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Finite-dimensional approximations of neutral-type conflict-controlled systems

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Abstract: The paper deals with a dynamical system described by a neutral-type functional differential equation in Hale's form which is controlled under conditions of unknown disturbances. This system is approximated by a controlled system of ordinary differential equations. An aiming procedure between the initial and approximating systems is elaborated. An application of such procedure is given for solving a control problem for linear neutral-type dynamical systems. An illustrative example is considered, simulation results are shown.

Keywords: control theory; differential games; neutral-type systems.

1. INTRODUCTION

The research of approximations of delay differential equations by ordinary differential equations have an extensive history. The convergence of such approximations for linear systems with constant delays was proved in Krasovskii (1964). In Repin (1965), this result was extended to nonlinear systems, and in Kurjanskii (1967) – to the case of variable delays. Later, similar approximations, their generalizations and applications to different kinds of problems were considered, for example, by Kryajimskii (1978); Banks and Burns (1978); Banks and Kappel (1979); Kunisch (1980); Fabiano (2013); including, approximations of neutral-type functional differential equations were considered in Kunisch (1980); Fabiano (2013). It was proposed in Krasovskii and Kotelnikova (2011) to use the approximating system of ordinary differential equations as a leader for the initial time-delay dynamical system controlled under conditions of disturbances or counteractions. In this case, an auxiliary problem arises of the aiming between the motion of the initial conflict-controlled system and the motion of the approximating system. A solution of this problem was given for conflict-controlled dynamical systems described by delay-type functional differential equations in Lukoyanov and Plaksin (2015), for systems described by linear neutral-type equations in Plaksin (2015), and for nonlinear neutral-type systems in Hale's form in Lukoyanov and Plaksin (2016). The present paper continues these investigations.

The paper is organized as follows. In Section 2 a conflict-controlled dynamical system described by a neutral-type functional differential equation in Hale's form is considered. In Section 3 a controlled approximating system described by ordinary differential equations is constructed, its useful properties are given. In order to provide the proximity of initial and approximating systems, in Section 4 a stable aiming procedure between these systems is elaborated. Effectiveness of the procedure is illustrated by an example in Section 5. In Section 6 an application of the aiming procedure to the following control problem is considered. A dynamical system controlled under conditions of unknown disturbances is described by a linear neutral-type functional differential equation. A quality index evaluates a motion history and realizations of control and disturbance actions. The goal of the control is to minimize the quality index. Within the game-theoretical approach of Krasovskii and Subbotin (1988) the problem of calculating the optimal guaranteed result of the control and constructing an optimal control scheme is posed. For its solution the auxiliary control problem for the approximating system is considered. It is shown that the optimal guaranteed result of the control in the auxiliary problem approximate the optimal guaranteed result in the initial control problem. The optimal control scheme is based on using optimal motions of the auxiliary control problem as a leader. In Section 7, an illustrative example is considered, simulation results are shown.

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2. CONFLICT-CONTROLLED SYSTEM

Consider a dynamical system described by the following neutral-type functional differential equation in Hale's form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x[t] - g\big(t, x_t[\cdot]\big)\Big) = f\big(t, x_t[\cdot], u[t], v[t]\big), \tag{1}$$

$$t \in [t_0, \vartheta], \quad x[t] \in \mathbb{R}^n, \quad u[t] \in \mathbb{U}, \quad v[t] \in \mathbb{V},$$

with the initial condition

$$x_{t_0}[\xi] = x[t_0 + \xi] = z[\xi], \quad \xi \in [-h, 0], \quad z[\cdot] \in Z. \tag{2}$$

Here t is the time variable; $x[t]$ is the value of the state vector at the time t; $h > 0$ is the delay constant; $x_t[\cdot]$

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is the motion history on the interval $[t-h,t]$, defined by $x_t[\xi]=x[t+\xi]$, $\xi\in[-h,0]$; $u[t]$ and $v[t]$ are respectively the current control and disturbance actions. It is assumed that times t_0, ϑ are given, $\mathbb{U}$, $\mathbb{V}$ are known compact sets of finite-dimensional spaces, Z is the set of absolutely continuous functions $z[\cdot]$ that satisfy the inequalities

$$\big\|z[\xi]\big\|\le R_0,\ \xi\in[-h,0], \tag{3}$$

$$\big\|\dot z[\xi]\big\|\le R_0 \text{ for almost all } \xi\in[-h,0],\quad R_0=\text{const}>0.$$

Hereinafter, the dot over a symbol denotes the derivative, double brackets $\|\cdot\|$ denote the Euclidean norm. Below, angle brackets $\langle\cdot,\cdot\rangle$ are used for the scalar product of vectors. The space of all continuous functions from $[-h,0]$ to $\mathbb{R}^n$ is denoted by $C=C([-h,0],\mathbb{R}^n)$ and is endowed with the supremum norm $\|\cdot\|_C$.

