Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-32 (2018) 213–218

A Maximum Principle for One Infinite Horizon Impulsive Control Problem ⋆

Dmitry V. Khlopin *

* *Krasovskii Institute of Mathematics and Mechanics, Yekaterinburg, 620990 Russia (e-mail: khlopin@imm.uran.ru).*
Ural Federal University, Yekaterinburg, 620083, Russia (e-mail: khlopin@imm.uran.ru)

Abstract: The paper is concerned with a nonlinear impulsive control system with trajectories of bounded variation. Necessary conditions of optimality in a form of the Maximum Principle are derived for a class of infinite horizon impulsive optimal control problems. For the overtaking optimality criterion under the assumption that all gradients of the payoff function are bounded, we construct a transversality condition for the adjoint variable in terms of limit points of the gradient of the payoff function. In the case when this limit point is unique, this condition supplements the system of the Maximum Principle and determines a unique solution of the adjoint system. This solution can be written explicitly with the use of the (Cauchy type) formula proposed earlier by S. M. Aseev and A. V. Kryazhimskii. The key idea of the proof is the application of the convergence of subdifferentials within Halkin's scheme.

Keywords: infinite horizon, maximum principle, impulsive control, necessary conditions, overtaking optimal, convergence of subdifferentials.

1. INTRODUCTION

Impulsive control problem is a useful mathematical idealization of the degenerate control processes. Everyday life examples of impulsive systems can also be found in mechanics, medicine, et cetera. The necessary conditions of optimality for these problems are well-known, see, for example, Blaquiere (1985), Rempala et al. (1988), Zavalischin et al. (2013), Silva et al. (1997), Miller et al. (2003), Dykhta et al. (2003), Goncharova et al. (2015).

In this paper we consider the possibility of transferring the Pontryagin Maximum Principle obtained in Silva et al. (1997) to infinite horizon. The general scheme of transferring the Pontryagin Maximum Principle for infinite horizon optimal control problems is well known, it was proposed in Halkin (1974); however, this scheme effectively disables the transversality condition on the adjoint variable, which leads to a much too broad family of solutions of the adjoint system. One of the means to explicitly select a unique solution of the Pontryagin Maximum Principle was proposed in Aseev at al. (2007) and involved the use of the (Cauchy type) formula. This method can be extended to a sufficiently broad class of control systems without any assumptions on the growth rate of any variable, payoff functions, or their gradients. In addition, every limit point of the gradient of the payoff function can be applied as the initial state of the adjoint system—the co-state arc at zero time. The key idea of this approach is the application of the convergence of subdifferentials.

As a whole, the results described below combine the Pontryagin Maximum Principle proved in Silva et al. (1997) with the application of the convergence of subdifferentials for an infinite horizon optimal control problem as proposed in Khlopin (2018). Two simplifying assumptions will be imposed, which may have been avoided: first, assume that all gradients of the payoff function are bounded; second, assume that the impulsive dynamic part is independent of the state variable x.

Note that the necessary conditions Blaquiere (1985), Rempala et al. (1988), Pereira et. al. (2011) of optimality for infinite horizon impulsive optimal control problem, which the author is aware of, contain the transversality conditions; however, these conditions work under assumption that the value function (or its gradients) are known. In this paper, the value function is not considered, moreover, the optimal value could be infinite everywhere.

The structure of the paper is as follows: first, we introduce the general statement, necessary definitions and notions, in particular, the variational analysis definitions. In Section 3, we consider the Pontryagin Maximum Principle, the corresponding adjoint system, and one limit modification of the transversality condition. In Section 4, we formulate the main result and discuss its correspondence with the Cauchy-type formula. In Section 5, we show the proof of the theorem.

2. ELEMENTARY DEFINITIONS AND STATEMENTS

Fix the state space of the initial control system—a certain finite-dimensional Euclidean space $\mathbb{R}^m$. Denote by $\mathfrak{M}$ the set of all regular Borel nonnegative-valued measures μ such that $\mu([0,T])$ is finite for all positive T.

Consider the infinite horizon minimization problem

⋆ This study was supported by the Russian Foundation for Basic Research, project no. 16-01-00505.

2405-8963 © 2018, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.11.383

$$\text{Minimize } l(b) + \int_0^\infty f_0(t,x,u)\,dt \tag{1a}$$

$$dx = f(t,x,u)\,dt + g(t)\,\mu(dt), \quad x(0) \in \mathcal{C}, \tag{1b}$$

$$u(t) \in U,\ \mu \in \mathfrak{M}. \tag{1c}$$

Here x is a state variable that assumes values from $\mathbb{R}^m$; the scalar functions l and f_0 and the vector functions f and g are assumed to be given; and the control consists of the measure $\mu \in \mathfrak{M}$ and a Borel measurable function u that assumes the values from a nonempty closed subset U of a finite-dimensional Euclidean space.

