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Numerical Stackelberg Solutions in a Class of Positional Differential Games

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Abstract: We consider a problem of numerical construction of a Stackelberg solution in a differential game with closed-loop information structure and terminal player payoffs. It is divided into two subproblems: the problem of computing of so-called "admissible" motions and the optimization problem on the set of admissible motions. We focus on the latter problem and consider two approaches: enumeration of the follower player payoffs and enumeration of the leader player payoffs. Algorithms implementing both approaches are presented and tested on a model system.

Keywords: Differential games, Stackelberg games, Numerical algorithms

1. INTRODUCTION

We consider a positional differential Stackelberg game with a hierarchy of two players: "the leader" and "the follower". The leader chooses and announces a strategy to the follower before the game and the follower then chooses a rational response. This construction allows us to pass from a game problem to an optimal control problem.

There are numerous works devoted to solutions in differential Stackelberg games, most of which consider either linear-quadratic games or games with open-loop information structure. A survey of classical settings and approaches to solving problems of these types can be found in Başar and Olsder (1999).

In this paper we consider a different class of games. Namely positional (closed-loop information structure) games with terminal player payoffs. In order to define player strategies and motions we follow the setting of the theory of (zero-sum) positional differential games developed by N.N. Krasovskii and his scientific school (see Krasovskii and Subbotin (1988)) and the theoretical apparatus for non-zero-sum games from Kleimenov (1993). This work continues Osipov (2007) and Kuvshinov and Osipov (2018) and is closely related to the latter. Here we refine the proposed numerical algorithm and provide new computation results.

The paper is organized as follows. Section 2 contains a brief description of the problem statement. Section 3 defines admissible motions. Section 4 describes an approach based upon follower payoff enumeration, or globally maximizing leader payoff as a function of follower payoff. Section 5 describes an alternative approach based upon leader payoff enumeration. Section 6 provides some numerical computation results on how the two approaches compare to each other for a model system.

2. PROBLEM STATEMENT

Let the dynamics of the differential game be described by the equation

$$\dot{x}(t) = f(t, x(t), u(t), v(t)), \quad x(t_0) = x_0, \quad t \in [t_0, \vartheta], \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is a phase vector, $\vartheta > t_0$ is a fixed final time, when the player payoffs are evaluated. Controls $u(t) \in P$ and $v(t) \in Q$ are handled by the leader and the follower respectively. Sets P and Q are compacts in some vector spaces.

Player payoffs are defined as

$$I_i = \sigma_i(x(\vartheta)), \quad i = 1, 2. \quad (2)$$

The leader maximizes I_1 while the follower maximizes I_2.

Let $G \subset [t_0, \vartheta] \times \mathbb{R}^n$ be a compact set such that $(t_0, x_0) \in G$ and every motion of (1) beginning in G lies in G. We assume the following conditions to hold.

(1) Both players have full information about the system and the current state $x(t)$.
(2) The function $f\colon G \times P \times Q \mapsto \mathbb{R}^n$ is continuous.
(3) There is $L > 0$ such that
$$\|f(t, x', u, v) - f(t, x'', u, v)\| \leq L\|x' - x''\|$$
for any $(t, x') \in G$, $(t, x'') \in G$, $u \in P$ and $v \in Q$.
(4) There is $\kappa > 0$ such that $\|f(t, x, u, v)\| \leq \kappa(1 + \|x\|)$ for any $(t, x) \in G$, $u \in P$ and $v \in Q$.
(5) For all $s \in \mathbb{R}^n$ and any $(t, x) \in G$ holds
$$\max_{u \in P} \min_{v \in Q} s^\top f(t, x, u, v) = \min_{v \in Q} \max_{u \in P} s^\top f(t, x, u, v).$$
(6) The set $\{ f(t, x, u, v) \mid u \in P, v \in Q \}$ is convex for each $(t, x) \in G$.
(7) Functions $\sigma_i\colon \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, 2$, are continuous.

