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On a Problem of Guarantee Optimization in Time-Delay Systems ⋆

Mikhail Gomoyunov * **Anton Plaksin** *

* *Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, S.Kovalevskaya str. 16, Ekaterinburg, 620990, Russia;*

Institute of Mathematics and Computer Sciences, Ural Federal University, Mira str. 19, Ekaterinburg, 620002, Russia,

(*e-mail: m.i.gomoyunov@gmail.com, a.r.plaksin@gmail.com*)

Abstract: A control problem under conditions of disturbances is considered for a linear time-delay dynamical system. The goal of the control is to minimize a non-terminal quality index that evaluates a motion history and realizations of control and disturbance actions. The control problem is posed within the game-theoretical approach. For calculating the optimal guaranteed result of the control and constructing a control scheme that ensures this result, two methods are proposed. The first one is based on an appropriate approximation of the quality index. The second one is based on a finite-dimensional approximation of the dynamical system. Both methods allow us to reduce the control problem to high-dimensional auxiliary differential games without delays and with terminal quality indices. An illustrative example is considered, simulations results are given.

Keywords: control theory; differential games; time-delay systems.

1. INTRODUCTION

The paper is devoted to a control problem under conditions of disturbances for a dynamical system described by a linear delay differential equation. The goal of the control is to minimize a non-terminal quality index that evaluates a motion history and realizations of control and disturbance actions. The control problem is posed within the game-theoretical approach of Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1995) (see also Osipov (1971) for time-delay systems).

Two methods are proposed for calculating the optimal guaranteed result of the control and constructing a control scheme that ensures this result. Both methods reduce the control problem to appropriate differential games without delays and with terminal quality indices. The first method is based on an approximation of the quality index. It uses a functional interpretation of the control process (see, e.g., Krasovskii (1959)) and certain predictions of system motions (see Lukoyanov and Reshetova (1998)). The second method is based on a finite-dimensional approximation of the dynamical system by a system of ordinary differential equations (see, e.g., Krasovskii (1964); Banks and Kappel (1979)). This approximating system is used as a leader (see, e.g., Krasovskii and Subbotin (1988)) for the initial system (see Lukoyanov and Plaksin (2013, 2015)). A solution to the obtained high-dimensional auxiliary differential games is constructed by the upper convex hulls method (see, e.g., Krasovskii (1987); Krasovskii and Krasovskii (1995); Lukoyanov (1994, 1998)).

The efficiency of the proposed solution methods is illustrated by an example. Results of numerical simulations are given.

2. STATEMENT OF THE PROBLEM

In the paper the following control problem is considered. A dynamical system is described by the delay differential equation

$$\dot{x}(t) = A(t)x(t) + A_h(t)x(t-h) + B(t)u(t) + C(t)v(t),$$
$$t_0 \le t < \vartheta, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r, \quad v \in \mathbb{R}^s, \qquad (1)$$

with the initial condition

$$x_{t_0}(\cdot) = \sigma(\cdot) \in C. \qquad (2)$$

Here t is the time variable, x is the state vector, $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$, u is the control vector, and v is the vector of unknown disturbances; t_0 and ϑ are respectively the initial and the terminal instants of time; matrix functions $A(t)$, $A_h(t)$, $B(t)$ and $C(t)$ are continuous; $h = \mathrm{const} > 0$ is the delay value; $x_t(\cdot)$ is the motion history on $[t-h, t]$ defined by $x_t(\xi) = x(t+\xi)$, $\xi \in [-h, 0]$; $C = C[-h, 0]$ is the set of all continuous functions from $[-h, 0]$ to $\mathbb{R}^n$.

It is assumed that admissible values of the control vector u and the disturbance vector v are restricted by the inclusions

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$$u \in P = \{u \in \mathbb{R}^r : \|u\| \le R\},$$
$$v \in Q = \{v \in \mathbb{R}^q : \|v\| \le R\},$$

where the symbol $\|\cdot\|$ denotes the Euclidian norm and the constant $R > 0$ is sufficiently large in order to we can use results of (Krasovskii and Krasovskii, 1995, p. 179) (see also Lukoyanov (1994)).