We assume the following conditions to hold:

- $\mathcal{C}$ is a closed subset of $\mathbb{R}^m$;
- l is a locally Lipschitz continuous scalar function of $x \in \mathbb{R}^m$;
- for all $u \in \mathcal{U}$, the functions $[0,\infty) \times \mathbb{R}^m \times U \ni (t,x,u) \mapsto f(t,x,u) \in \mathbb{R}^m$ and $[0,\infty) \times \mathbb{R}^m \times U \ni (t,x,u) \mapsto f_0(t,x,u) \in \mathbb{R}$ and their derivatives with respect to x are $\mathcal{L} \times \mathcal{B}$-measurable in (t,u) and Lipschitz continuous in x; also, f satisfies the sublinear growth condition with respect to x.
- the function $g\colon [0,\infty) \to \mathbb{R}^m$ is continuous.

Denote by $\mathcal{U}$ the set of all bounded on every compact Borel measurable mappings $u\colon [0,\infty) \to U$. Also, denote by $BV^+([0,\infty);\mathbb{R}^m)$ the space of all right continuous on $(0,\infty)$ functions $x\colon [0,\infty) \to \mathbb{R}^m$ such that for each positive $T > 0$ the total variation of $x|_{[0,T]}$ is bounded.

Following Silva et al. (1997), we say that a function $x \in BV^+([0,\infty);\mathbb{R}^m)$ is a robust solution to (1b) for an initial condition $b \in \mathbb{R}^m$, a control $u \in \mathcal{U}$, and a measure μ if

$$x(\tau) = b + \int_0^\tau f(t,x(t),u(t))\,dt + \int_{(0,\tau]} g(t)\,\mu(dt)\ \forall \tau \geq 0.$$

Denote this solution of system (1b) by $y(b,0,u,\mu;\cdot)$. It is easy to see that under the assumptions above it can always be extended to the whole half-line $[0,\infty)$.

Note that, for fixed u and μ, if, for two arbitrary initial conditions $b_1, b_2 \in \mathbb{R}^m$, we have $y(b_1,0,u,\mu;t) = y(b_2,0,u,\mu;t)$, then $b_1 = b_2$. Then, for a positive θ, some $b \in \mathbb{R}^m$, a control $u \in \mathcal{U}$, and a measure μ, there exists a unique solution $y(b,\theta,u,\mu;\cdot)$ of (1b) with the initial condition $x(\theta) = b$.

Let us now introduce a scalar function J as follows: for all $b \in \mathbb{R}^m$, $u \in \mathcal{U}$, and $\theta \geq 0, T > \theta$,

$$J(b,\theta;u,\mu,T) = \int_\theta^T f_0\big(t, y(b,\theta,u,\mu;t), u(t)\big)\,dt.$$

Call a triplet $(x,u,\mu) \in BV^+([0,\infty),\mathbb{R}^m) \times \mathfrak{U} \times \mathfrak{M}$ an admissible process if $x(0) \in \mathcal{C}$, $x(\cdot) = y(x(0),0,u,\mu;\cdot)$. Call an admissible process $(\tilde{x},\tilde{u},\tilde{\mu})$ overtaking optimal (see Carlson (1990)) for problem (1a)–(1c) if, for every admissible process (x,u,μ), it holds that

$$\liminf_{T\to\infty} \Big(l(x(0)) - l(\tilde{x}(0)) + \int_0^T [f_0(t,x(t),u(t)) - f_0(t,\tilde{x}(t),\tilde{u}(t))]\,dt \Big) \geq 0.$$

Hereinafter assume that a certain admissible process $(\tilde{x},\tilde{u},\tilde{\mu})$ is overtaking optimal for problem (1a)–(1c). For brevity, let us also introduce

$$\tilde{J}(b;T) = J(b,0;\tilde{u},\tilde{\mu},T) \qquad \forall T > 0, b \in \mathbb{R}^m,$$
$$\tilde{y}(b;T) = y(b,0;\tilde{u},\tilde{\mu},T) \qquad \forall T > 0, b \in \mathbb{R}^m.$$

We will also need certain elementary definitions of the variational analysis Mordukhovich (2006).

Consider a lower semicontinuous function $h\colon \mathbb{R}^m \to \mathbb{R}$; define $\operatorname{epi} h(\cdot)$ by the rule

$$\operatorname{epi} h(\cdot) \stackrel{\triangle}{=} \{(\xi,r) \,|\, \forall \xi \in \mathbb{R}^m, r \geq h(\xi)\}.$$

Further, for all $\xi \in \mathbb{R}^m$, by $\hat{\partial} h(\xi)$ denote the Fréchet subdifferential of this function at the point $\xi \in \mathbb{R}^m$; it consists of all gradients $\bar{h}'(\xi) \in (\mathbb{R}^m)^*$ of a Fréchet differentiable function $\bar{h}\colon \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ such that $h(\xi) = \bar{h}(\xi)$ and $\bar{h}(\xi') \leq h(\xi')$ hold for all $\xi' \in \mathbb{R}^m$. Denote by $\partial h(\xi)$ the limiting subdifferential of h at ξ; it consists of all $\zeta \in (\mathbb{R}^m)^*$ such that, for some sequences of $y_n \in \mathbb{R}^m, \zeta_n \in \hat{\partial} h(y_n)$, it holds that

$$y_n \to \xi, \zeta_n \to \zeta, h(y_n) \to h(\xi).$$

We will also need the singular limiting subdifferential. Denote by $\partial^\infty h(\xi)$ the singular limiting subdifferential of h at ξ; it consists of all $\zeta \in (\mathbb{R}^m)^*$ such that, for some sequences of $\lambda_n > 0, y_n \in \mathbb{R}^m, \zeta_n \in \hat{\partial} h(y_n)$, it holds that

$$\lambda_n \to 0, y_n \to \xi, \lambda_n \zeta_n \to \zeta, h(y_n) \to h(\xi).$$

Denote by $N^{\mathcal{C}}(\xi)$ the limiting normal cone of $\mathcal{C}$ at ξ.