Following Kleimenov (1993) we define (pure) leader strategy as a pair of functions $U = (u(t, x, \varepsilon), \beta_1(\varepsilon))$, $u\colon G \times (0, +\infty) \mapsto P$, $\beta_1\colon (0, +\infty) \mapsto (0, +\infty)$. The function $\beta_1(\varepsilon)$ is continuous, monotone and $\lim_{\varepsilon \to 0} \beta_1(\varepsilon) = 0$.

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The follower strategy is defined analogously as a pair $V = (v(t, x, \varepsilon), \beta_2(\varepsilon))$.

Functions β_i, $i = 1, 2$, determine upper bounds for time steps used by the players when building broken line (approximate) motions. Here we will skip description of the corresponding limit motions, details may be found in Krasovskii and Subbotin (1988) and Kleimenov (1993).

The following definition is schematic. Denote $V(U)$ a follower strategy that maximizes I_2 (2) when the leader follows the strategy U. Denote U^S a leader strategy that maximizes I_1 when the follower follows the strategy $V^S = V(U^S)$. Stackelberg solutions are defined then as pairs of strategies (U^S, V^S).

We call two different solutions "equivalent" if both players' payoffs on these solutions are the same.

3. ADMISSIBLE MOTIONS

Let us introduce an auxiliary zero-sum positional differential game Γ_2. The dynamics of this game is defined by (1). Here the follower maximizes payoff I_2 (2) while the leader minimizes I_2. Due to Krasovskii and Subbotin (1988) it is known that Γ_2 has a "universal saddle point"

$$(u^{(2)}(t, x, \varepsilon), v^{(2)}(t, x, \varepsilon)) \tag{3}$$

and continuous value $\gamma_2(t, x)$ where $(t, x) \in G$ is considered an initial position. The word "universal" means that these controls are optimal for any position $(t, x) \in G$ taken as an initial position.

Problem 1. Find measurable controls $u\colon [t_0, \vartheta] \mapsto P$ and $v\colon [t_0, \vartheta] \mapsto Q$ maximizing I_1 given that the corresponding solution $x(t)$ of (1) satisfies

$$\gamma_2(t, x(t)) \le \gamma_2(\vartheta, x(\vartheta)) = \sigma_2(x(\vartheta)), \quad t \in [t_0, \vartheta]. \tag{4}$$

Theorem 2. (Theorem 1.10 (Kleimenov, 1993, p. 35)). Let all assumptions listed above hold then solutions of problem 1 exist.

Let $u^*(t)$ and $v^*(t)$ be some player controls and let $x^*(t)$ be the corresponding solution of (1). Now consider the player strategies $U^0 = (u^0(t, x, \varepsilon), \beta_1^0(\varepsilon))$, $V^0 = (v^0(t, x, \varepsilon), \beta_2^0(\varepsilon))$, where

$$\begin{aligned} u^0(t, x, \varepsilon) &= \begin{cases} u^*(t) \text{ if } \|x - x^*(t)\| < \varepsilon\varphi(t), \\ u^{(2)}(t, x, \varepsilon) \text{ otherwise}, \end{cases} \\ v^0(t, x, \varepsilon) &= \begin{cases} v^*(t) \text{ if } \|x - x^*(t)\| < \varepsilon\varphi(t), \\ v^{(2)}(t, x, \varepsilon) \text{ otherwise}. \end{cases} \end{aligned} \tag{5}$$

Here $\varphi(t)$ is some estimate of accumulated broken line error. Functions $\beta_i^0(\cdot)$, $i = 1, 2$, are to be selected in order to prevent violation of the condition $\|x - x^*(t)\| < \varepsilon\varphi(t)$ when both players are faithfully following motion $x^*(t)$.