Admissible control and disturbance realizations are Borel measurable functions

$$u[t_0[\cdot]\vartheta) = \{u(t) \in P,\, t_0 \le t < \vartheta\},$$
$$v[t_0[\cdot]\vartheta) = \{v(t) \in Q,\, t_0 \le t < \vartheta\}.$$

Such realizations uniquely generate a motion of system (1)

$$x[t_0 - h[\cdot]\vartheta] = \{x(t) \in \mathbb{R}^n,\, t_0 - h \le t \le \vartheta\}$$

that is an absolutely continuous function, which satisfies initial condition (2) and, together with $u(t)$ and $v(t)$, satisfies equation (1) for almost all $t \in [t_0, \vartheta]$. The triple $\{x[t_0-h[\cdot]\vartheta], u[t_0[\cdot]\vartheta), v[t_0[\cdot]\vartheta)\}$ is called a control process realization. The quality of this realization is evaluated by the index

$$\gamma = \left(\int_{\vartheta-h}^{\vartheta} \|x(t)\|^2 \mathrm{d}t\right)^{1/2} + \int_{t_0}^{\vartheta} \Big[\langle u(t), \Phi(t)u(t)\rangle - \langle v(t), \Psi(t)v(t)\rangle\Big]\mathrm{d}t. \tag{3}$$

Here the symbol $\langle\cdot,\cdot\rangle$ denotes the scalar product of vectors; $\Phi(t)$ and $\Psi(t)$ are symmetric continuous matrix functions such that the quadratic forms $\langle u, \Phi(t)u\rangle$ and $\langle v, \Psi(t)v\rangle$ are positive definite for $t \in [t_0, \vartheta]$.

The goal of the control is to minimize quality index (3). Let us note that, since disturbance actions are unknown, the worst-case may occur when disturbances maximize (3).

According to Osipov (1971); Krasovskii and Krasovskii (1995) the control problem (1)–(3) is posed as follows.

A control strategy $U(\cdot)$ is an arbitrary function

$$U(\cdot) = \{U(t, x_t(\cdot), \varepsilon) \in P,\, (t, x_t(\cdot)) \in [t_0, \vartheta] \times C,\, \varepsilon > 0\},$$

where $\varepsilon > 0$ is the accuracy parameter. The strategy $U(\cdot)$ acts onto system (1) in the discrete time scheme on the basis of a partition of the control interval $[t_0, \vartheta]$:

$$\Delta_\delta = \{\tau_j : \tau_1 = t_0,\, 0 < \tau_{j+1} - \tau_j \le \delta,\, j = \overline{1,k},\, \tau_{k+1} = \vartheta\}. \tag{4}$$

A triple $\{U(\cdot), \varepsilon, \Delta_\delta\}$ defines a control law that forms a piecewise constant control realization according to the following step-by-step rule:

$$u(t) = U(\tau_j, x_{\tau_j}(\cdot), \varepsilon), \quad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{1,k}.$$

Let us denote by $\Omega = \Omega(U(\cdot), \varepsilon, \Delta_\delta)$ the set of all control process realizations $\{x[t_0 - h[\cdot]\vartheta], u[t_0[\cdot]\vartheta), v[t_0[\cdot]\vartheta)\}$ such that $v[t_0[\cdot]\vartheta)$ is an admissible disturbance realization; $u[t_0[\cdot]\vartheta)$ is the control realization formed according to the law $\{U(\cdot), \varepsilon, \Delta_\delta\}$; $x[t_0 - h[\cdot]\vartheta]$ is the system motion generated by these realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$.

Let us define

$$\Gamma = \sup\Big\{\gamma : \{x[t_0 - h[\cdot]\vartheta], u[t_0[\cdot]\vartheta), v[t_0[\cdot]\vartheta)\} \in \Omega\Big\}.$$

Then, the optimal guaranteed result of the control is the following value:

$$\Gamma^0 = \inf_{U(\cdot)} \limsup_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \sup_{\Delta_\delta} \Gamma. \tag{5}$$

According to this definition, the value Γ^0 is the infimum of quality index values that can be ensured when the control scheme described above is used.

It is known (see, e.g., Osipov (1971)) that the infimum in (5) is attained. The corresponding strategy $U^0(\cdot)$ is called the optimal control strategy. Let us note that according to (5) the following property of the strategy $U^0(\cdot)$ is valid.

For any number $\zeta > 0$, there exist a number $\varepsilon^0 > 0$ and a function $\delta^0(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon^0]$, such that, for any value $\varepsilon \in (0, \varepsilon^0]$ and any partition Δ_δ (4) with $\delta \le \delta^0(\varepsilon)$, the control law $\{U^0(\cdot), \varepsilon, \Delta_\delta\}$ ensures the inequality

$$\gamma \le \Gamma^0 + \zeta \tag{6}$$

for any admissible disturbance realization $v[t_0[\cdot]\vartheta)$.

