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Polynomial Time Approximation Scheme for the Minimum-weight k-Size Cycle Cover Problem in Euclidean space of an arbitrary fixed dimension

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Abstract: We study the Min-k-SCCP on the partition of a complete weighted digraph by k vertex-disjoint cycles of minimum total weight. This problem is the generalization of the well-known traveling salesman problem (TSP) and the special case of the classical vehicle routing problem (VRP). It is known that the problem Min-k-SCCP is strongly NP-hard and remains intractable even in the geometric statement. For the Euclidean Min-k-SCCP in $\mathbb{R}^d$, we construct a polynomial-time approximation scheme, which generalizes the approach proposed earlier for the planar Min-2-SCCP. For any fixed $c > 1$, the scheme finds a $(1 + 1/c)$-approximate solution in time of $O\big(n^{d+1}(k \log n)^{(O(\sqrt{d}c))^{d-1}} 2^k\big)$.

Keywords: cycle cover of size k, traveling salesman problem (TSP), NP-hard problem, polynomial-time approximation scheme (PTAS).

1. INTRODUCTION

The Cycle Cover Problem (CCP) is a combinatorial optimization problem, which is to find an optimal cover of a given graph by a set of vertex-disjoint cycles. To the best of our knowledge, for the first time, this problem was introduced in the seminal paper by Sahni and Gonzales (Sahni and Gonzales, 1976). Since that time, the CCP and its various modifications were extensively studied in numerous publications (see, e.g. (Bläser and Manthey, 2005; Bläser et al., 2006; Bläser and Siebert, 2001; Chandran and Ram, 2007; Manthey, 2008, 2009; Szwarcfiter and Wilson, 1979)). Since each cycle in a cover can be considered as a tour of some vehicle visiting an appropriate set of clients, the CCP is closely related to the Vehicle Routing Problem (VRP). Moreover, the studied problem is a natural generalization of the well-known Traveling Salesman Problem (TSP), since the Min-1-SCCP is equivalent to the TSP. Below, we propose a brief overview of the related problems and previous work.

The Traveling Salesman Problem (TSP) (Garey and Johnson, 1979) is a classic combinatorial optimization problem, which deals with finding the minimum-cost salesman tour (Hamiltonian cycle) in a given complete weighted graph. There are several special cases of the problem, and some of them, such as the Metric TSP and the Euclidean TSP, are of particular interest. An instance of the Metric TSP is defined by an undirected weighted graph so that edge weights satisfy the triangle inequality. Furthermore, in Euclidean TSP the nodes of the given graph are points in d-dimensional space (for some $d > 1$), and edge weights are Euclidean distances between the incident nodes. It is known (Papadimitriou, 1977) that the TSP is NP-hard even in the Euclidean case; i.e., the optimal solution can not be found in polynomial time unless $P = NP$. Although the TSP is hardly approximable (Sahni and Gonzales, 1976) in the general setting, there are polynomial-time approximation algorithms for some special cases. For instance, the Metric TSP (Christofides, 1975) can be approximated in polynomial time with a ratio of 3/2, and, for the Euclidean TSP, a polynomial-time approximation scheme (Arora, 1998) and a randomized asymptotically correct algorithm (Gimadi, 2008) are developed.

The Vehicle Routing Problem (VRP) (Dantzig and Ramser, 1959) deals with servicing a number of clients (customers) with a fleet of vehicles. In the simplest case (which is also known as the Multiple Traveling Salesmen Problem or the mTSP (Bektas, 2006)), an instance of the VRP is defined by n client locations and one dedicated location (depot). The goal is to find a minimum-cost set of vehicle routes visiting every client only once so that any route starts and finishes at the depot. Surveys of the recent results concerning polynomial-time approximation algorithms and heuristics for several modifications of this problem can be found in (Toth and Vigo, 2001; Golden et al., 2008; Kumar and Panneerselvam, 2012).

The m-Peripatetic Salesmen Problem (m-PSP) (De Kort, 1991; Krarup, 1975) is related to searching for several edge-disjoint salesmen routes optimizing a given objective function (e.g. the total weight of routes, maximum weight, etc.) As for the TSP, so the m-PSP is intractable and

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hardly approximable in the general case, but, for the Euclidean case, polynomial-time asymptotically correct algorithms are known (Gimadi, 2008; Baburin et al., 2009).