Leader's strategy U^0 may be interpreted as follows. Follow motion $x^*(t)$ using $u^*(t)$. If at some $t^* \in [t_0, \vartheta)$ the follower refuses to follow $x^*(t)$ switch to the "penalty strategy" $u^{(2)}(t, x, \varepsilon)$ (3) in order to minimize the follower's payoff, which then can't be greater than

$$\max\{\, \gamma_2(t^*, x) \mid (t^*, x) \in G \wedge \|x - x^*(t^*)\| \le \varepsilon\varphi(t^*) \,\}.$$

A summary of theorems 1.11, 1.12, 1.14 and 1.15 from (Kleimenov, 1993, pp. 36–40) may be presented in the form of the following theorem.

Theorem 3. Let all assumptions listed above hold. Suppose that controls $u^*(t)$ and $v^*(t)$ constitute a solution of problem 1. Then a pair of strategies (U^0, V^0) (5) is a Stackelberg solution. Inversely: for every Stackelberg solution there exists an equivalent Stackelberg solution in the form (5) where controls $u^*(t)$ and $v^*(t)$ constitute a solution of problem 1.

Thus we can search Stackelberg solutions in the form (5). In order to do this we have to solve problem 1. A pair of possible controls $u\colon [t_0, \vartheta] \mapsto P$ and $v\colon [t_0, \vartheta] \mapsto Q$ such that the corresponding solution $x(t)$ of (1) satisfies the inequality (4) is "admissible controls". The motion $x(t)$ is "admissible motion". Any motion produced by some Stackelberg solution is an admissible motion maximizing the leader's payoff I_1 (2).

Define sets

$$M_i^c = \{\, x \in \mathbb{R}^n \mid \sigma_i(x) \ge c \,\}, \quad i = 1, 2. \tag{6}$$

Then a set

$$W_2^c = \{\, (t, x) \in G \mid \gamma_2(t, x) \ge c \,\}$$

is a "maximal stable bridge" in a pursuit-evasion game (for details see Krasovskii and Subbotin (1988)) with dynamics (1) and the target set M_2^c (6) where the follower tries to steer $x(\vartheta)$ into M_2^c, while the leader opposes. There are a variety of works on (approximate or exact) computation of maximal stable bridges. To list a few: Ganebny et al. (2012) and Kamneva and Patsko (2016) (linear dynamics), Tarasyev et al. (1992) (non-linear dynamics). Below we assume that we can build W_2^c for any reasonable $c \ge \gamma_2(t_0, x_0)$.

In Osipov (2007) inequality (4) was interpreted in terms of maximal stable bridges in the form of the following proposition.

Proposition 4. A motion $x(t)$ is admissible if and only if there exists $c \in \mathbb{R}$ such that $x(\vartheta) \in M_2^c$ and $(t, x(t)) \notin \operatorname{int} W_2^c$ for all $t \in [t_0, \vartheta)$.

4. ENUMERATION OF FOLLOWER PAYOFFS

Denote $\underline{c}_2 = \gamma_2(t_0, x_0)$ and $\overline{c}_2 = \max\{\, \sigma_2(x) \mid (\vartheta, x) \in G \,\}$. Denote $D = \{\, x(\vartheta) \mid x(\cdot) \text{ is admissible motion} \,\}$. So

$$D^{c_2} = \{\, x \in D \mid \sigma_2(x) = c_2 \,\}, \quad c_2 \in [\underline{c}_2, \overline{c}_2],$$

comprises endpoints of all admissible motions providing the follower payoff c_2. Now we may formally introduce the function

$$c_1^{max}(c_2) = \max\{\, \sigma_1(x) \mid x \in D^{c_2} \,\}. \tag{7}$$

This function gives the maximal possible leader payoff when the follower payoff is fixed to be c_2. Now Stackelberg solution may be found by maximizing $c_1^{max}(c_2)$ on $[\underline{c}_2, \overline{c}_2]$. But we have to point out that in general:

(1) We do not have an explicit analytic formula representing $c_1^{max}(c_2)$.
(2) Function $c_1^{max}(c_2)$ is not unimodal.
(3) Function $c_1^{max}(c_2)$ is not Lipschitz continuous (Karasev et al. (2017)).