The problem under consideration is to find the value of the optimal guaranteed result and to construct a control scheme that ensures inequality (6).

In Sections 4 and 5 two solution methods for this problem are given. In both methods the control problem (1)–(3) is reduced to an auxiliary differential game of a special type. So, before we describe these methods let us consider in the next Section a differential game of this type.

3. AUXILIARY DIFFERENTIAL GAME

In this Section one of the classical problems of the positional differential games theory is considered. An effective solution to this problem is described. Detailed exposition can be found in Lukoyanov (1994); Krasovskii and Krasovskii (1995).

A zero-sum two-person differential game is described by the dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)u(t) + \mathbf{C}(t)v(t),$$
$$t_0 \le t < \vartheta, \quad \mathbf{x} \in \mathbb{R}^{\mathbf{n}}, \quad u \in P, \quad v \in Q, \tag{7}$$

with the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{8}$$

and the quality index

$$\boldsymbol{\gamma} = \|\mathbf{D}\mathbf{x}(\vartheta)\| + \int_{t_0}^{\vartheta} \Big[\langle u(t), \Phi(t)u(t)\rangle - \langle v(t), \Psi(t)v(t)\rangle\Big]\mathrm{d}t. \tag{9}$$

Here $\mathbf{x}$ is the state vector, u is the control vector of the first player, and v is the control vector of the second player; matrix functions $\mathbf{A}(t)$, $\mathbf{B}(t)$ and $\mathbf{C}(t)$ are piecewise-continuous; $\mathbf{D}$ is a constant $(\mathbf{d} \times \mathbf{n})$-matrix $(1 \le \mathbf{d} \le \mathbf{n})$. For the meanings of the other symbols see Section 2.

The first player aims to minimize index (9), while the second player aims to maximize it.

A pair of admissible realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$ uniquely generate a motion of system (7)

$$\mathbf{x}[t_0[\cdot]\vartheta] = \{\mathbf{x}(t) \in \mathbb{R}^{\mathbf{n}},\ t_0 \le t \le \vartheta\}$$

that is an absolutely continuous function, which satisfies initial condition (8) and, together with $u(t)$ and $v(t)$, satisfies equation (7) for almost all $t \in [t_0, \vartheta]$.

It is known that the differential game (7)–(9) has a value ρ^0 and a saddle point which consists of the optimal minimax $u^0(t, \mathbf{x}, \varepsilon)$ and maximin $v^0(t, \mathbf{x}, \varepsilon)$ strategies, $(t, \mathbf{x}) \in [t_0, \vartheta] \times \mathbb{R}^{\mathbf{n}}$, $\varepsilon > 0$. In particular, this means that, for any number $\zeta > 0$, there exist a number $\varepsilon^* > 0$ and a function $\delta^*(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon^*]$, such that, for any value $\varepsilon \in (0, \varepsilon^*]$ and any partition Δ_δ (4) with $\delta \le \delta^*(\varepsilon)$, the following statement is valid.

On the one hand, the step-by-step control law $\{u^0(\cdot), \varepsilon, \Delta_\delta\}$ of the first player that forms the piecewise constant realization

$$u(t) = u^0(\tau_j, \mathbf{x}(\tau_j), \varepsilon), \quad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{1, k},$$

ensures the inequality

$$\boldsymbol{\gamma} \le \rho^0 + \zeta$$

for any admissible realization $v[t_0[\cdot]\vartheta)$, and on the other hand, the control law $\{v^0(\cdot), \varepsilon, \Delta_\delta\}$ of the second player that forms the realization

$$v(t) = v^0(\tau_j, \mathbf{x}(\tau_j), \varepsilon), \quad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{1, k},$$

ensures the inequality

$$\boldsymbol{\gamma} \ge \rho^0 - \zeta$$

for any admissible realization $u[t_0[\cdot]\vartheta)$.

The value ρ^0 and the optimal strategies $u^0(\cdot)$ and $v^0(\cdot)$ in the differential game (7)–(9) can be effectively calculated by the upper convex hulls method. The resulting formulas are given below.

