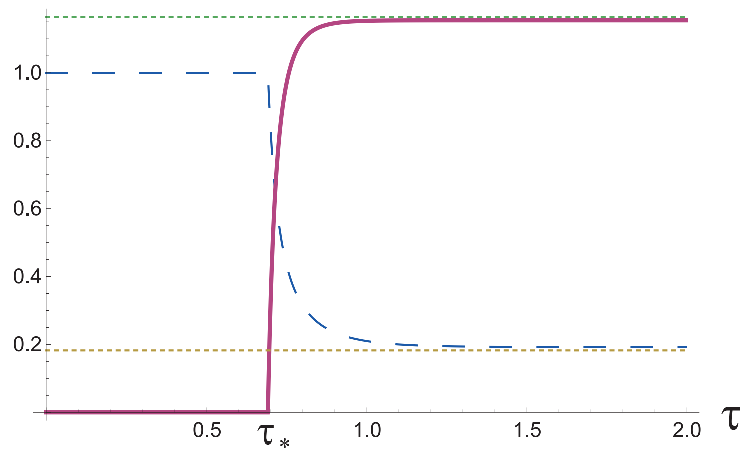

Figure 3. The graphs of the functions $\bar{w}^1(\tau, s)$ (the continuous line) and $\bar{w}^2(\tau, s)$ (the dashed line) for $\arctan 2 < s < \pi/2$.

4. Construction of a suboptimal control in the initial problem

Let us now derive the solution of problem (2.1), (2.3), (2.5) basing one of reduced problems.

Extremals $x(t, s), y(t, s)$ cover R^2. Therefore, for every $(x_0^1, x_0^2) \in R^2 \setminus (0, 0)$, there exists a pair (τ_0, s_0) such that $x^1(\tau_0, s_0) = x_0^1$, $x^2(\tau_0, s_0) = x_0^2$. Further, in the system (3.1), for every fixed $(x_0^1, x_0^2) \neq 0$, an optimal control is unique, which implies the uniqueness of the value τ_0 (while corresponding values of s_0 may be not unique, but one can choose any of them).

Then $T(x_0^1, x_0^2) = \tau_0$ is the transition time and

$$\bar{v}^1(t) = v^1(\tau_0 - t,\, s_0), \quad \bar{v}^2(t) = v^2(\tau_0 - t,\, s_0)$$

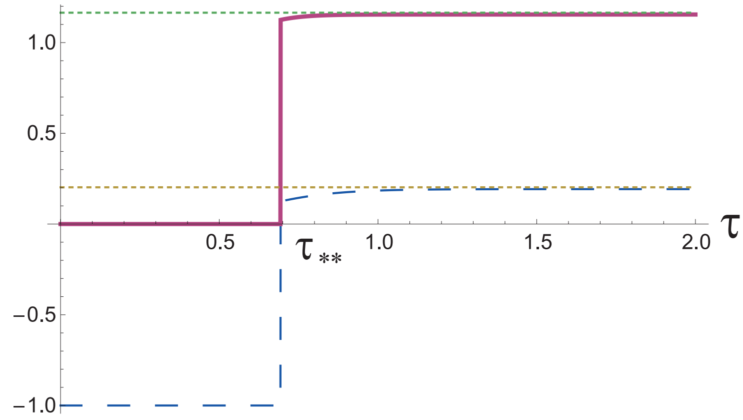

Figure 4. The graphs of the functions $\bar{w}^1(\tau,s)$ (the continuous line) and $\bar{w}^2(\tau,s)$ (the dashed line) for $-\pi/2 < s < \arctan 2$.

is the suboptimal control for (2.1).

Let us now consider system (2.5). For an initial point (y_k^0, y_{3k}^0), the corresponding trajectory $(y_k(t), y_{3k}(t))$ satisfies the condition

$$y_k(T_k) = y_{3k}(T_k) = 0,$$

where

$$T_k(\mu_k) = \frac{1}{k^2}\Big(\frac{k^2}{\mu_k}y_k^0,\ \frac{k^2}{\mu_k}y_{3k}^0\Big).$$

The constructed synthesis implies that T_k is monotonically decreasing in μ_k, and it is easy to see that $T_k \to 0$ as $\mu_k \to +\infty$ and $T_k \to \infty$ as $\mu_k \to 0$. Therefore, for every k, there exists a unique value μ_k^* such that $T_k(\mu_k^*)$ is the same for all k. Moreover, μ_k^* can be chosen satisfying the condition $\sum \mu_k^* = v_0$. One can easily see that

$$\frac{\alpha}{k^2} \le \mu_k^* \le \frac{\beta}{k^2}$$

for some positive α and β.