This makes the problem of finding a global maximum of $c_1^{max}(c_2)$ with arbitrary precision hard (or even unsolvable in general case).

Of course, we can try any numerical derivative-free scalar optimization method if we can (approximately) compute $c_1^{max}(c_2)$. In Osipov (2007) it was done by grid search enumerating $c_2 \geq \underline{c}_2$ with some small step, while $c_1^{max}(c_2)$ was computed approximately using polygonal approximations of $W_2^{c_2}$ and D^{c_2} with computational geometry procedures (the case of linear dynamics in plane).

However, in $\mathbb{R}^n$, $n > 2$, it may be difficult to explicitly build geometric approximation of D^{c_2} due to scalability issues of non-convex polyhedra and software complexity.

Problem 5. Given c_1 and c_2 find an admissible motion on which the leader payoff is at least c_1 and the follower payoff is exactly c_2.

Assume that we can compute the following function and the corresponding admissible motion if one exists (in the case of linear dynamics this can be done using only convex polyhedra as described in Kuvshinov and Osipov (2018)):

$$S(c_1, c_2) = \begin{cases} 1 \text{ if problem 5 has a solution for } c_1,\, c_2, \\ 0 \text{ otherwise.} \end{cases} \tag{8}$$

Now we can approximate $c_1^{max}(c_2)$ to arbitrary precision using binary search (Kuvshinov and Osipov (2018), lemma 1).

Let $A^c \subset [t_0, \vartheta] \times G$ be a reachability set of (1) with phase constraint $(t, x(t)) \notin \operatorname{int} W_2^c$ for $t \in [t_0, \vartheta)$. Denote $A^c(t) = \{\, x \in \mathbb{R}^n \mid (t, x) \in A^c \,\}$.

Geometric sense of $S(c_1, c_2)$ is revealed by lemma 6.

Lemma 6. For $c_1 \in [\underline{c}_1, \overline{c}_1]$ and $c_2 \in [\underline{c}_2, \overline{c}_2]$ we have $S(c_1, c_2) = 1$ if and only if $(M_1^{c_1} \cap M_2^{c_2}) \cap A^{c_2}(\vartheta) \neq \varnothing$.

Proof. ($\Rightarrow$) $S(c_1, c_2) = 1$ means there is admissible $x(t)$ such that $\sigma_1(x(\vartheta)) \geq c_1$ and $\sigma_2(x(\vartheta)) = c_2$ so $x(\vartheta) \in M_1^{c_1} \cap \partial M_2^{c_2}$ and $x(t) \in A^{c_2}(t)$ due to definition of A^{c_2}. That means the set $M_1^{c_1} \cap M_2^{c_2} \cap A^{c_2}(\vartheta)$ contains at least $x(\vartheta)$ and thus it is not empty. ($\Leftarrow$) Let $x^* \in M_1^{c_1} \cap M_2^{c_2} \cap A^{c_2}(\vartheta)$. Due to definition of A^{c_2} there is at least one admissible motion $x(t)$ such that $x(\vartheta) = x^*$. Then we have $\sigma_i(x(\vartheta)) \geq c_i$, $i = 1, 2$. But due to proposition 4 we have $A^{c_2} \cap \operatorname{int} W_2^{c_2} = \varnothing$ so $A^{c_2}(\vartheta) \cap \operatorname{int} M_2^{c_2} = \varnothing$ and we have $\sigma_2(x(\vartheta)) \leq c_2$. That means $x(t)$ solves problem 5 and $S(c_1, c_2) = 1$.