Let us denote

$$\widehat{\mathbf{B}}(t) = \mathbf{DX}(\vartheta, t)\mathbf{B}(t), \quad \widehat{\mathbf{C}}(t) = \mathbf{DX}(\vartheta, t)\mathbf{C}(t),$$

$$\mathbf{K}(t) = \frac{1}{4}\int_t^{\vartheta} \Big[\widehat{\mathbf{C}}(\xi)\Psi^{-1}(\xi)\widehat{\mathbf{C}}^{\top}(\xi) - \widehat{\mathbf{B}}(\xi)\Phi^{-1}(\xi)\widehat{\mathbf{B}}^{\top}(\xi)\Big]\mathrm{d}\xi,$$

$$\lambda^0(t) = \max_{\tau \in [t, \vartheta]} \lambda(\tau), \quad t \in [t_0, \vartheta].$$

Here $\mathbf{X}(\vartheta, t)$ is a fundamental solution matrix for the equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ such that $\mathbf{X}(t, t) = E$, where E is the identity matrix; the superscript $\top$ denotes transposition; $\Phi^{-1}(\xi)$ and $\Psi^{-1}(\xi)$ are the inverse matrices of $\Phi(\xi)$ and $\Psi(\xi)$, respectively; $\lambda(\tau)$ is the largest eigenvalue of the matrix $\mathbf{K}(\tau)$.

Then the following representation formulas are valid

$$\rho^0 = \max_{\mathbf{l} \in G} \Big[\langle \mathbf{l}, \mathbf{DX}(\vartheta, t_0)\mathbf{x}_0 + \mathbf{K}(t_0)\mathbf{l}\rangle - \lambda^0(t_0)\big(\|\mathbf{l}\|^2 - 1\big)\Big],$$

$$\begin{aligned} u^0(t, \mathbf{x}, \varepsilon) &= -\frac{1}{2}\Phi^{-1}(t)\widehat{\mathbf{B}}^{\top}(t)\mathbf{l}^{(u)}(t, \mathbf{x}, \varepsilon), \\ v^0(t, \mathbf{x}, \varepsilon) &= \frac{1}{2}\Psi^{-1}(t)\widehat{\mathbf{C}}^{\top}(t)\mathbf{l}^{(v)}(t, \mathbf{x}, \varepsilon), \end{aligned} \tag{10}$$

$$(t, \mathbf{x}) \in [t_0, \vartheta] \times \mathbb{R}^{\mathbf{n}}, \quad \varepsilon > 0,$$

where

$$\mathbf{l}^{(u)}(t, \mathbf{x}, \varepsilon) \in \operatorname*{argmax}_{\mathbf{l} \in G}\Big[\langle \mathbf{l}, \mathbf{DX}(\vartheta, t)\mathbf{x} + \mathbf{K}(t)\mathbf{l}\rangle - \lambda^0(t)\|\mathbf{l}\|^2 - r(t, \varepsilon)\sqrt{1 + \|\mathbf{l}\|^2}\,\Big],$$

$$\mathbf{l}^{(v)}(t, \mathbf{x}, \varepsilon) \in \operatorname*{argmax}_{\mathbf{l} \in G}\Big[\langle \mathbf{l}, \mathbf{DX}(\vartheta, t)\mathbf{x} + \mathbf{K}(t)\mathbf{l}\rangle - \lambda^0(t)\|\mathbf{l}\|^2 + r(t, \varepsilon)\sqrt{1 + \|\mathbf{l}\|^2}\,\Big],$$

$$G = \{\mathbf{l} \in \mathbb{R}^{\mathbf{d}} : \|\mathbf{l}\| \le 1\}, \quad r(t, \varepsilon) = \sqrt{(1 + t - t_0)\varepsilon}.$$

4. APPROXIMATION OF QUALITY INDEX

The first method for solving the control problem (1)–(3) is based on an approximation of the quality index.