Cycle covers of graphs are spanning subgraphs consisting of vertex-disjoint simple cycles and maybe isolated vertices. In (Bläser et al., 2006; Chandran and Ram, 2007), it is shown that cycle covers provide an efficient tool for approximating the TSP, the VRP, and their modifications. Among others, L-cycle covers are mostly discovered (Bläser and Siebert, 2001; Bläser and Manthey, 2005; Bläser et al., 2006; Manthey, 2009; Chandran and Ram, 2007). For a given subset $L \subset \mathbb{N}$, a cycle cover is called L-cycle cover if every containing cycle has the number of edges, which belongs to L. While computing L-covers of the minimum (or maximum) weight is NP-hard (Manthey, 2008), approximation results for multiple restricted cases of the problem are known. For instance, the Min-k-UCC(1, 2) problem where $L = \{k, k+1, \ldots\}$ and edge weights restricted to 1 and 2, can be approximated polynomially within a factor 7/6 for any k (Bläser and Siebert, 2001). Further, in (Manthey, 2009) it is shown that the Metric Min-L-UCC problem, for any fixed L, is polynomial-time approximable within a constant factor (depending on L).

In (Khachay and Neznakhina, 2015), the Minimum-weight k-Size Cycle Cover Problem (Min-k-SCCP) is introduced. This problem is closely related to both the Traveling Salesman, the Vehicle Routing, and the Minimum L-cycle cover problems. In contrast to the Min-L-UCC problem, we restrict not the length of cycles covering the graph but the number of cycles (size of the cover) itself.

Actually, in the Min-k-SCCP, for a fixed natural number k and a given complete weighted digraph (with loops) $G = (V, E, w)$, it is required to find a minimum-weight cover of the set V by k vertex-disjoint cycles. In (Khachay and Neznakhina, 2015), it is shown that the Min-k-SCCP is strongly NP-hard and preserves its intractability even in the geometric statement. For the Metric Min-k-SCCP 2-approximation polynomial-time algorithm is proposed, its approximation ratio and running time do not depend on k. In (Khachai and Neznakhina, 2015a,b), for the Euclidean Min-2-SCCP in $\mathbb{R}^2$, polynomial-time approximation scheme (PTAS) is developed using the approach extending the famous result proposed in (Arora, 1998) for the Euclidean TSP.

In the paper we consider the Euclidean Min-k-SCCP, where the graph G is supposed to be undirected and the weights of its edges are defined by the Euclidean distances between vertices in $\mathbb{R}^d$. Generalizing our earlier result for the Euclidean Min-2-SCCP in $\mathbb{R}^2$ (see, e.g. (Khachay and Neznakhina, 2015)), we propose PTAS for the Min-k-SCCP for any arbitrary fixed dimension $d > 1$ and $k = O(\log n)$.

2. PTAS FOR THE EUCLIDEAN MIN-K-SCCP

It is generally believed that a combinatorial optimization problem has a polynomial-time approximation scheme (PTAS) if, for any fixed $c > 1$, there exists an algorithm, finding a $(1 + 1/c)$-approximate solution of the problem in time bounded by some polynomial of the instance length. Generally speaking, the order and coefficients of this polynomial can be dependent on c.

The general idea of our algorithm generalizes the approach proposed in (Khachai and Neznakhina, 2015a) when developing the PTAS for the Min-2-SCCP on the plane and consists of the following stages

(i). Decomposition of the problem considered into $m \le k$ independent subproblems. Actually, we construct a partition of the given graph G into vertex-disjoint subgraphs $G_1, \ldots, G_m$. Then, for each subgraph G_i, we consider Euclidean Min-q_i-SCCP for some appropriate number q_i such that $\sum_{i=1}^m q_i = k$.
(ii). Reducing each of the subproblems obtained to corresponding *well-rounded* Euclidean Min-q_i-SCCP.
(iii). For each well-rounded instance, constructing a recursive partition of the enclosing hypercube.
(iv). Proof of the claim that, for any $c > 0$, with high probability where exists an $(1 + 1/c)$-approximate k-size cycle cover.
(v). Deterministic construction of $(1 + 1/c)$-approximate k-size cycle cover by means of dynamic programming standard derandomization scheme.