The following algorithm may be proposed as an alternative way to do grid maximum search. We compute $c_1^{max}(c_2^i)$ on a grid with nodes $\{\, c_2^i \,\}_{i=0}^N$ for some $N \in \mathbb{N}$. All $c^{max}(c_2^i)$ are evaluated by binary search in lockstep, at each binary search step all nodes with upper bounds not higher than the current maximal lower bound are removed.

Algorithm 1. Grid search.

(1) Input: set $C = \{\, c_2^i \,\}_{i=0}^N \subset [\underline{c}_2, \overline{c}_2]$, precision parameter tol > 0.
(2) Let lo and hi be dictionaries, mapping numbers to numbers.
(3) Assign gap $\leftarrow \overline{c}_1 - \underline{c}_1$.
(4) For all $c \in C$:
  (a) Assign $lo[c] \leftarrow \underline{c}_1$.
  (b) Assign $hi[c] \leftarrow \overline{c}_1$.
(5) While gap $> tol$:
  (a) For all $c \in C$:
    (i) Let $m = (lo[c] + hi[c])/2$.
    (ii) If $S(m, c) = 1$, then $lo[c] = m$.
    (iii) Else $hi[c] = m$.
  (b) Let $L = \max\{\, lo[c] \mid c \in C \,\}$.
  (c) Assign $C \leftarrow \{\, c \in C \mid hi[c] > L \,\}$.
  (d) Assign gap $\leftarrow$ gap/2.
(6) Let $L = \max\{\, lo[c] \mid c \in C \,\}$.
(7) Let $F = \max\{\, c \in C \mid lo[c] = L \,\}$.
(8) Output: payoffs (L, F).

5. ENUMERATION OF LEADER PAYOFFS

It is possible to determine interval $[\underline{c}_1, \overline{c}_1]$ containing leader payoff on a Stackelberg solution just analogously to $[\underline{c}_2, \overline{c}_2]$: $\underline{c}_1 = \gamma_1(t_0, x_0)$ where $\gamma_1(t, x)$ is a value function of a zero-sum game Γ_1 where the leader maximizes I_1 (2) while the follower opposes, $\overline{c}_1 = \max\{\, \sigma_1(x) \mid (\vartheta, x) \in G \,\}$.

Problem 7. Given some $c_1 \in [\underline{c}_1, \overline{c}_1]$ find $c_2 \in [\underline{c}_2, \overline{c}_2]$ such that $S(c_1, c_2) = 1$ (8) or determine that $S(c_1, c_2) = 0$ for all $c_2 \in [\underline{c}_2, \overline{c}_2]$.

It should be pointed out that $S(\underline{c}_1, \underline{c}_2) = 1$.

Lemma 8. There is $c_1^* \in [\underline{c}_1, \overline{c}_1]$ such that problem 7 has solution for any $c_1 \in [\underline{c}_1, c_1^*]$ and does not have solutions for any $c_1 \in (c_1^*, \overline{c}_1]$.

Proof. Admissible motions exist (theorem 2) and contain a motion providing a Stackelberg solution (theorem 3) that means that maximum of leader payoff on admissible motions exist. Let such a motion provide the leader with payoff c_1^* and the follower with payoff c_2^* then obviously $S(c_1, c_2^*) = 1$ for any $c_1 \in [\underline{c}_1, c_1^*]$ (see problem 5) and $S(c_1, c_2) = 0$ for any $c_1 \in (c_1^*, \overline{c}_1]$ and any $c_2 \in [\underline{c}_2, \overline{c}_2]$.

If we can solve problem 7 then due to lemma 8 we can find an admissible motion (and hence a solution in form (5)), on which the leader payoff is arbitrarily close to that on a Stackelberg solution using binary search.

Denote

$$c_2^m(c_1, c_2) = \max\{\, \sigma_2(x) \mid x \in M_1^{c_1} \cap A^{c_2}(\vartheta) \,\}. \tag{9}$$

Lemma 9. Let $c_1 \in [\underline{c}_1, \overline{c}_1]$ and $c_2 \in [\underline{c}_2, \overline{c}_2]$ are such that $M_1^{c_1} \cap A^{c_2}(\vartheta) \neq \varnothing$. Then $c_2^m(c_1, c_2)$ can be approximated to any given arbitrary precision.