3. THE PONTRYAGIN MAXIMUM PRINCIPLE AND ITS ADDITIONAL CONDITIONS

Let the Hamilton–Pontryagin function $H\colon \mathbb{R}^m \times (\mathbb{R}^m)^* \times U \times [0,\infty)^2 \mapsto \mathbb{R}$ be given by the following: for all $(x,\psi,u,\lambda,t) \in \mathbb{R}^m \times (\mathbb{R}^m)^* \times U \times [0,\infty)^2$,

$$H(x,\psi,u,\lambda,t) \stackrel{\triangle}{=} \psi f\big(t,x,u\big) - \lambda f_0\big(t,x,u\big).$$

Let us introduce the relations of the Pontryagin Maximum Principle:

$$dx(t) = f\big(t,x(t),\tilde{u}(t)\big)\,dt + g(t)\,\tilde{\mu}(dt); \tag{2a}$$

$$-\dot{\psi}(t) = \frac{\partial H}{\partial x}\big(x(t),\psi(t),\tilde{u}(t),\lambda,t\big); \tag{2b}$$

$$\sup_{u\in U} H\big(x(t),\psi(t),u,\lambda,t\big) \tag{2c}$$

$$= H\big(x(t),\psi(t),\tilde{u}(t),\lambda,t\big) \quad a.e.\ t \geq 0. \tag{2d}$$

We will also need the following relations concerning the impulsive component:

$$\psi(t)g(t) \leq 0 \qquad \forall t \geq 0; \tag{3}$$

$$\psi(t)g(t) \geq 0 \qquad \forall t \in \operatorname{supp} \mu. \tag{4}$$

The conditions imposed above are sufficient to guarantee Silva et al. (1997) that relations (2a)–(4) are necessary for impulsive problems on any finite horizon $[\theta,T] \subset [0,\infty)$. Applying the Halkin scheme (see Halkin (1974)) directly, it is easy to show that the relations (2a)–(2c) are also necessary for infinite horizon optimal control problem (1a)–(1c) for every overtaking optimal process. However, the system that will be obtained in such a way will have no boundary condition on the adjoint variable that would correspond to the transversality condition at infinity. For impulsive problems, such a condition was announced e.g. in Pereira et. al. (2011), where the key requirement was the uniqueness of the ω-point of the optimal trajectory and the total controllability assumption at this point. Under this requirement, the co-state variable is the subgradient of the value function. A similar transversality condition was constructed Cannarsa (2018), Khlopin (2017a), Sagara (2010) for infinite horizon control problem.

In this paper, we use a slightly different approach: we study the gradients of the payoff function in the state variable for fixed optimal control. For infinite horizon control problem, this approach was realized in Khlopin (2018) for a payoff function that was Lipschitz continuous in x. To this end, we consider the following definition Khlopin (2015):

Call a nontrivial solution $(\tilde{x}, \tilde{\psi}, \tilde{\lambda})$ of system (2a)–(2b) an exact limiting solution iff for some sequences of $y_n \in \mathbb{R}^m, t_n \geq 0, \lambda_n > 0$, it holds that

$$\begin{aligned} &t_n \to \infty,\ y_n \to \tilde{x}(0), \qquad \lambda_n \to \tilde{\lambda}, \\ &-\lambda_n \frac{\partial \tilde{J}}{\partial x}(y_n; t_n) \to \tilde{\psi}(0),\ \tilde{J}(y_n; t_n) - \tilde{J}(\tilde{x}(0); t_n) \to 0. \end{aligned} \tag{5}$$

As it is easy to see, it will be worthwhile to check the existence of $\frac{\partial \tilde{J}}{\partial x}$ and $\frac{\partial \tilde{y}}{\partial x}(\xi; T)$ under the above-mentioned conditions. Let us confine ourselves to checking whether $\frac{\partial \tilde{y}}{\partial x}(\xi; T)$ exists for all $\xi \in \mathbb{R}^m, T \geq 0$.

Indeed, fix certain $x_0 \in \mathbb{R}^m, T \geq 0$. Evidently, $[0, \infty) \ni t \mapsto r(t) \stackrel{\triangle}{=} \tilde{y}(\xi; t) - \tilde{x}(t)$ is a solution of the Cauchy problem

$$\begin{aligned} \frac{dx}{dt}(t) &= f\big(t, x(t) + \tilde{x}(t), \tilde{u}(t)\big) - f\big(t, \tilde{x}(t), \tilde{u}(t)\big), \\ r(0) &= \xi - \tilde{x}(0). \end{aligned}$$

Since this is a system of ordinary differential equations that satisfies the Carathéodori conditions and its right-hand side is sufficiently smooth in x, its solution smoothly depends on ξ. But then the map $\xi \mapsto \tilde{y}(\xi; t)$ is the same. Moreover, all the desired derivatives can now be expressed explicitly.