2.1 Decomposition of the problem

Our constructions are based on the well-known geometric Jungs's inequality, establishing the dependence between the diameter D of a bounded set in d-dimensional space and the radius R of a minimal sphere enclosing this set.

$$\frac{1}{2}D \le R \le \left(\frac{d}{2d+2}\right)^{\frac{1}{2}} D.$$

Center of such a sphere is called a *Chebyshev center*.

Using a modification of Kruskal's algorithm (Khachai and Neznakhina, 2015b), we construct a k-minimum spanning forest (k-MSF) consisting of k trees $T_1, T_2, \ldots, T_k$. To each tree T_i, we assign diameter D_i, a Chebyshev center c_i, and the appropriate radius R_i. Then, we clusterize trees into $m \le k$ clusters using single-linkage (Nearest Neighbour) algorithm (see, e.g. (Gan et al., 2007)). Actually, we assign trees T_i and T_j to the same cluster iff

$$\|c_i - c_j\|_2 \le (2k+1)R$$

for $R = \max\{R_i\colon\ i \in \mathbb{N}_k\}$. Hereinafter, we do not distinguish i-th cluster and its vertex-set, which we denote by S_i. By construction, $S_1 \cup \ldots \cup S_m = V$ and $S_i \cap S_j = \varnothing$ for any $i \ne j$.

Theorem 1. Any minimum-weight k-size cycle cover has no cycles intersecting more than one cluster.

In addition, for any cluster S_i, its diameter D_{S_i} does not exceed $\left(\frac{d}{2d+2}\right)^{1/2}(2k^2 - k + 1)OPT$.

Proof. 1. Assume the contrary. Suppose, some minimum-weight k-size cycle cover contains a cycle including vertices from different clusters S_i and S_j.

Denote this cycle by P. By assumption, P contains at least two edges, which span vertices of different clusters; the total length of these edges is greater than $2(2k-1)R$. The cycle P visits multiple trees among $T_1, \ldots, T_k$. Moreover, vertices belonging to different trees can alternate. Let $\{u, v\}$ be arbitrary edge of P such that $u \in S_i, v \in S_j$. Fixing the order of traversal for P: $u \to u_{i_1} \to \ldots \to v$, we construct a partition of P into a collection of connected fragments such that any fragment is a maximal one among fragments starting and finishing at the same tree. Remove edges of P connecting the vertices from different fragments.

We denote the number of cycles and the number of fragments related to the tree T_i by l_i $(0 \le l_i \le k-1)$ and p_i $(0 \le p_i \le 1)$, respectively. By construction,

$$\sum_{i=1}^{k} l_i = k-1 \qquad \text{and} \qquad 2 \le \sum_{i=1}^{k} p_i \le k. \tag{1}$$

Denote the number of trees, for which $l_i + p_i = 0$ by q; consider two cases $q = 0$ and $q \ge 1$. In the first case, we construct k-size cycle cover as follows. For the tree T_i, the assigned l_i cycles and the fragment of P must be combined in one cycle. To make such a transformation, we should add at most $(l_i + p_i)$ new edges. Summarizing, the total weight of the added edges is at most

$$2R\sum_{i=1}^{k}(l_i + p_i) \le 2(2k-1)R \tag{2}$$

and the total weight of the removed edges exceeds (2). Therefore, we construct a lighter k-size cycle cover than the initial cover, which contradicts its optimality.

In the case of $q \ge 1$, $(k-1)$ cycles and at least two fragments of P relate to $(k-q)$ tree. For any such tree, $l_i + p_i > 0$. By construction, any tree can be assigned to at most one fragment of P. Then, at least $(q+1)$ cycles are assigned to trees, which already be assigned to either a fragment of P or another cycle. Exclude from consideration any q cycles (along with vertices covered by them) such that the number of trees, for which $l_i + p_i = 0$, is not changed. We obtain $(k-q-1)$ cycles and fragments of P, which are assigned to $(k-q)$ trees. The following relations are correct:

$$l_i + p_i > 0, \qquad i = 1, \ldots, k-q,$$
$$\sum_{i=1}^{k-q} l_i = k-q-1 \qquad \text{and} \qquad 2 \le \sum_{i=1}^{k-q} p_i \le k-q. \tag{3}$$