Proof. Let x be an arbitrary point in $M_1^{c_1} \cap A^{c_2}(\vartheta)$, then $\sigma_2(x) \leq c_2^m(c_1, c_2) \leq c_2$. Given some precision $\zeta > 0$ we can use binary search to find $c_2^* \in [\sigma_2(x), c_2]$ such that $M_1^{c_1} \cap M_2^{c_2^*} \cap A^{c_2}(\vartheta) \neq \varnothing$ and $M_1^{c_1} \cap M_2^{c_2^*+\zeta} \cap A^{c_2}(\vartheta) = \varnothing$. It follows then that $c_2^m(c_1, c_2) \in [c_2^*, c_2^* + \zeta]$.

Lemma 10. (Lemma 2 in Kuvshinov and Osipov (2018)). Let $c_1 \in [\underline{c}_1, \overline{c}_1]$ find $c_2 \in [\underline{c}_2, \overline{c}_2]$ are such that $M_1^{c_1} \cap A^{c_2}(\vartheta) = \varnothing$. Then $S(c_1, c_2') = 0$ for all $c_2' \in (c_2^*, c_2]$, where $c_2^* = c_2^m(c_1, c_2) < c_2$.

Lemma 10 paves the way to the following algorithm.

Algorithm 2. Problem 7 solver.

(1) Input: $c_1 \in [\underline{c}_1, \overline{c}_1]$.
(2) Assign $c_2 \leftarrow \overline{c}_2$.
(3) If $c_2 < \underline{c}_2$ or $M_1^{c_1} \cap A^{c_2}(\vartheta) = \varnothing$, then exit: $S(c_1, \cdot) \equiv 0$.

(4) If $(M_1^{c_1} \cap M_2^{c_2}) \cap A^{c_2}(\vartheta) \neq \varnothing$, then exit: $S(c_1, c_2) = 1$.
(5) Assign $c_2 \leftarrow c_2^m(c_1, c_2)$.
(6) Repeat from (3).

Lemma 11. (Lemma 3 in Kuvshinov and Osipov (2018)). If problem 7 does not have a solution then algorithm 2 establishes this fact after a finite number of iterations.

Lemma 12. (Lemma 4 in Kuvshinov and Osipov (2018)). Suppose there is $c_2' \in [\underline{c}_2, \overline{c}_2]$ such that $S(c_1, c_2') = 1$ and $S(c_1, c_2) = 0$ for all $c_2 \in (c_2', \overline{c}_2]$. Then the sequence of values c_2 computed by algorithm 2 converges to c_2'.

It seems improbable that in general case algorithm 2 will reach the solution (c_2' from lemma 12) in finite number of iterations. Consider two possibilities:

(1) $M_2^{c_2} \cap A^{c_2}(\vartheta) = \varnothing$. That implies $W_2^{c_2} \cap A^{c_2} = \varnothing$, thus step 5 of the algorithm may return the maximal c_2 reachable, in which case we'll immediately get the solution. Or we may fall into the second possibility.
(2) $M_2^{c_2} \cap A^{c_2}(\vartheta) \neq \varnothing$. Due to convergence we can reach $M_1^{c_1-\eta} \cap M_2^{c_2} \cap A^{c_2}(\vartheta) \neq \varnothing$ for any $\eta > 0$ in finite number of iterations.

The results presented in this section make it possible to propose algorithm 3. For convenience we assume a Boolean function trace(T, W) is defined, which tells whether there is a motion $x(t)$ (1) such that $x(\vartheta) \in T$ and $(t, x(t)) \notin \operatorname{int} W$ for any $t \in [t_0, \vartheta)$. If it is true then we have this motion available as $x(t)$. Due to lemma 6 we can write $S(c_1, c_2) = \text{trace}(M_1^{c_1} \cap M_2^{c_2}, W_2^{c_2})$.