Denote by $\mathbb{L}$ the linear space of all real $m \times m$ matrices. For all $\xi \in \mathbb{R}^m$, consider the Cauchy problem

$$\frac{dA(\xi; t)}{dt} = \frac{\partial f}{\partial x}\big(\tilde{y}(\xi; t), \tilde{u}(t)\big) A(\xi; t),\ A(\xi; 0) = 1_{\mathbb{L}}. \tag{6}$$

and its solution $A(\xi; \cdot) \in C([0, \infty), \mathbb{L})$. Then, for all $\xi \in \mathbb{R}^m, T \geq 0$, we obtain

$$\frac{\partial \tilde{y}}{\partial x}(\xi; T) = A(\xi; T), \tag{7}$$

$$\frac{\partial \tilde{J}}{\partial x}(\xi; T) = \int_0^T \frac{\partial f_0}{\partial x}\big(t, \tilde{y}(\xi; t), \tilde{u}(t)\big)\, A(\xi; t)\, dt, \tag{8}$$

further, for every positive λ, the corresponding solution (x, ψ) of system (2a)–(2b) satisfies the following Cauchy formula:

$$\psi(t) A(x(0); t) - \psi(0) = \lambda \frac{\partial \tilde{J}}{\partial x}(x(0); t) \quad \forall t \geq 0. \tag{9}$$

4. THE MAIN RESULT

Theorem 1. Let the process $(\tilde{x}, \tilde{u}, \tilde{\mu})$ be overtaking optimal for (1a)–(1c).

Assume that, for every bounded neighborhood Ξ of $\tilde{x}(0)$, for all $T > 0, \xi \in \Xi$, the gradients $\frac{\partial \tilde{J}}{\partial x}(\xi; T) = \frac{\partial J}{\partial x}(\xi, 0; \tilde{u}, T)$ are uniformly bounded.

Then, there exists an exact limiting solution $(\tilde{x}, \tilde{\psi}, 1)$ of the Pontryagin Maximum Principle (2a)-(4) such that

$$\tilde{\psi}(0) \in \partial l(\tilde{x}(0)) + N^{\mathcal{C}}(\tilde{x}(0)). \tag{10}$$

In particular, $-\tilde{\psi}(0)$ is a partial limit of $\frac{\partial J}{\partial x}(\xi, 0; \tilde{u}, \tilde{\mu}, T)$ as $\xi \to \tilde{x}(0), T \to \infty$.

Moreover, for any choice of the unboundedly increasing sequence of times t_n, there exists an exact limit solution $(\tilde{x}, \tilde{\psi}, 1)$ of the Pontryagin Maximum Principle (2a)–(2c) that satisfies (3),(4), (10) such that $-\tilde{\psi}(0)$ is a partial limit of the gradients $\frac{\partial J}{\partial x}(\xi, 0; \tilde{u}, \tilde{\mu}, t_n)$ as $\xi \to \tilde{x}(0), n \to \infty$.

In paper Aseev at al. (2007), and then in Aseev at al. (2012), Khlopin (2013), Aseev at al. (2014), Khlopin (2015), and Belyakov (2015), for optimal control problems, there was obtained a series of assumptions on the asymptotics of the functions f, f_0, J and their derivatives under which the solution of the Pontryagin Maximum Principle is uniquely (for a given process) determined by the rules

$$\begin{aligned} -\tilde{\psi}(0) &= \lim_{T \to \infty} \frac{\partial \tilde{J}}{\partial x}(\tilde{x}(0); T) \\ &= \int_0^\infty \frac{\partial f_0}{\partial x}\big(t, \tilde{x}(t), \tilde{u}(t)\big)\, A(\tilde{x}(0); t)\, dt, \\ \tilde{\lambda} &= 1. \end{aligned} \tag{11}$$

Note that, for this formula to be correct, in (11), there must, at least, exist the limit (as $T \uparrow \infty$) of $\frac{\partial \tilde{J}}{\partial x}(\tilde{x}(0), T)$. However, the validity of the formula in (11) does not guarantee its consistency with the Pontryagin Maximum Principle. An elementary example of such a control problem was considered in Khlopin (2017b). We have to impose a stronger condition: let, as in Khlopin (2018), there be only the continuity in ξ of the limit (as $T \uparrow \infty$) of $\tilde{J}(\xi, T)$ at $\xi = \tilde{x}(0)$, that is,

$$\lim_{\xi \to \tilde{x}(0), T \to \infty} \frac{\partial J}{\partial x}(\xi, 0; \tilde{u}, \tilde{\mu}, T) \in \mathbb{R}^m. \tag{12}$$

In view of this condition, (5) implies (11). So, we obtain

Corollary 2. Under conditions of the theorem, there also exists a finite limit (12). Then, ψ, λ are uniquely determined by (2b),(11) and also satisfy (2c), (3), (4), (10).