Similarly to the first case, we carry out the modification for these $(k-q)$ trees. To this end, we obtain the new $(k-q)$-size cycle cover for the subgraph constructed, whose weight exceeds the weight of the initial one at most

$$2R\sum_{i=1}^{k-q}(l_i + p_i) \le 2(2k-2q-1)R.$$

Add to this cover q cycles which was excluded earlier. Thus, we construct a lighter k-size cycle cover for the initial graph, which contradicts to our assumption again. The first claim is proved.

2. Now we proceed with estimation of diameters the clusters obtained. In $\mathbb{R}^d$, the diameter of a cluster will be maximal if the Chebyshev centers for the constructed k-MSF trees belong to one straight line. By construction, any cluster S_i contains at most k trees. Therefore, the following inequality holds:

$$\max_{i \in \mathbb{N}_m} D_{S_i} \le (k-1)(2k+1)R + 2R.$$

Applying the right-hand side of Jungs's inequality and taking into account the straightforward bound

$$D \le MSF \le OPT,$$

for $D = \max\{D_i\}$ and MSF — the weight of k-MSF we have

$$R \le \left(\frac{d}{2d+2}\right)^{1/2} D \le \left(\frac{d}{2d+2}\right)^{1/2} OPT.$$

Hence,

$$\max_{i \in \mathbb{N}_m} D_{S_i} \le \left(\frac{d}{2d+2}\right)^{1/2} (2k^2 - k + 1)OPT.$$

Theorem is proved.

Thus, if $m = k$, Euclidean Min-k-SCCP decomposes into k independent traveling salesman subproblems. In particular, PTAS for the initial problem can be composed of PTASes for these k subproblems.

If the number of clusters is strictly less than k, then it is necessary to consider all possible combinations of k cycles by m clusters and for each one to solve m independent smaller cycle cover subproblems, after that choose an optimal one. The number of such combinations is equal to the binomial coefficient $\binom{k-1}{m-1}$.

Hereinafter we consider the particular case of the problem Euclidean Min-k-SCCP, for which $m = 1$, since it is the worst one in terms of computational complexity.

2.2 Well-rounded problem

To approximate Euclidean Min-k-SCCP in $\mathbb{R}^d$, it is sufficient to have an efficient approximation algorithm for the special case of the problem in question, which is called *well-rounded*. In particular, PTAS for the well-rounded Min-k-SCCP induces PTAS for the general case with the same bound on the running time.

An instance of Euclidean Min-k-SCCP is called well-rounded, if the following conditions are valid

(i) any node i of the input graph G has integral coordinates $x_i, y_i \in \mathbb{N}^0_{O(cn\sqrt{d})}$;
(ii) for each edge $e \in E$, $w(e) \ge 4$.

As it follows from Theorem 1, for any given instance of the Euclidean Min-k-SCCP, there exists an enclosing hypercube $\mathcal{S}$ with side-length L such that

$$L = \left(\frac{d}{2d+2}\right)^{1/2}(2k^2 - k + 1)MSF \le \left(\frac{d}{2d+2}\right)^{1/2}(2k^2 - k + 1)OPT.$$

W.o.l.g. we assume that the origin coincides with one of corners of $\mathcal{S}$.

In order to well-round the given instance, it is sufficient to carry out the following transformations.

(i). Place a grid of granularity $L/8cn\sqrt{d}$ in $\mathcal{S}$ and move each node to its nearest gridpoint. This changes the total cost of k-size cycle cover by at most $L/4c$.
(ii). Rescale distances by $32cn\sqrt{d}/L$. Then, the minimum nonzero internode distance is 4 units and the size of the bounding cube $\mathcal{S}$ is $O(cn\sqrt{d})$.

Further, we describe PTAS for well-rounded Euclidean Min-k-SCCP in $\mathbb{R}^d$.