Let us fix a natural number m, define

$$\Delta h = h/m, \quad \vartheta_i = \vartheta - h + i\Delta h, \quad i = \overline{1, m},$$

and approximate the first integral in quality index (3) by the right rectangles formula:

$$\int_{\vartheta - h}^{\vartheta} \|x(t)\|^2 \mathrm{d}t \approx \sum_{i=1}^{m} \|x(\vartheta_i)\|^2 \Delta h. \tag{11}$$

Similarly to Section 2, let us pose a control problem for system (1), initial condition (2) and the approximating quality index

$$\begin{aligned} \gamma^{(m)} = {} & \Big(\sum_{i=1}^{m} \|x(\vartheta_i)\|^2 \Delta h\Big)^{1/2} \\ & + \int_{t_0}^{\vartheta} \Big[\langle u(t), \Phi(t)u(t)\rangle - \langle v(t), \Psi(t)v(t)\rangle\Big]\mathrm{d}t. \end{aligned} \tag{12}$$

Let us denote by $\Gamma^{(m)}$ and $U^{(m)}(\cdot)$ the corresponding value of the optimal guaranteed result and the optimal control strategy.

By the scheme proposed in Lukoyanov (1998) the following result can be established.

Theorem 1. For any number $\zeta > 0$ there exists a number $M > 0$ such that, for any natural number $m \ge M$, the following statements are valid:

i) The inequality $|\Gamma^0 - \Gamma^{(m)}| \le \zeta$ holds.

ii) There exist a number $\varepsilon_1^{(m)} > 0$ and a function $\delta_1^{(m)}(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_1^{(m)}]$, such that, for any value $\varepsilon \in (0, \varepsilon_1^{(m)}]$ and any partition Δ_δ (4) with $\delta \le \delta_1^{(m)}(\varepsilon)$, the control law $\{U^{(m)}(\cdot), \varepsilon, \Delta_\delta\}$ ensures inequality (6) for any admissible disturbance realization $v[t_0[\cdot]\vartheta)$.

According to Theorem 1, in order to solve the initial control problem with quality index (3) it is sufficient to solve the control problem with approximating quality index (12). For solving the last one let us apply the approach given in Lukoyanov and Reshetova (1998).

A pair $(t, x_t(\cdot)) \in [t_0, \vartheta] \times C$ is called a position of system (1). For a position $(t, x_t(\cdot))$, let us define the following system motion predictions to the instants ϑ_i under zero control and zero disturbance realizations:

$$w_i(t, x_t(\cdot)) = F(\vartheta_i, t)x(t) + \int\limits_t^{t+h} F(\vartheta_i, \xi)A_h(\xi)x(\xi - h)\mathrm{d}\xi$$

for $t \in [t_0, \vartheta_i)$, and $w_i(t, x_t(\cdot)) = x(\vartheta_i)$ for $t \in [\vartheta_i, \vartheta]$.

Here $F(\eta, \xi)$ is an $(n \times n)$-matrix satisfying the following conditions. If $\eta < \xi$ then $F(\eta, \xi) = 0$, if $\eta = \xi$ then $F(\eta, \eta) = E$, and if $t_0 \le \xi < \eta \le \vartheta$ then $F(\eta, \xi)$ is a solution of the functional differential equation

$$\frac{\partial F(\eta, \xi)}{\partial \eta} = A(\eta)F(\eta, \xi) + A_h(\eta)F(\eta - h, \xi).$$

For the position $(t, x_t(\cdot))$, from the vectors $w_i(t, x_t(\cdot))$, $i = \overline{1, m}$, let us compose an informational image

$$W(t, x_t(\cdot)) = \big\{w_i(t, x_t(\cdot)),\ i = \overline{1, m}\big\} \in \mathbb{R}^{mn}. \quad (13)$$

This notation means that the first n coordinates of $W(t, x_t(\cdot))$ are set to be identical with the coordinates of $w_1(t, x_t(\cdot))$, the next n coordinates are set to be identical with the coordinates of $w_2(t, x_t(\cdot))$, and so on, and the last n coordinates of $W(t, x_t(\cdot))$ are set to be identical with the coordinates of $w_m(t, x_t(\cdot))$.

Let us consider an auxiliary dynamical system described by the ordinary differential equation

$$\dot{Z}(t) = \widetilde{B}(t)u(t) + \widetilde{C}(t)v(t), \qquad t_0 \le t < \vartheta, \quad Z \in \mathbb{R}^{mn}, \quad u \in P, \quad v \in Q, \quad (14)$$

with the initial condition

$$Z(t_0) = W(t_0, \sigma(\cdot)). \quad (15)$$

Here the matrices $\widetilde{B}(t)$ and $\widetilde{C}(t)$ are composed from the matrices

$$B_i(t) = F(\vartheta_i, t)B(t), \quad C_i(t) = F(\vartheta_i, t)C(t)$$

according to the following rule. The first n rows of $\widetilde{B}(t)$ are set to be identical with the rows of $B_1(t)$, the next n rows of $\widetilde{B}(t)$ are set to be identical with the rows of $B_2(t)$, and so on, and the last n rows of $\widetilde{B}(t)$ are set to be identical with the rows of $B_m(t)$. The matrix $\widetilde{C}(t)$ is constructed by the same rule.