The following conditions are assumed:

(C.1) The mapping $[t_0,\vartheta]\times C\times\mathbb{U}\times\mathbb{V}\ni(t,w[\cdot],u,v)\mapsto f=f(t,w[\cdot],u,v)\in\mathbb{R}^n$ is continuous.

(C.2) There exists a constant $\alpha_f>0$ such that

$$\big\|f\big(t,w[\cdot],u,v\big)\big\|\le\alpha_f\big(1+\big\|w[\cdot]\big\|_C\big)$$

for all $t\in[t_0,\vartheta]$, $w[\cdot]\in C$, $u\in\mathbb{U}$, and $v\in\mathbb{V}$.

(C.3) For any compact set $D\subset C$ there exists a number $\lambda_f(D)>0$ such that

$$\big\|f\big(t,w[\cdot],u,v\big)-f\big(t,r[\cdot],u,v\big)\big\|\le\lambda_f(D)\big\|w[\cdot]-r[\cdot]\big\|_C$$

for all $t\in[t_0,\vartheta]$, $u\in\mathbb{U}$, $v\in\mathbb{V}$, and $w[\cdot],r[\cdot]\in D$.

(C.4) There exist numbers $\lambda_g>0$ and h_i, $i=\overline{1,k}$, $0<h_1<h_2<\ldots<h_k=h$ such that

$$\big\|g\big(t,w[\cdot]\big)-g\big(t,r[\cdot]\big)\big\|$$

$$\le\lambda_g\bigg(\int\limits_{-h}^{0}\big\|w[\xi]-r[\xi]\big\|\,\mathrm{d}\xi+\sum_{i=1}^{k}\big\|w[-h_i]-r[-h_i]\big\|\bigg),$$

for all $t\in[t_0,\vartheta]$ and $w[\cdot],r[\cdot]\in C$.

(C.5) There exists a constant $\alpha_g>0$ such that

$$\big\|g\big(t,w[\cdot]\big)-g\big(\xi,w[\cdot]\big)\big\|\le\alpha_g\Big(1+\big\|w[\cdot]\big\|_C\Big)|t-\xi|$$

for all $t,\xi\in[t_0,\vartheta]$ and $w[\cdot]\in C$.

Measurable functions $u\colon[t_0,\vartheta)\mapsto\mathbb{U}$ and $v\colon[t_0,\vartheta)\mapsto\mathbb{U}$ are called admissible realizations of actions $u[t]$ and $v[t]$ and denoted by $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$, respectively.

Under conditions (C.1)–(C.5), we can show (see also Lukoyanov and Plaksin (2016)) that, for any admissible realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$ and any function $z[\cdot]\in Z$, there exists a unique solution $x[t_0-h,\vartheta]$ of problem (1), (2), which is an absolutely continuous function on $[t_0-h,\vartheta]$, satisfying initial condition (2) and, together with $u[t]$ and $v[t]$, almost everywhere satisfying equation (1).

3. APPROXIMATING SYSTEM

Let $m\in\mathbb{N}$, $m\ge2$, $\Delta h=h/m$. Applying an approximation idea (see Lukoyanov and Plaksin (2015), Section 3) to the initial system (1), (2) we construct the following system of ordinary differential equations:

$$\begin{cases}\dot y^{[0]}[t]=f\big(t,S\big(Y[t]\big)[\cdot],p[t],q[t]\big),\\ \dot y^{[1]}[t]=\big(y^{[0]}[t]+g\big(t,S\big(Y[t]\big)[\cdot]\big)-y^{[1]}[t]\big)/\Delta h,\\ \dot y^{[i]}[t]=\big(y^{[i-1]}[t]-y^{[i]}[t]\big)/\Delta h,\ i=\overline{2,m},\end{cases} \tag{4}$$

$$t\in[t_0,\vartheta],\quad y^{[i]}[t]\in\mathbb{R}^n,\ i=\overline{0,m},\quad p[t]\in\mathbb{U},\quad q[t]\in\mathbb{V},$$

with the state vector $Y[t]=(y^{[0]}[t],y^{[1]}[t],\ldots,y^{[m]}[t])$ and the initial condition

$$y^{[i]}[t_0]=y_0^{[i]}\in\mathbb{R}^n,\quad i=\overline{0,m}, \tag{5}$$

$$Y_0=\big(y_0^{[0]},y_0^{[1]},\ldots,y_0^{[m]}\big)\in\mathbb{R}^{(m+1)n}.$$

Here $S(Y[t])[\cdot]$ denote the linear spline on the interval $[-h,0]$ with notes $-i\Delta h$, $i=\overline{0,m}$, such that

$$\begin{aligned}S\big(Y[t]\big)[0]&=y^{[1]}[t],\\ S\big(Y[t]\big)[-i\Delta h]&=y^{[i]}[t],\quad i=\overline{1,m}.\end{aligned} \tag{6}$$

By conditions (C.1)–(C.5), for any admissible realizations $p[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$ and any value $Y_0\in\mathbb{R}^{(m+1)n}$, there exists a unique solution $Y[t_0[\cdot]\vartheta]$ of problem (4), (5), which is an absolutely continuous function on $[t_0,\vartheta]$ satisfying initial condition (5) and, together with $p[t]$ and $q[t]$, almost everywhere satisfying system of equations (4).