2.3 Geometric partition

We construct a geometric partition of an enclosing hypercube $\mathcal{S}$ for the well-rounded Min-k-SCCP using 2^d-ary tree, which appers to be a natural extension of the well-known concept of the quadtree. Let L be some power of 2. We view partition of $\mathcal{S}$ as a 2^d-ary tree, whose root is $\mathcal{S}$. Each hypercube of the tree is partitioned into 2^d equal child sub-hypercubes. We stop the recursive partitioning if the current hypercube has inside at most one node.

The constructed hypercubes form a hierarchy, the enclosing hypercube is at level 0, its 2^d children form the level 1, and so on. Note that 2^d-ary tree contains $O(2^d n)$ leaves, $O(\log L) = O(\log(cn\sqrt{d}))$ levels and thus $O(2^d n \log(cn\sqrt{d}))$ hypercubes in all.

If $a_1, a_2, \ldots, a_d \in \mathbb{N}_L$, then the $(a_1, a_2, \ldots, a_d)$-shift of the dissection is defined by shifting the all coordinates of all lines by $a_1, \ldots, a_n$ and then reducing modulo L. The nodes in the Min-k-SCCP instance do not move. We denote 2^d-ary tree with $(a_1, a_2, \ldots, a_d)$-shift by $T(a_1, a_2, \ldots, a_d)$.

Let m be a positive integer. An orthogonal lattice in the $d-1$-dimensional hypercube consisting of its 2^{d-1} vertices and m uniformly distributed points is called an m-regular portal set. For hypercube of level i, the spacing between the nearest portals is $O(L/2^i m^{1/(d-1)})$. We denote union of m-regular portal sets for each node of $T(a_1, a_2, \ldots, a_d)$ (excluding the root) by $P(a_1, \ldots, a_d; m)$.

2.4 Structure theorem

Let us prove the existence of a k-size cycle cover for a given graph G, having a number of properties.

Let C be an arbitrary simple cycle in the graph G in $\mathbb{R}^d$. We denote the set of vertices C by $V(C)$. The closed continuous piecewise linear route $l(C)$ is called an $(m; r)$-approximation of the cycle C if

(i) $l(C)$ bends only at points of $V(C) \bigcup P(a_1, \ldots, a_d; m)$;
(ii) the nodes of $V(C)$ are visited by $l(C)$ in the same order as by C;
(iii) for any facet of any node of $T(a_1, a_2, \ldots, a_d)$, the route $l(C)$ crosses this facet at points of $P(a_1, \ldots, a_d; m)$ and at most r times.

In what follows, we need the main result of (Arora, 1998) one of its equivalent formulations is presented below.

Theorem 2. (Structure Theorem). Let an instance of the well-rounded TSP in $\mathbb{R}^d$ be given by the graph G, let L be the side-length of the enclosing hypercube $\mathcal{S}$, and let constants $c > 1$ is fixed. If the stochastic variables $a_1, a_2, \ldots, a_d$ are distributed uniformly in $\mathbb{N}_L$, then, for any $\eta \in (0, 1)$ there $D_1, D_2 > 0$ such that for $r = \lceil (D_1\sqrt{d}c)^{d-1} \rceil$, $m = \lceil (D_2 dc \log L)^{d-1} \rceil$ for any simple cycle C in graph G of weight $W(C)$, there exists an (m, r)-approximation $l(C)$ of weight $W(l(C)) \leq (1 + 1/c)W(C)$ with a probability at least $1 - \eta$.

Following Arora's approach to approximate Hamiltonian tours by portal-respecting (m, r)-approximations, we define a similar construction for our problem.

Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be an arbitrary k-size cycle cover in the graph G and let $l(C_i)$ be some (m, r)-approximation of the cycle C_i. Then, the family $\mathcal{L}(\mathcal{C}) = l(C_1), \ldots, l(C_k)$ is called a cycle (m, r, k)-cover in the graph G.

We generalize the result of Theorem 5 of (Khachay and Neznakhina, 2015) for the Euclidean Min-k-SCCP in $\mathbb{R}^d$ and prove the following theorem.

Theorem 3. Let a constant $c > 1$ be fixed, let L be the side-length of the enclosing hypercube $\mathcal{S}$ for the given instance of the well-rounded Min-k-SCCP in $\mathbb{R}^d$. If the discrete stochastic variables $a_1, a_2, \ldots, a_d$ are distributed uniformly in $\mathbb{N}_L$ and $m = (O(dc \log L))^{d-1}$, $r = (O(\sqrt{d}c))^{d-1}$ then, for the graph G, there exists a cycle (m, r, k)-cover of weight at most $(1+1/c)OPT$ with a probability at least 1/2.