5. THE PROOF

Proof. Since, for every bounded neighborhood Ξ of the point $\tilde{x}(0)$, the mappings

$$\begin{aligned} &\Xi \ni \xi \mapsto \frac{\partial \tilde{J}}{\partial x}(\xi; T) && \forall T > 0, \\ &\Xi \ni \xi \mapsto \frac{\partial \tilde{J}}{\partial x}(\xi; T) - \frac{\partial \tilde{J}}{\partial x}(\tilde{x}(0); T) && \forall T > 0 \end{aligned}$$

are uniformly (in $T > 0$) bounded, the mappings $\Xi \ni \xi \mapsto \tilde{J}(\xi; T) - \tilde{J}(\tilde{x}(0); T) (\forall T > 0)$ share a common Lipschitz constant L; they are also uniformly equicontinuous. Since all these mappings become zero at $\xi = \tilde{x}(0)$, they are also uniformly bounded, therefore, the family of these mappings is precompact. Hence, the closure of

$$\{\mathbb{R}^m \ni \xi \mapsto \tilde{J}(\xi;T) - \tilde{J}(\tilde{x}(0);T)\,|\,T>0\}$$

is compact in the compact-open topology.

Fix an arbitrary unboundedly increasing sequence of positive t_n. Removing some elements if necessary, it is safe to assume that the mappings $\mathbb{R}^m \ni \xi \mapsto \tilde{J}(\xi;t_n) - \tilde{J}(\tilde{x}(0);t_n)$ converge to a certain locally Lipschitz continuous mapping uniformly on every compact.

Note that, for all $\xi\in\mathbb{R}^m, T\geq 0, t_k>T$, it holds that

$$\tilde{J}(\xi;T) = \tilde{J}(\xi,t_k) - J(\tilde{y}(\xi,T),T;\tilde{u},\tilde{\mu},t_k).$$

Since for all $\xi\in\mathbb{R}^m, t\geq 0$ there exists a vector $\xi_1 = y(\xi,t,\tilde{u},\tilde{\mu};0)\in\mathbb{R}^m$ such that $\xi=\tilde{y}(\xi_1,t)$, we obtain

$$\begin{aligned} &J(\xi,t;\tilde{u},\tilde{\mu},t_k) - J(\tilde{x}(t),t;\tilde{u},\tilde{\mu},t_k)\\ &= J(\xi_1,0;\tilde{u},\tilde{\mu},t_k) - J(\xi_1,0;\tilde{u},\tilde{\mu},t)\\ &\quad -J(\tilde{x}(0),0;\tilde{u},\tilde{\mu},t_k) + J(\tilde{x}(0),0;\tilde{u},\tilde{\mu},t)\\ &= \tilde{J}(y(\xi,t,\tilde{u},\tilde{\mu};0);t_k) - \tilde{J}(\tilde{x}(0);t_k)\\ &\quad -\big(\tilde{J}(y(\xi,t,\tilde{u},\tilde{\mu};0);t) - \tilde{J}(\tilde{x}(0);t)\big). \end{aligned}$$

Now, for all positive t and vector $\xi\in\mathbb{R}^m$, in view of the choice of t_k, there exists a finite limit

$$J_*(\xi,t) \stackrel{\triangle}{=} \lim_{k\to\infty}\big[J(\xi,t;\tilde{u},\tilde{\mu},t_k) - J(\tilde{x}(t),t;\tilde{u},\tilde{\mu},t_k)\big]. \quad (13)$$

Moreover, this limit is uniform in every compact subset of the set $\Xi\subset\mathbb{R}^m$, and the mapping $\xi\mapsto J_*(\xi,t)$ is bounded and locally Lipschitz continuous because for every $\tau\geq t$ the mapping $\xi\mapsto J\big(y(\xi,t,\tilde{u},\tilde{\mu};0),0;\tilde{u},\tilde{\mu};\tau\big)$ is the same.

Further, for all $\xi\in\mathbb{R}^m, T>0$, for all sufficiently large t_k, (13) implies the following equality:

$$\begin{aligned} &\tilde{J}(\xi;T) - \tilde{J}(\tilde{x}(0);T)\\ &= \tilde{J}(\xi;t_k) - J(\tilde{y}(\xi,T),T;\tilde{u},\tilde{\mu},t_k)\\ &\quad -\big(\tilde{J}(\tilde{x}(0);t_k) - J(\tilde{x}(T),T;\tilde{u},\tilde{\mu},t_k)\big)\\ &= J_*(\xi,0) - J_*(\tilde{y}(\xi,T),T). \end{aligned}$$

Consider the limiting subdifferential $\partial_x J_*(\xi;0)$ of $\xi\mapsto J_*(\xi;0)$. Since the maps $\xi\to\tilde{J}(\xi;T)$ are continuously differentiable, (Mordukhovich, 2006, Proposition 1.107(ii)) yields the following: for all $\xi\in\mathbb{R}^m, T>0$,

$$\partial_x J_*(\xi,0) = \frac{\partial\tilde{J}}{\partial x}(\xi;T) + \partial_\xi\Big(J_*(\tilde{y}(\xi,T),T)\Big).$$

Recall that the mappings $\mathbb{R}^m\ni\xi\to\tilde{y}(\xi;T)$ are continuously differentiable, and their derivatives $\frac{\partial}{\partial x}\tilde{y}(\xi;T) = A(\xi;T)$ are surjective operators since these are solutions of linear adjoint system (6) with the condition $A(\xi;0)=1_{\mathbb{L}}$. Hence, in view of the chain rule (Mordukhovich, 2006, Proposition 1.112(i)), we obtain

$$\partial_x J_*(\xi,0) = \frac{\partial\tilde{J}}{\partial x}(\xi;T) + \partial_x J_*(\tilde{y}(\xi,T),T)A(\xi;T) \quad (14)$$

for all $\xi\in\mathbb{R}^m, T>0$.