Algorithm 3. Stackelberg motion approximation.

(1) Input: precision parameter tol > 0, parameters $\zeta > 0$ and $\eta \in (0, \text{tol})$.
(2) Assign $(l, u) \leftarrow (\underline{c}_1, \overline{c}_1)$.
(3) While $u - l > \text{tol}$, do:
  (a) Assign $c_1 \leftarrow (l + u)/2$.
  (b) Assign $c_2 \leftarrow \max\{\, \sigma_2(x) \mid (\vartheta, x) \in G \wedge \sigma_1(x) \geq c_1 \,\}$.
  (c) Loop:
    (i) If $c_2 < \underline{c}_2$ or not trace$(M_1^{c_1}, W_2^{c_2})$, then assign $u \leftarrow c_1$, exit loop.
    (ii) If trace$(M_1^{c_1} \cap M_2^{c_2}, W_2^{c_2})$, then assign $l \leftarrow c_1$ (now we have the next approximation $x(t)$ with payoffs c_1, c_2), exit loop.
    (iii) Assign $c_2^{next} \leftarrow c_2^m(c_1, c_2)$.
    (iv) If $c_2 - c_2^{next} < \zeta$, then assign $c_1 \leftarrow c_1 - \eta$.
    (v) If $c_1 \leq l$, then exit while.
    (vi) Assign $c_2 \leftarrow c_2^{next}$.
(4) The last approximation $x(t)$ is the result.

An alternative may be to decrease c_2 instead. That is change (iv) to

(iv) If $c_2 - c_2^{next} < \zeta$, then assign $c_2^{next} \leftarrow c_2 - \eta$.

We denote the first variant of the algorithm "Leader-c_1" and the alternative variant "Leader-c_2".

6. MODEL SYSTEM

Consider the system with the following dynamics:

$$\dot{z}_1(t) = (\vartheta - t)(F_{11}(t)\cos\varphi(t) - F_{12}(t)\sin\varphi(t) + F_{21}(t)),$$
$$\dot{z}_2(t) = (\vartheta - t)(F_{11}(t)\sin\varphi(t) + F_{12}(t)\cos\varphi(t) + F_{22}(t)),$$
$$z_1(0) = -\rho_0, \quad z_2(0) = 0.$$

Here the leader chooses $(F_{11}(t),\ F_{12}(t))$ given $F_{11}^2(t) + F_{12}^2(t) \leq 1$, and the follower chooses $(F_{21}(t),\ F_{22}(t))$ and $\varphi(t)$, given $F_{21}^2(t) + F_{22}^2(t) \leq 1$ and $|\varphi(t)| \leq \varphi_0$.

Player payoffs are defined as:

$$I_1 = -(z_1(\vartheta) - a_1)^2 - (z_2(\vartheta) - a_2)^2,$$
$$I_2 = -z_1(\vartheta)^2 - z_2(\vartheta)^2.$$

Thus the leader's objective is to get as close as possible to point (a_1, a_2).

The set of admissible motions in this system can be described analytically, see Osipov (2007) for details.

We consider three numerical algorithms:

- "Brent": maximization of approximately computed $c_1^{max}(c_2)$, $c_2 \in [\underline{c}_2, \overline{c}_2]$ using Brent's method (a general derivative-free scalar optimization method, see Brent (1973)).
- "Grid": maximization of $c_1^{max}(c_2^i)$, $c_2^i = \underline{c}_2 + \delta i$, $i = \overline{0, N}$, where $\delta = (\overline{c}_2 - \underline{c}_2)/N$ by algorithm 1 proposed in section 4.
- "Leader" is algorithm 3 proposed in section 5, $c_2^m(c_1, c_2)$ is approximated using binary search. It has two variants: "Leader-c_1" and "Leader-c_2".