Proof. Indeed, consider an arbitrary optimal solution $C^* = \{C_1^*, \ldots, C_k^*\}$ of the well-rounded Euclidean Min-k-SCCP. Then, $OPT = \sum_{i=1}^k W(C_i^*)$ where $W(C_i^*)$ is a weight of the cycle C_i^*. As it follows from Theorem 2, for $\eta = 1/(2k)$, $m = O((dc \log L)^{d-1})$ and $r = O((\sqrt{d}c)^{d-1})$, with probability at least $1-\eta$, for the cycle C_i^*, there exists a (m, r)-approximation $l(C_i^*)$ such that

$$W(l(C_i^*)) \leq (1 + 1/c)W(C*).$$

Since the random variables $a_1, a_2, \ldots, a_d$ are distributed uniformly, with probability at least 1/2, there exists a cycle (m, r, k)-cover $l(C_1), l(C_2), \ldots, l(C_k)$ such that

$$\sum_{i=1}^k W(l(C_i^*)) \leq (1 + 1/c) \sum_{i=1}^k W(C_i^*) = (1 + 1/c)OPT.$$

Theorem is proved.

2.5 Dynamic programming

The proposed search procedure for minimum weight cycle (m, r, k)-cover $\{l_1, \ldots, l_k\}$ of the well-rounded Euclidean Min-k-SCCP is based on the dynamic programming algorithm, extends the approach proposed in (Khachai and Neznakhina, 2015b), and has upper time complexity bound of $O(n(\log n)^{O(\sqrt{d}c)^{d-1}})$.

For any node S of 2^d-ary tree, the goal of the inner subtask is to find a minimum-weight part of a cycle (m, r, k)-cover belonging to the hypercube S and visiting all nodes locating inside S.

The procedure starts from the leaves of the 2^d-ary tree $T(a_1, \ldots, a_d)$. By construction, any leaf of this tree contains at most one node of the initial graph G. Therefore, in this case the inner subtask can be solved trivially in time $O(dr)$.

Time complexity of the inner subtask for other (not-a-leaf) node of the 2^d-ary tree has the upper bound

$$O((m + 2^{d-1})^{2dr}(dr)^{2dr}(2dr)!).$$

Taking into account that the total node-number of the tree $T(a_1, \ldots, a_d)$ is of $O(2^d n \log(cn\sqrt{d}))$, we obtain an upper time complexity bound for the given offsets $a_1, \ldots, a_d$

$$O(2^d n \log(cn\sqrt{d}) \times (m + 2^{d-1})^{4dr}((2dr)!)^2(dr)^{2dr} \times k^{dr}). \quad (4)$$

Since, by construction,

$$m = O((dc\log(cn\sqrt{d}))^{d-1}),\ r = O((\sqrt{d}c)^{d-1}),$$

where c and d are constants and the number of assignments of k cycles to m clusters is of $O(2^k)$, the bound (4) is reduced to

$$O(n2^k(k\log n)^{O(\sqrt{d}c)^{d-1}})$$

Further, applying the standard derandomization scheme consisting of the exhaustive search for all possible offsets $(a_1, \ldots, a_d)$ and having running time of $O(n^d)$ we prove the following Theorem.

Theorem 4. The Euclidean Min-k-SCCP in $\mathbb{R}^d$ has PTAS with time complexity

$$O(n^{d+1}2^k(k\log n)^{O(\sqrt{d}c)^{d-1}}).$$

3. CONCLUSION

This paper introduces a new polynomial time approximation scheme for the Euclidean Min-k-SCCP in d-dimensional Euclidean space for any fixed $d > 1$ and $k = O(\log n)$. It is known that this problem has no FPTAS unless $P = NP$. Therefore, to date the PTAS proposed appears to be a state-of-the-art approximation result for the problem in question. Nevertheless, some questions remain open. For instance, it would be interesting to develop PTASes having polynomial or even quasi-polynomial time complexity bounds in n and k.

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