We assume that initial functions $z[\cdot]$ from (2) and initial vectors $Y_0\in\mathbb{R}^{(m+1)n}$ from (5) are connected by the following condition.

(C.6) The following inequalities are valid:

$$\big\|y_0^{[0]}-z[0]+g\big(t_0,z[\cdot]\big)\big\|\le R_*\Delta h,$$

$$\big\|y_0^{[i]}-z[-i\Delta h]\big\|\le R_*\Delta h,\quad i=\overline{1,m},\quad R_*=\text{const}>0.$$

By a solution $Y[t_0[\cdot]\vartheta]$ of problem (4), (5), we define function $y[t_0-h[\cdot]\vartheta]$:

$$y[t]=\begin{cases}y^{[0]}[t]+g\big(t,S\big(Y[t]\big)[\cdot]\big),\ t\in[t_0,\vartheta],\\ \tilde y[t],\ t\in[t_0-\Delta h,t_0),\\ S\big(Y[t_0]\big)[t-t_0],\ t\in[t_0-h,t_0-\Delta h),\end{cases} \tag{7}$$

where $\tilde y[t]=y[t_0]+(y[t_0]-y^{[1]}[t_0])(t-t_0)/\Delta h$.

Two Lemmas below talk about approximating properties of system (4), (5). This properties help to prove Theorem 3 in the next Section.

Lemma 1. There exists a compact set $D_y\subset C$ such that, for any natural number $m\ge2$, function $z[\cdot]\in Z$, initial value $Y_0\in\mathbb{R}^{(m+1)n}$, and any admissible realizations $p[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$, under conditions (C.1)–(C.6), the following inclusions holds for the solution $Y[t_0[\cdot]\vartheta]$ of problem (4), (5) and the corresponding function $y[t_0-h[\cdot]\vartheta]$ defined in (7):

$$y_t[\cdot]\in D_y,\quad S\big(Y[t]\big)[\cdot]\in D_y,\quad t\in[t_0,\vartheta].$$

Lemma 2. There exists a number $K > 0$ such that, for any natural $m \geq 2$, any function $z[\cdot] \in Z$, and initial value $Y_0 \in \mathbb{R}^{(m+1)n}$, and admissible realizations $p[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$, under conditions (C.1)–(C.6) the solution $Y[t_0[\cdot]\vartheta]$ of problem (4), (5) and the corresponding function $y[t_0 - h[\cdot]\vartheta]$ defined in (7) satisfy the inequality

$$\big\|y_t[\cdot] - S\big(Y[t]\big)[\cdot]\big\|_C \leq K\sqrt{m}, \quad t \in [t_0, \vartheta].$$

The detailed proof of these Lemmas is given in Lukoyanov and Plaksin (2016).

4. AIMING PROCEDURE

The following condition is assumed:

(C.7) For all $t \in [t_0, \vartheta]$, $w[\cdot] \in C$, and $s \in \mathbb{R}^n$ the following equality holds

$$\min_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \big\langle f\big(t, w[\cdot], u, v\big), s\big\rangle = \max_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \big\langle f\big(t, w[\cdot], u, v\big), s\big\rangle.$$

In the differential games theory this condition is called "Isaacs condition" (see, e.g., Isaacs (1965)) or "the saddle point condition in a small game" (see, e.g., Krasovskii and Subbotin (1988)). This assumption does not play a crucial role and can be omitted (see, Remark 4, below).