Define the constant map $z_n\in C(\mathbb{R},\mathbb{R})$ by the rule

$$z_n(t) = J_*(\tilde{x}(t_n),t_n) \qquad \forall t\geq 0.$$

Since $(\tilde{x},\tilde{u},\tilde{\mu})$ is an overtaking optimal process, we obtain

$$\liminf_{n\to\infty}\Big[l(b) + J(b,0;u,\mu,t_n) - \tilde{J}(\tilde{x}(0);t_n)\Big] \geq l(\tilde{x}(0))$$

for all $u\in\mathcal{U}, b\in\mathcal{C}, \mu\in\mathfrak{M}$, in particular, for all $n\in\mathbb{N}$, $u\in\mathcal{U}$, and $\mu\in\mathfrak{M}$, we have $u|_{[t_n,\infty)} = \tilde{u}|_{[t_n,\infty)}$ $\mu|_{\mathcal{B}([t_n,\infty))} = \tilde{\mu}|_{\mathcal{B}([t_n,\infty))}$. Now,

$$\begin{aligned} l(\tilde{x}(0)) &\leq \liminf_{k\to\infty}\Big[l(b) + J(b,0;u,\mu,t_n)\\ &\qquad + J(\tilde{y}(b;t_n),t_n;\tilde{u},\tilde{\mu},t_k) - \tilde{J}(\tilde{x}(0);t_k)\Big]\\ &= l(b) + J(b,0;u,\mu,t_n)\\ &\quad -\tilde{J}(\tilde{x}(0);t_n) + J_*(\tilde{y}(b;t_n),t_n). \end{aligned}$$

holds for all $u\in\mathcal{U}$, $b\in\mathcal{C}$, $n\in\mathbb{N}, \mu\in\mathfrak{M}$. Now, for all $n\in\mathbb{N}$, the optimal value of the problem

$$\begin{aligned} &\text{Minimize } l(x(0)) + J_*(x(t_n),t_n)\\ &+\int_0^{t_n}\Big[f_0\big(t,x(t),u(t)\big) - f_0\big(t,\tilde{x}(t),\tilde{u}(t)\big)\Big]dt\\ &dx = f(t,x,u)\,dt + g(t)\,\mu(dt), \quad x(0)\in\mathcal{C},\\ &u(t)\in U,\ \mu\in\mathfrak{M} \end{aligned}$$

is not less than $l(\tilde{x}(0))$. Thus, the process $(\tilde{x},\tilde{u},\tilde{\mu})$ is optimal in this problem for every natural n.

Then, for every $n\in\mathbb{N}$, $(\tilde{x},z_n,\tilde{u},\tilde{\mu})$ is optimal in the following problem:

$$\begin{aligned} &\text{Minimize } l(x(0)) + +z(t_n)\\ &+\int_0^{t_n}\Big[f_0\big(t,x(t),u(t)\big) - f_0\big(t,\tilde{x}(t),\tilde{u}(t)\big)\Big]dt\\ &dx = f(t,x,u)\,dt + g(t)\,\mu(dt), \quad \dot{z}=0,\\ &(x(0),z(0))\in\mathcal{C}\times\mathbb{R}, \quad (x(t_n),z(t_n))\in\operatorname{epi} J_*(\cdot,t_n)\\ &u(t)\in U,\ \mu\in\mathfrak{M}. \end{aligned}$$

Note that the Hamilton–Pontryagin function for the new problem coincides with the previously considered H for every $n\in\mathbb{N}$; now, by the Pontryagin Maximum Principle (Silva et al., 1997, Theorem 4.2), there exist $\psi_n\in C(\mathbb{R}_{\geq 0},(\mathbb{R}^m)^*), \phi_n\in\mathbb{R}, \lambda_n\in\{0,1\}$ such that

$$||\psi_n(0)|| + |\phi_n| + |\lambda_n| > 0$$

and every triple $(\tilde{x}_{[0,t_n]},\psi_n|_{[0,t_n]},\lambda_n)$ satisfies the Pontryagin Maximum Principle (2a)-(4) almost everywhere in $[0,t_n]$ with the boundary conditions

$$\begin{aligned} (\psi_n(0),\phi_n) &\in \lambda_n\partial l(\tilde{x}(0))\times\{0\} + N(\tilde{x}(0);\mathcal{C}),\\ -(\psi_n(t_n),\phi_n) &\in (0,\lambda_n) + N(\tilde{x}(t_n),z_n(t_n);\operatorname{epi} J_*(\cdot,t_n)). \end{aligned}$$