Finally, we consider the initial problem (2.1), (2.3), (2.5). Let $u_0(x) = \sum_{k=1}^{\infty} u_k^0 \sin x$ be the Fourier expansion of the initial function $u_0(x)$. Taking (y_k^0, y_{3k}^0), $k \in Q$, as an initial point for system (2.6), we find

$$\bar{w}_k^0(t) = \frac{1}{\mu_k}\bar{v}_k^1(t), \quad \bar{w}_{3k}^0(t) = \frac{1}{\mu_k}\bar{v}_k^2(t), \quad k \in Q.$$

Thus, the following statement holds.

Theorem 1. *The function*

$$\bar{v}(t,x) = \sum_{k=1}^{\infty} \bar{w}_k^0(t) \sin kx$$

is a suboptimal control in problem (2.1) *for the initial state* $u_0(x)$*.*

5. Conclusion

The paper is devoted to the time-optimal problem for the process of heat conductivity in a rod when the control parameter is the intensity of external heat sources. A suboptimal control is constructed by the combination of the Chernous'ko approach with the method of grouping terms of the Fourier expansion.

This method may be applied to the time-optimal control problem for other systems given in an evolutionary form.

The following question naturally arises: how effective is the method of grouping? First of all, let us bring general considerations. The set of all admissible controls in the initial problem (1.1)–(1.2) can be identified with the subset

$$U_{Initial} = \Big\{ u \in l_2 \mid \sup_{0 \le x \le \pi} \big| \sum_{k=0}^{\infty} u_k \sin kx \big| \le v_0 \Big\}.$$

As noted in Section 1, Chernous'ko restricted the set of controls using

$$U_{Ch} = \Big\{ u \in l_2 \mid |u_k| \le U_k, \;\; k = 1, 2, 3, \dots \Big\},$$

where U_k is a sequence chosen from the condition $\sum U_k \le v_0$ and guaranteeing the equality $u(t,x) \equiv 0$ for some $T = T_{Ch} > 0$.

The considerations in this paper are based on the set

$$U_{gr} = \Big\{ u \in l_2 \mid \max_{0 \le x \le \pi} |u_k \sin kx + u_{3kx} \sin 3kx| \le U_k \Big\}$$

taken as a region of admissible controls.

One can easily see that

$$U_{gr} \subset U_{Ch} \subset U_{Initial}.$$

These relations imply $T_2 \le T_1 \le T_0$ for optimal and suboptimal times of transition respectively.

If one takes an initial point of the form $(0, 0, \dots, x_m^0, 0, \dots, 0)$, i.e., in terms of the initial problem (1.1)–(1.2), $\varphi(x) = (0, 0, \dots, x_m^0 \sin mx, 0, \dots, 0)$, then, obviously, $U_{Initial} = U_{Ch} = U_{gr}$ and, thus, $T_2 = T_1 = T_0$. But if an initial point is taken in the form $(0, 0, \dots, 0, x_k^0, 0, \dots, 0, x_{3k}^0, 0, \dots)$, then $U_{In} = U_{gr}$ while $U_{In} \supset U_{Ch}$ and, thus, $T_2 = T_0 < T_1$. Table 3 contains values for specific cases.

Table 3

Initial point	T_1	$T_0 = T_2$
$(1, 0, 1, 0, 0, 0, \dots)$	0.7	0.64
$(1, 0, 2, 0, 0, 0, \dots)$	0.67	0.61
$(1, 0, -1, 0, 0, 0, \dots)$	0.72	0.65
$(1, 0, -2, 0, 0, 0, \dots)$	0.69	0.62

Obviously, $x_k^0 \ne 0$ at least for three values of the index k when $T_2 < T_0 < T_1$.

The final note is that the method of grouping can be applied only if there some algebraic relations between the eigenfunctions of the operator A.

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