Let us describe an aiming procedure between systems (1) and (4). We assume that instead of exact values $x[t]$ and $Y[t] = (y^{[0]}[t], \ldots, y^{[m]}[t])$ we known only approximate values $x_*[t]$ and $Y_*[t] = (y_*^{[0]}[t], \ldots, y_*^{[m]}[t])$ such that

$$\big\|x[t] - x_*[t]\big\| \leq \eta, \tag{8}$$

$$\big\|y^{[i]}[t] - y_*^{[i]}[t]\big\| \leq \eta, \quad i = \overline{0, m}, \quad t \in [t_0, \vartheta].$$

The procedure is based on a partition of the control interval $[t_0, \vartheta]$:

$$\Delta_\delta = \big\{t_j : 0 < t_{j+1} - t_j < \delta,\ j = \overline{0, J-1},\ t_J = \vartheta\big\}. \tag{9}$$

Realizations $u[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$ are formed according to the following feedback rule:

$$u[t] = u_j^\circ, \quad q[t] = q_j^\circ, \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, J-1}, \tag{10}$$

where

$$u_j^\circ \in \arg\min_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \big\langle f\big(t_j, x_{*t_j}[\cdot], u, v\big),\ s_*[t_j]\big\rangle, \tag{11}$$

$$q_j^\circ \in \arg\max_{q\in\mathbb{V}} \min_{p\in\mathbb{U}} \big\langle f\big(t_j, S\big(Y_*[t_j]\big)[\cdot], p, q\big),\ s_*[t_j]\big\rangle,$$

$$s_*[t] = x_*[t] - g\big(t, x_{*t}[\cdot]\big) - y_*^{[0]}[t].$$

Theorem 3. Let conditions (C.1)–(C.7) be valid. Then, for any number $\zeta > 0$, there exist numbers $M > 0$, $\delta > 0$ and $\eta > 0$ such that, for any natural number $m \geq M$, any function $z[\cdot] \in Z$, and any admissible realizations $p[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$, if realizations $u[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$ are formed according to aiming procedure (8)–(11), then, for the solution $x[t_0 - h[\cdot]\vartheta]$ of problem (1), (2) and function $y[t_0 - h[\cdot]\vartheta]$ (7) constructed by the solution $Y[t_0[\cdot]\vartheta]$ of problem (4), (5) the following inequality holds:

$$\big\|x[t] - y[t]\big\| \leq \zeta, \quad t \in [t_0, \vartheta].$$

The proof is carried out by the scheme of the proof of Theorem 3 from Lukoyanov and Plaksin (2016).

Remark 4. Theorem 3 is valid without condition (C.7) if we change the aiming procedure (8)–(11) as follows:

$$u[t] = u_j^\circ, \quad q[t] = q_j^\circ(p[t]), \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, J-1},$$

where

$$u_j^\circ \in \arg\min_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \big\langle f\big(t_j, x_{t_j}[\cdot], u, v\big),\ s_*[t_j]\big\rangle,$$

$$q_j^\circ(p) \in \arg\max_{q\in\mathbb{V}} \big\langle f\big(t_j, S\big(Y[t_j]\big)[\cdot], p, q\big),\ s_*[t_j]\big\rangle.$$

5. EXAMPLE 1

Consider a dynamical system described by the following neutral-type functional differential equation in Hale's form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x[t] - \sin\big(x[t-1]\big)\Big) =$$

$$-2\int_{t-1}^{t} x[\xi]e^{-0.3|x[\xi]|}\,\mathrm{d}\xi + u[t] + v[t]x[t-1],$$

$$t \in [0, 5], \quad x[t] \in \mathbb{R}, \quad \big|u[t]\big| \leq 1, \quad \big|v[t]\big| \leq 1,$$

with the initial condition

$$x[\xi] = \xi\cos(6\xi), \quad \xi \in [-1, 0].$$

For this system, according to (4), (5) we construct the approximating ordinary differential system, perform the aiming procedure (8)–(11) and simulate the situation when realizations $v[t_0[\cdot]\vartheta)$ and $p[t_0[\cdot]\vartheta)$ counteract realizations $u[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$, respectively. The results of simulations are shown in Figures 1 and 2.

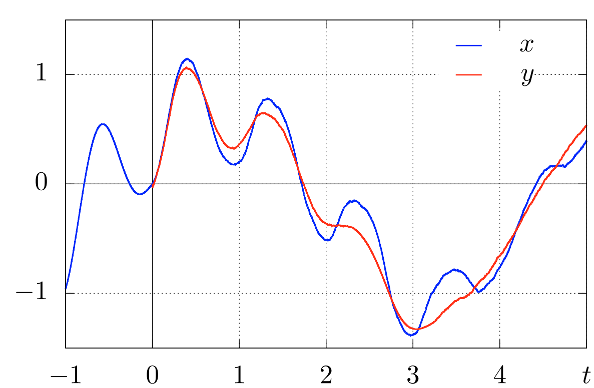

Fig. 1. Simulation results for $m = 50$, $\delta = 0.001$, $\eta = 0.1$.