It follows from the first boundary condition that $\phi_n=0$; moreover, the following equations hold:

$$\lambda_n\in\{0,1\}, \quad ||\psi_n(0)|| + |\lambda_n| > 0, \quad (15)$$

$$\psi_n(0)\in\lambda_n\partial l(\tilde{x}(0)) + N(\tilde{x}(0);\mathcal{C}), \quad (16)$$

$$\begin{aligned} -(\psi_n(t_n),\lambda_n) &\in N(\tilde{x}(t_n),z_n(t_n);\operatorname{epi} J_*(\cdot,t_n)) \quad (17)\\ &= N(\tilde{x}(t_n),J_*(\tilde{x}(t_n),t_n);\operatorname{epi} J_*(\cdot,t_n)). \end{aligned}$$

By the definition of the limiting subdifferential, from (17) it follows that

$$\text{either}\quad -\psi_n(t_n) \in \lambda_n \partial_x J_*(\tilde{x}(t_n), t_n), \lambda_n > 0,$$
$$\text{or}\quad -\psi_n(t_n) \in \partial_x^\infty J_*(\tilde{x}(t_n), t_n), \lambda_n = 0.$$

However, for a Lipschitz continuous function $J_*(\cdot, t_n)$, we have $\partial_x^\infty J_*(\cdot, t_n) \equiv \{0\}$. Then, in view of (15), we obtain

$$-\psi_n(t_n) \in \partial_x J_*(\tilde{x}(t_n), t_n), \lambda_n = 1.$$

Further, ψ_n, as a solution of (2b), satisfies the Cauchy formula (see (9)), and, by sequential application of (9),(18), and (14), we obtain

$$\begin{aligned}
-\psi_n(0) &= -\psi_n(0)/\lambda_n \\
&= -\psi_n(t_n) A(\tilde{x}(0); t_n)/\lambda_n + \frac{\partial \tilde{J}}{\partial x}(\tilde{x}(0); t_n) \\
&\in \partial_x J_*(\tilde{x}(t_n), t_n) A(\tilde{x}(0); t_n) + \frac{\partial \tilde{J}}{\partial x}(\tilde{x}(0); t_n) \\
&= \partial_x J_*(\tilde{x}(0), 0) - \frac{\partial \tilde{J}}{\partial x}(\tilde{x}(0); t_n) + \frac{\partial_x \tilde{J}}{\partial x}(\tilde{x}(0); t_n) \\
&= \partial J_*(\tilde{x}(0), 0).
\end{aligned}$$

Thus, for each natural $n \in \mathbb{N}$ we obtain

$$-\psi_n(0) \in \partial_x J_*(\tilde{x}(0), 0).$$

Since J_* is locally Lipschitz continuous in x, we have proved the boundedness of the vectors $\psi_n(0)$. Passing from the sequence of t_n to its certain subsequence if necessary, we can assume that the sequence of $\psi_n(0)$ converges. Hence, by the theorem on continuous dependence of differential equations' solutions on initial conditions, the sequence of ψ_n converges in $[0, \infty)$ to a certain solution $\tilde{\psi}$ of adjoint system (2b), and this convergence is uniform in arbitrary compact time intervals. But, consequently, the triple $(\tilde{x}, \tilde{\psi}, 1)$ also satisfies relations (2a)–(4) on the whole $[0, \infty)$; moreover, now, for $\tilde{\psi}$, condition (10) for $\tilde{\psi}$ is implied by (16), and $-\psi_n(0) \in \partial_x J_*(\tilde{x}(0), 0)$ yields

$$-\tilde{\psi}(0) \in \partial_x J_*(\tilde{x}(0), 0).$$

It remains to prove that $(\tilde{x}, \tilde{\psi}, 1)$ is an exact limiting solution of (2a)–(2b). Recall that J_* is a limit of the sequence of mappings $\xi \mapsto \tilde{J}(\xi; t_n)$ that is uniform in a certain neighborhood of the point $\tilde{x}(0)$. As it was showed in (Ledyaev et al., 2012, Theorem 6.1), this means that every element from the Fréchet subdifferential $\hat{\partial}_x J_*(z, 0)$ (for all $z \in \mathbb{R}^m$) can be rendered as a limit of $\frac{\partial \tilde{J}}{\partial x}(\xi_i; t_{n(i)}) = \frac{\partial J}{\partial x}(\xi_i, 0; \tilde{u}, t_{n(i)})$ for certain sequences $\xi_i \to z, n(i) \to \infty$. By the definition of the limiting subdifferential, every element from $\partial_x J_*(z, 0)$ (for all $z \in \mathbb{R}^m$) can be expressed—in view of a certain converging to z sequence of ξ_i—as a limit of elements from $\hat{\partial}_x J_*(\xi_i, 0)$; however, it implies that every element of $\partial_x J_*(z, 0)$ is a limit of $\frac{\partial \tilde{J}}{\partial x}(\xi_i; t_{n(i)}) = \frac{\partial J}{\partial x}(\xi_i, 0; \tilde{u}, t_{n(i)})$ for certain subsequences of $\xi_i \to z$, $n(i) \to \infty$. By $-\tilde{\psi}(0) \in \partial_x J_*(\tilde{x}(0), 0)$, there exist a sequence of ξ_i that converges to $\tilde{x}(0)$ and an unboundedly increasing sequence of natural $n(i)$ such that $-\psi^*(0) = \lim_{i\to\infty} \frac{\partial J}{\partial x}(\xi_i, 0; t_{n(i)})$. Since the mappings $\Xi \ni \xi \mapsto \tilde{J}(\xi; t)$ have a common Lipschitz constant in a certain neighborhood $\Xi \subset \mathbb{R}^m$, from $||\xi_i - \tilde{x}(0)|| \to 0$, it automatically follows that $|J(\xi_i, 0; t_{n(i)}) - J(\tilde{x}(0), 0; t_{n(i)})| \to 0$. Thus, the triple $(\tilde{x}, \tilde{\psi}, 1)$ is an exact limiting solution of the Pontryagin Maximum Principle, which is what we wanted to prove.