Let $\rho_0 = 0.5$, $\varphi_0 = \pi/4$, $\vartheta = 1$. We choose $\underline{c}_1 = -a_1^2 - a_2^2$, $\underline{c}_2 = \gamma_2(0, (z_1(0), z_2(0))) = (1 - \cos\varphi_0)\vartheta^2/2 - \rho_0 \approx -0.353553$, $\overline{c}_1 = \overline{c}_2 = 0$. Binary search precision in all cases is 10^{-4}, in algorithm 3 parameters tol $= \eta = \zeta = 10^{-4}$. Grid size $N = \lceil 10^4(\overline{c}_2 - \underline{c}_2) \rceil = 3536$, so that the grid step is approximately 10^{-4} too.

We consider three choices of (a_1, a_2) listed in table 1. Value I_1^S is the actual leader's payoff on a Stackelberg solution. All numerical methods except "Leader-c_1" give the same $I_1^N \approx I_1^S$ value in each case. This is not strange because each method ultimately uses binary search to approximate I_1^S with the same bounds $[\underline{c}_1, \overline{c}_1]$ and the same precision. In each case absolute error $I_1^S - I_1^N < 10^{-4}$ as it should be.

Table 1. Solution precision: most methods

Case	a_1	a_2	I_1^N	I_1^S	$I_1^S - I_1^N$
1	0.3	0.25	−0.050912	−0.050832	$8 \cdot 10^{-5}$
2	0.3	0.27	−0.056265	−0.056210	$5.5 \cdot 10^{-5}$
3	0.25	0	−0.021301	−0.021247	$5.4 \cdot 10^{-5}$

"Leader-c_1" produces a bit different results presented in table 2.

Table 2. Solution precision: "Leader-c_1"

Case	a_1	a_2	I_1^N	I_1^S	$I_1^S - I_1^N$
1	0.3	0.25	−0.050879	−0.050832	$4.8 \cdot 10^{-5}$
2	0.3	0.27	−0.056300	−0.056210	$9 \cdot 10^{-5}$
3	0.25	0	−0.021340	−0.021247	$9.3 \cdot 10^{-5}$

Table 3 presents follower's payoffs and total amount of evaluations of function trace ("trace" column) and functions $c_1^{max}(c_2)$ or $c_2^m(c_1, c_2)$ ("evaluations" column) for all combinations of a numerical method and (a_1, a_2) choice. These functions contribute to trace evaluations as well due to being approximated by binary search.

If we don't remove nodes with too low upper bounds in "Grid" method, then we get 42432 trace evaluations in each case, which is several times worse than the proposed variant. Still "Grid" method in this example is much more computationally complex than other tested methods, while "Brent" (a good general optimization method) is the obvious winner here. This may result from the fact that in this example $c_1^{max}(c_2)$ is actually a smooth unimodal function, while our "Leader" method is designed in order to provide the solution without any special assumptions about properties of $c_1^{max}(c_2)$.

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Table 3. Numerical methods

Case	I_2^N	trace	evaluations
Brent-1	−0.345917	156	(c_1^{max}) 13
Grid-1	−0.344155	7426	—
Leader-c_1-1	−0.345062	1159	(c_2^m) 145
Leader-c_2-1	−0.344196	990	(c_2^m) 125
Brent-2	−0.353303	216	(c_1^{max}) 18
Grid-2	−0.352254	6650	—
Leader-c_1-2	−0.352042	3208	(c_2^m) 510
Leader-c_2-2	−0.352245	415	(c_2^m) 72
Brent-3	−0.231224	180	(c_1^{max}) 15
Grid-3	−0.230370	6643	—
Leader-c_1-3	−0.230397	4351	(c_2^m) 362
Leader-c_2-3	−0.230335	864	(c_2^m) 76