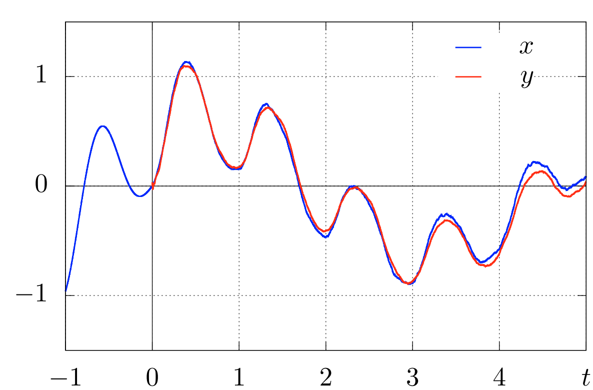

Fig. 2. Simulation results for $m = 500$, $\delta = 0.001$, $\eta = 0.1$.

6. APPLICATION TO A CONTROL PROBLEM

Similarly to Gomoyunov and Plaksin (2015) let us apply aiming procedure (11) for solving the following control problem. A dynamical system is described by the linear neutral-type functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x[t] - A_\tau[t]x[t-\tau]\Big) \tag{12}$$
$$= A[t]x[t] + A_h[t]x[t-h] + B[t]u[t] + C[t]v[t],$$
$$t \in [t_0, \vartheta], \quad x[t] \in \mathbb{R}^n, \quad u[t] \in \mathbb{R}^{n_1}, \quad v[t] \in \mathbb{R}^{n_2},$$

with the initial condition

$$x_{t_0}[\cdot] = z[\cdot] \in Z. \tag{13}$$

Here $\tau, h = \text{const} > 0$ and $0 < \tau \leq h$; matrix function $A_\tau[t]$ is Lipschitz continuous and $\|A_\tau[t]\| < 1$, $t \in [t_0, \vartheta]$; matrix functions $A[t]$, $A_h[t]$, $B[t]$ and $C[t]$ are continuous; The set Z is defined by (3).

It is assumed that admissible values of the control actions $u[t]$ and the disturbance actions $v[t]$ are restricted by the inclusions

$$u[t] \in \mathbb{U} = \big\{u \in \mathbb{R}^{n_1} : \|u\| \leq E\big\},$$
$$v[t] \in \mathbb{V} = \big\{v \in \mathbb{R}^{n_2} : \|v\| \leq E\big\},$$

where the constant $E > 0$ is sufficiently large in order to results of Krasovskii and Subbotin (1988) (see also Lukoyanov (1994)) can be used.

It can be noted that the system (12) satisfies conditions (C.1)–(C.5). Consequently (see Section 1), for any admissible realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$ and any function $z[\cdot] \in Z$, there exists a unique solution $x[t_0 - h[\cdot]\vartheta]$ of problem (12), (13). The triple $\{x[t_0 - h[\cdot]\vartheta], u[t_0[\cdot]\vartheta), v[t_0[\cdot]\vartheta)\}$ is called a control process realization. The quality of this realization is evaluated by the index

$$\gamma = \bigg(\int\limits_{\vartheta - h}^{\vartheta} \|x[\xi]\|^2 \,\mathrm{d}\xi\bigg)^{1/2} + \int\limits_{t_0}^{\vartheta} \Big[\langle u[\xi], \Phi[\xi]u[\xi]\rangle - \langle v[\xi], \Psi[\xi]v[\xi]\rangle\Big]\,\mathrm{d}\xi. \tag{14}$$

Here $\Phi[\xi]$ and $\Psi[\xi]$ are symmetric continuous matrix functions such that the quadratic forms $\langle u, \Phi[\xi]u\rangle$ and $\langle v, \Psi[\xi]v\rangle$ are positive definite for $\xi \in [t_0, \vartheta]$.

The goal of the control is to minimize quality index (14). Let us note that, since the disturbance is unknown, the worst-case may occur when the disturbance maximize (14).

According to Gomoyunov, Lukoyanov and Plaksin (2016) (see also Krasovskii (1985)) the control problem (12)–(14) is posed as follows.

A control strategy $u(\cdot)$ is an arbitrary function

$$u(t, w[\cdot], \varepsilon) \in \mathbb{U}, \quad t \in [t_0, \vartheta], \quad w[\cdot] \in C, \quad \varepsilon > 0,$$

where $\varepsilon > 0$ is the accuracy parameter.

Let number $\varepsilon > 0$ and partition Δ_δ (9) be chosen. A triple $\{u(\cdot), \varepsilon, \Delta_\delta\}$ defines a control law that forms a piecewise constant control realization according to the following step-by-step rule:

$$u[t] = u\big(t_j, x_{t_j}[\cdot], \varepsilon\big), \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, J-1}.$$

Let us denote by $\Omega = \Omega(u(\cdot), \varepsilon, \Delta_\delta)$ the set of all control process realizations $\{x[t_0 - h[\cdot]\vartheta], u[t_0[\cdot]\vartheta), v[t_0[\cdot]\vartheta)\}$ such that $v[t_0[\cdot]\vartheta)$ is an admissible disturbance realization; $u[t_0[\cdot]\vartheta)$ is the control realization formed according to the law $\{u(\cdot), \varepsilon, \Delta_\delta\}$; $x[t_0 - h[\cdot]\vartheta]$ is the solution of problem (12), (13), corresponding to realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$.