6. CONCLUSION

Halkin's scheme is a classical method for proving the Maximum Principle in infinite horizon control problems. But, usually, the necessary relations obtained by this method have no boundary condition on the adjoint variable that would correspond to the transversality condition at infinity. In this paper, the appropriate condition has been obtained as direct consequence of the well-known theorem on convergence of subdifferentials in the case of infinite horizon impulsive optimal control problems with bounded gradients of the payoff function. Apparently, this way can be applied in the general case, without any boundedness conditions.

REFERENCES

S.M. Aseev, A.V. Kryazhimskii. The Pontryagin Maximum Principle and problems of optimal economic growth. *Proc. Steklov Inst. Math*, 257:1–255, 2007.

S.M. Aseev, A.V. Kryazhimskii, and K. Besov. Infinite-horizon optimal control problems in economics. *Russ. Math. Surv*, 2012, 67:195–253, 2012.

S.M. Aseev, V. Veliov. Needle variations in infinite-horizon optimal control. In: G. Wolansky, A.J. Zaslavski, editors, *Variational and optimal control problems on unbounded domains*, page 1–17. AMS, Providence, 2014.

A.O. Belyakov. Necessary conditions for infinite horizon optimal control problems revisited. *arXiv preprint arXiv*:1512.01206, 2015.

A. Blaquiere. Impulsive optimal control with finite or infinite time horizon. *J. Optim. Theory Appl*, 46(4):431–439, 1985

P. Cannarsa, H. Frankowska. Value function, relaxation, and transversality conditions in infinite horizon optimal control. *J Math Anal Appl.* 457:1188-1217, 2018.

D.A. Carlson. Uniformly overtaking and weakly overtaking optimal solutions in infinite–horizon optimal control: when optimal solutions are agreeable. *J. Optim. Theory Appl*, 64(1):55–69, 1990.

V.A. Dykhta, O.N. Samsonyuk. *Optimalnoe impulsnoe upravlenie s prilozheniyami*, Fizmatlit, Moscow, 2003

E.V. Goncharova, M.V. Staritsyn. Optimal impulsive control problem with state and mixed constraints: The case of vector-valued measure. *Automation and Remote Control*, 76(3), 377–384, 2015.

H. Halkin. Necessary conditions for optimal control problems with infinite horizons. *Econometrica*, 42:267–272, 1974

D.V. Khlopin Necessity of vanishing shadow price in infinite horizon control problems. *J. Dyn. Con. Sys.*, 19(4):519–552, 2013.

D.V. Khlopin Necessity of limiting co-state arc in Bolza-type infinite horizon problem. *Optimization*, 64(11):2417–2440, 2015.

D.V. Khlopin On boundary conditions at infinity for infinite horizon control problem // In: *Constructive Nonsmooth Analysis and Related Topics (dedicated to the memory of V.F. Demyanov)(CNSA)*. pages 139-142. VVM Publ, St. Petersburg, 2017. , D.V. Khlopin On necssary limit gradients in control problems with infinite horizon. *Trudy instituta matematiki i mekhaniki UrO RAN*, 24:1, 247–256, 2018.

D.V. Khlopin On lipschitz continuity of value functions for infinite horizon problem. *Pure and Applied Functional Analysis*, 2(3):535–552, 2017.

Y.S. Ledyaev, J.S. Treiman Sub-and supergradients of envelopes, semicontinuous closures, and limits of sequences of functions. *Russ Math Surv*, 67:345–373, 2012

B. Miller, E. Rubinovich. *Impulsive Control in Continuous and Discrete-Continuous Systems.* Kluwer Academic / Plenum Publishers, New York, 2003.

B.S. Mordukhovich *Variational Analysis and Generalized Differentiation I. Basic Theory.* Springer, Berlin, 2006.

F.L. Pereira, G.N. Silva. A maximum principle for constrained infinite horizon dynamic control systems. *IFAC Proceedings Volumes* 44(1):10207-10212, 2011.

R.Rempala, J.Zabczyk. On the maximum principle for deterministic impulsive control problems. *J. Optim. Theory Appl*, 59(2):281-288, 1988.

N. Sagara Value functions and transversality conditions for infinite-horizon optimal control problems. *Set-Valued Var Anal*, 18:1–28, 2010.

G.N. Silva, R.B. Vinter. Necessary conditions for optimal impulsive control problems. *SIAM J. Cont. Optim*, 35:1829–1846, 1997.

S.T. Zavalischin, A.N. Sesekin. *Dynamic impulse systems: theory and applications*, Springer, Berlin, 2013.