Let us note that auxiliary system (14) describes the evolution of informational image (13) with respect to initial system (1). Namely, the following statement holds. Let motions $x[t_0 - h[\cdot]\vartheta]$ and $Z[t_0[\cdot]\vartheta]$ be generated by the same admissible realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$. Then, the equality is valid

$$Z(\vartheta) = W(\vartheta, x_\vartheta(\cdot)).$$

Therefore, taking into account the equalities $w_i(\vartheta, x_\vartheta(\cdot)) = x(\vartheta_i)$, $i = \overline{1, m}$, in accordance with (12) let us consider the quality index

$$\gamma_2^{(m)} = \sqrt{\Delta h}\|Z(\vartheta)\| + \int\limits_{t_0}^{\vartheta} \Big[\langle u(t), \Phi(t)u(t)\rangle - \langle v(t), \Psi(t)v(t)\rangle\Big]\mathrm{d}t. \quad (16)$$

For system (14), initial condition (15) and quality index (16), an auxiliary differential game is considered (see Section 3 for details). Let us denote by $\rho_1^{(m)}$ and $u_1^{(m)}(t, Z, \varepsilon)$ the value and the optimal minimax strategy in this game.

The next theorem establishes a connection between the control problem (1), (2), (12) and the auxiliary differential game (14)–(16) (see Lukoyanov and Reshetova (1998)).

Theorem 2. For the value of the optimal guaranteed result $\Gamma^{(m)}$ and the optimal control strategy $U^{(m)}(\cdot)$ in the control problem (1), (2), (12), the following representation formulas are valid

$$\Gamma^{(m)} = \rho_1^{(m)},$$
$$U^{(m)}(t, x_t(\cdot), \varepsilon) = u_1^{(m)}\big(t, W(t, x_t(\cdot)), \varepsilon\big),$$
$$(t, x_t(\cdot)) \in [t_0, \vartheta] \times C, \quad \varepsilon > 0.$$

Thus, in this Section the following solution to the initial control problem (1)–(3) is obtained. Let us fix a sufficiently large natural number m and construct the value $\rho_1^{(m)}$ and the optimal minimax strategy $u_1^{(m)}(\cdot)$ in the auxiliary differential game (14)–(16) according to (10). Then, the value $\rho_1^{(m)}$ approximates the value of the optimal guaranteed result Γ^0 and, for an appropriate value ε and partition Δ_δ (4), the control law $\{U^{(m)}(\cdot), \varepsilon, \Delta_\delta\}$ that forms the realization

$$u(t) = u_1^{(m)}\big(\tau_j, W(\tau_j, x_{\tau_j}(\cdot)), \varepsilon\big), \qquad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{1, k}, \quad (17)$$

ensures inequality (6).

5. APPROXIMATION OF DYNAMICAL SYSTEM

The second method for solving the control problem (1)–(3) is based on an approximation of the dynamical system.

As in the previous Section, let us fix a natural number m and define $\Delta h = h/m$. Let $x[t_0[\cdot]\vartheta]$ be a motion of system (1) generated by admissible realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$. Let us consider the functions

$$x^{[i]}(t) = x(t - i\Delta h), \quad t \in [t_0, \vartheta], \quad i = \overline{0, m}.$$

Due to (1), for almost all $t \in [t_0, \vartheta]$, we have

$$\dot{x}^{[0]}(t) = A(t)x^{[0]}(t) + A_h(t)x^{[m]}(t) + B(t)u(t) + C(t)v(t).$$

For every $i = \overline{1, m}$, let us approximate the derivatives $\dot{x}^{[i]}(t)$ by the divided differences formula:

$$\dot{x}^{[i]}(t) \approx \big(x^{[i-1]}(t) - x^{[i]}(t)\big)/\Delta h, \quad t \in [t_0, \vartheta].$$