Let us define

$$\Gamma = \sup\Big\{\gamma : \{x[t_0 - h[\cdot]\vartheta], u[t_0[\cdot]\vartheta), v[t_0[\cdot]\vartheta)\} \in \Omega\Big\}.$$

Then, the optimal guaranteed result of the control is the following value:

$$\Gamma^\circ = \inf_{u(\cdot)} \limsup_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \sup_{\Delta_\delta} \Gamma. \tag{15}$$

According to this definition, the value Γ° is the infinum of quality index values that can be ensured when the control scheme described above is used.

Below, a method of the approximate calculation of the value Γ° is presented. A control procedure is given, which ensures the achievement of this value with the prescribed accuracy.

Applying the approximation from Section 2 to system (12), (13), we get the following linear system of ordinary differential equations:

$$\begin{cases} \dot{y}^{[0]}[t] \,=\, A[t]y^{[0]}[t] + A_h[t]y^{[m]}[t-h] \\ \qquad\qquad + \, B[t]p[t] + C[t]q[t], \\ \dot{y}^{[1]}[t] \,=\, \big(y^{[0]}[t] + A_\tau[t]y^{[m_\tau]} - y^{[1]}[t]\big)/\Delta h, \\ \dot{y}^{[i]}[t] \,=\, \big(y^{[i-1]}[t] - y^{[i]}[t]\big)/\Delta h, \ i = \overline{2, m}, \end{cases} \tag{16}$$

$$t \in [t_0, \vartheta], \quad y^{[i]}[t] \in \mathbb{R}^n, \ i = \overline{0, m}, \quad p[t] \in \mathbb{U}, \quad q[t] \in \mathbb{V},$$

with the initial condition

$$\begin{aligned} y^{[0]}[t_0] &= z[0] - A_\tau[t_0]z[-\tau], \\ y^{[i]}[t_0] &= z[-i\Delta h], \quad i = \overline{1, m}. \end{aligned} \tag{17}$$

Here $m_\tau = \tau/\Delta h$. For simplicity we assume that $m_\tau \in \mathbb{N}$.

Similarly to the aiming procedure from Section 3, considering quality index (14), the following aiming procedure between systems (12) and (16) is obtained. Let realizations $u[t_0[\cdot]\vartheta)$ in neutral-type system (12) and $q[t_0[\cdot]\vartheta)$ in approximating system (16) be constructed on the basis of a partition Δ_δ (9) by the step-by-step rule:

$$u[t] \,=\, u_j \in \arg\min_{u \in \mathbb{U}} \Big[\langle B[t_j]u, s[t_j]\rangle + \langle u, \Phi[t_j]u\rangle \tilde{s}[t_j]\Big]$$
$$q[t] \,=\, q_j \in \arg\max_{q \in \mathbb{V}} \Big[\langle C[t_j]q, s[t_j]\rangle - \langle q, \Psi[t_j]q\rangle \tilde{s}[t_j]\Big]$$
$$t \in [t_j, t_{j+1}), \quad j = \overline{0, J-1}, \tag{18}$$

where

$$s[t] \,=\, x[t] - A_\tau[t]x[t-\tau] - y^{[0]}[t],$$
$$\tilde{s}[t] \,=\, \int\limits_{t_0}^{t} \Big[\langle u[\xi], \Phi[\xi]u[\xi]\rangle - \langle v[\xi], \Psi[\xi]v[\xi]\rangle\Big]\,\mathrm{d}\xi - \int\limits_{t_0}^{t} \Big[\langle p[\xi], \Phi[\xi]p[\xi]\rangle - \langle q[\xi], \Psi[\xi]q[\xi]\rangle\Big]\,\mathrm{d}\xi.$$

By analogy with the proof of Theorem 3 we can proof the following theorem:

Theorem 5. For any number $\zeta > 0$, there exist numbers $M > 0$ and $\delta > 0$ such that, for any natural $m \geq M$, any function $z[\cdot] \in Z$, and any admissible realizations $p[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$, if realizations $u[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$ are formed according to the aiming procedure (18), then, for the solution $x[t_0 - h[\cdot]\vartheta]$ of problem (12), (13) and function $y[t_0 - h[\cdot]\vartheta]$ (7) constructed by the solution $Y[t_0[\cdot]\vartheta]$ of problem (16), (17) the following inequalities hold:

$$\big\|x[t] - y[t]\big\| \leq \zeta, \quad \big\|x[t - i\Delta h] - y^{[i]}[t]\big\| \leq \zeta, \quad i = \overline{1, m},$$

$$\big\|s[t]\big\| \leq \zeta, \quad \big|\tilde{s}[t]\big| \leq \zeta, \quad t \in [t_0, \vartheta].$$

Taking into account Theorem 5 and the quality index (14), let us define the approximating quality index

$$\gamma_m = \bigg(\sum_{i=1}^{m} \|y^{[i]}(\vartheta)\|^2 \Delta h\bigg)^{1/2} + \int_{t_0}^{\vartheta} \Big[\langle p[\xi], \Phi[\xi] p[\xi]\rangle - \langle q[\xi], \Psi[\xi] q[\xi]\rangle\Big]\, \mathrm{d}\xi. \tag{19}$$

For system (16), initial condition (17) and quality index (19), let us consider an auxiliary control problem (see Krasovskii (1985)). In this problem $p[t]$ and $q[t]$ are treated as control and disturbance actions, respectively.

A control strategy $p_m(\cdot)$ is an arbitrary function

$$p_m(t, Y, \varepsilon) \in \mathbb{U}, \quad t \in [t_0, \vartheta], \quad Y \in \mathbb{R}^{(m+1)n}, \quad \varepsilon > 0.$$

Let number $\varepsilon > 0$ and partition Δ_δ (9) be chosen. A triple $\{p_m(\cdot), \varepsilon, \Delta_\delta\}$ defines a control law that forms a piecewise constant control realization according to the following step-by-step rule:

$$p[t] = p_m\big(t_j, Y[t_j], \varepsilon\big), \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, J-1}. \tag{20}$$

Let us denote by $\Omega_m = \Omega_m(p_m(\cdot), \varepsilon, \Delta_\delta)$ the set of all control process realizations $\{Y[t_0[\cdot]\vartheta], p[t_0[\cdot]\vartheta), q[t_0[\cdot]\vartheta)\}$ such that $q[t_0[\cdot]\vartheta)$ is an admissible disturbance realization; $p[t_0[\cdot]\vartheta)$ is the control realization formed according to the law $\{p_m(\cdot), \varepsilon, \Delta_\delta\}$; $Y[t_0[\cdot]\vartheta]$ is the solution of problem (16), (17), corresponding to realizations $p[t_0[\cdot]\vartheta)$ and $q[t_0[\cdot]\vartheta)$.

Let us define

$$\Gamma_m = \sup\Big\{\gamma : \big\{Y[t_0[\cdot]\vartheta], p[t_0[\cdot]\vartheta), q[t_0[\cdot]\vartheta)\big\} \in \Omega_m\Big\}.$$

Then, the optimal guaranteed result of the control is the following value:

$$\Gamma_m^\circ = \inf_{p_m(\cdot)} \limsup_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \sup_{\Delta_\delta} \Gamma_m. \tag{21}$$

It is known (see, e.g., Krasovskii (1985)) that the infinum in (21) is attained. Moreover, the value Γ_m° and the corresponding optimal control strategy $p_m^\circ(\cdot)$ can be effectively calculated by the upper convex hulls method. Details and resulting formulas can be found in Lukoyanov (1994); Krasovskii and Krasovskii (1995).

Theorem 5 and the results of Lukoyanov and Plaksin (2016); Gomoyunov, Lukoyanov and Plaksin (2016) allow us to establish the following theorem.

Theorem 6. For any number $\zeta > 0$, there exists a number $M > 0$ such that, for any natural number $m \geq M$, the following statements are valid:

i) The inequality $|\Gamma^\circ - \Gamma_m^\circ| \leq \zeta$ holds.

ii) There exist a number $\varepsilon^* > 0$ and a function $\delta^*(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon^*]$, such that, for any value $\varepsilon \in (0, \varepsilon^*]$ and any partition Δ_δ (9) with $\delta \leq \delta^*(\varepsilon)$, control scheme (18), (20) with $p_m(\cdot) = p_m^\circ(\cdot)$ ensures inequality

$$\gamma < \Gamma^\circ + \zeta. \tag{22}$$

for any admissible disturbance realization $v[t_0[\cdot]\vartheta)$.

Thus, for control problem (12)–(14), the following solution method is obtained. Let us fix a sufficiently large number m and construct the value Γ_m° and the optimal control strategy $p_m^\circ(\cdot)$ in auxiliary control problem (16), (17), (19). Then, the value Γ_m° approximates the value of the optimal guaranteed result Γ° and, for an appropriate value ε and partition Δ_δ (9), control scheme (18), (20) with $p_m(\cdot) = p_m^\circ(\cdot)$ ensures inequality (22).

7. EXAMPLE 2

The following control problem is considered. A dynamical system is described by the linear neutral-type functional differential equation

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(r_1[t] - 0.4 r_2[t-0.5]\Big) = \\ \quad -\ 2r_1[t] - 0.4\dot{r}_1[t] + 0.02 r_2[t] - r_1[t-1] \\ \quad -\ 0.4\dot{r}_1[t-1] + 0.4 r_2[t-1] - \dot{r}_2[t-1] \\ \quad +\ (5-t)u_1[t] + 2v_1[t], \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(r_2[t] - 0.5 r_1[t-0.5]\Big) \\ \quad =\ 0.01 r_1[t] - r_2[t] - 0.1\dot{r}_2[t] - 0.3 r_1[t-1] \\ \quad +\ 0.7\dot{r}_1[t-1] - 0.4 r_2[t-1] + 0.5\dot{r}_2[t-1] \\ \quad +\ (4-0.5t)u_2[t] + 3v_2[t], \end{cases}$$

$$t \in [0, 4], \quad x[t] = (r_1[t], \dot{r}_1[t], r_2[t], \dot{r}_2[t]) \in \mathbb{R}^4,$$

$$u[t] = (u_1[t], u_2[t]) \in \mathbb{R}^2, \quad v[t] = (v_1[t], v_2[t]) \in \mathbb{R}^2,$$

with the initial condition

$$z[\xi] = \big(\sin(\xi), \cos(\xi), \cos(\xi), -\sin(\xi)\big), \quad \xi \in [-1, 0].$$

The quality of a control process realization is evaluated by the index

$$\gamma = \bigg(\int_3^4 \Big[r_1^2[\xi] + r_2^2[\xi] + \dot{r}_1^2[\xi] + \dot{r}_2^2[\xi]\Big]\, \mathrm{d}\xi\bigg)^{1/2} + \int_0^4 \Big[u_1^2[\xi] + u_2^2[\xi] - v_1^2[\xi] - v_2^2[\xi]\Big]\, \mathrm{d}\xi.$$

For this control problem, the solution method proposed in Section 6 is tested for different values m, ε and the uniform partitions Δ_δ (9) with the diameter δ.

For the value of the optimal guaranteed result Γ^0, the following approximating values are obtained:

$$\Gamma^\circ_{10} = 2.178, \quad \Gamma^\circ_{50} = 1.905, \quad \Gamma^\circ_{100} = 1.872.$$

In order to illustrate the workability of the proposed control schemes disturbance actions are simulated in the following way. Within the game-theoretical approach, for system (12), initial condition (13) and quality index (14), similarly to the initial control problem let us consider the problem of forming the worst-case disturbances aimed to maximize index (14). For this problem, results which are similar to Theorems 5 and 6 can be established and the corresponding scheme of forming worst-case disturbance actions can be applied.

Using control scheme (18), (20) with $p_m(\cdot) = p^\circ_m(\cdot)$ and the corresponding worst-case disturbances the following values of quality index (14) are obtained:

m	ε	δ	γ
10	0.1	0.01	1.295
50	0.02	0.002	1.869
100	0.01	0.001	1.856

The results of simulations for $m = 100$, $\varepsilon = 0.01$, $\delta = 0.001$ are shown in Figures 3 and 4.

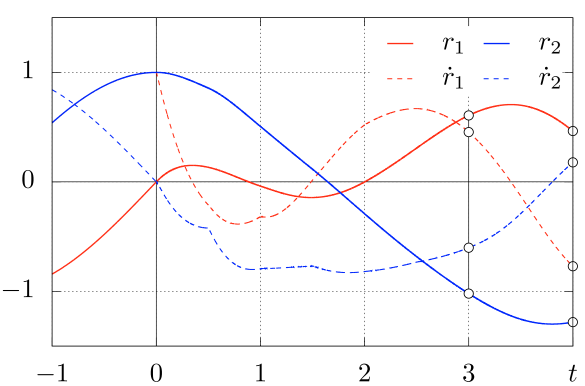

Fig. 3. The motion realization.

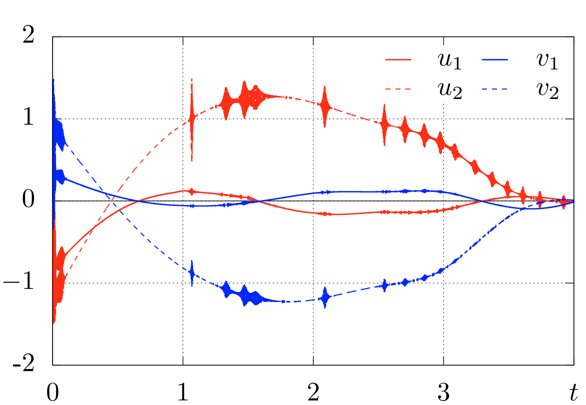

Fig. 4. The control and disturbance realizations.

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