These arguments lead us to the following approximating dynamical system described by the ordinary differential equations

$$\dot{y}^{[0]}(t) = A(t)y^{[0]}(t) + A_h(t)y^{[m]}(t) + B(t)\widetilde{u}(t) + C(t)\widetilde{v}(t),$$
$$\dot{y}^{[i]}(t) = \big(y^{[i-1]}(t) - y^{[i]}(t)\big)/\Delta h, \quad i = \overline{1, m}, \quad (18)$$
$$t_0 \le t < \vartheta, \quad y^{[i]} \in \mathbb{R}^n, \quad i = \overline{0, m}, \quad \widetilde{u} \in P, \quad \widetilde{v} \in Q,$$

with the initial condition

$$y^{[i]}(t_0) = \sigma(-i\Delta h), \quad i = \overline{0, m}. \quad (19)$$

The state vector $Y = \big\{y^{[i]},\ i = \overline{0, m}\big\} \in \mathbb{R}^{(m+1)n}$ of system (18) is composed from the vectors $y^{[i]}$, $i = \overline{0, m}$, similarly to (13).

The desired approximating property between systems (1) and (18) is provided by the following aiming procedure. Let realizations $u[t_0[\cdot]\vartheta)$ in initial system (1) and $\widetilde{v}[t_0[\cdot]\vartheta)$ in approximating system (18) be constructed on the basis of a partition Δ_δ (4) by the step-by-step rule:

$$\begin{aligned} u(t) = u_j \in \operatorname*{argmin}_{u\in P}\Big[&\langle B(\tau_j)u, x(\tau_j) - y^{[0]}(\tau_j)\rangle \\ &+\langle u, \Phi(\tau_j)u\rangle\alpha(\tau_j)\Big], \\ \widetilde{v}(t) = \widetilde{v}_j \in \operatorname*{argmax}_{\widetilde{v}\in Q}\Big[&\langle C(\tau_j)\widetilde{v}, x(\tau_j) - y^{[0]}(\tau_j)\rangle \\ &-\langle \widetilde{v}, \Psi(\tau_j)\widetilde{v}\rangle\alpha(\tau_j)\Big], \\ t \in [\tau_j, \tau_{j+1}),&\quad j = \overline{1,k}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \alpha(t) = &\int_{t_0}^{t}\Big[\langle u(\xi), \Phi(\xi)u(\xi)\rangle - \langle v(\xi), \Psi(\xi)v(\xi)\rangle\Big]\mathrm{d}\xi \\ &- \int_{t_0}^{t}\Big[\langle \widetilde{u}(\xi), \Phi(\xi)\widetilde{u}(\xi)\rangle - \langle \widetilde{v}(\xi), \Psi(\xi)\widetilde{v}(\xi)\rangle\Big]\mathrm{d}\xi. \end{aligned} \tag{21}$$

The following theorem can be proved (see Lukoyanov and Plaksin (2013)).

Theorem 3. For any number $\zeta > 0$ there exist numbers $M > 0$ and $\delta > 0$ such that, for any natural number $m \geq M$ and any partition Δ_δ (4), the following statement holds. Let in initial (1) and in approximating (18) systems realizations $u[t_0[\cdot]\vartheta)$ and $\widetilde{v}[t_0[\cdot]\vartheta)$ be formed according to aiming procedure (20). Then, for any admissible realizations $v[t_0[\cdot]\vartheta)$, $\widetilde{u}[t_0[\cdot]\vartheta)$ and the corresponding motions $x[t_0 - h[\cdot]\vartheta]$, $Y[t_0[\cdot]\vartheta]$, the inequalities are valid

$$|x(t-i\Delta h) - y^{[i]}(t)| \leqslant \zeta,\ \ i = \overline{0,m},\ \ |\alpha(t)| \leqslant \zeta,\ \ t \in [t_0, \vartheta].$$

Taking into account Theorem 3 and approximation (11), let us define the quality index

$$\begin{aligned} \gamma_2^{(m)} = &\left(\sum_{i=1}^{m}\|y^{[m-i]}(\vartheta)\|^2\Delta h\right)^{1/2} \\ &+ \int_{t_0}^{\vartheta}\Big[\langle \widetilde{u}(t), \Phi(t)\widetilde{u}(t)\rangle - \langle \widetilde{v}(t), \Psi(t)\widetilde{v}(t)\rangle\Big]\mathrm{d}t. \end{aligned} \tag{22}$$

For system (18), initial condition (19) and quality index (22), let us consider an auxiliary differential game (see Section 3 for details). Let us denote by $\rho_2^{(m)}$ and $u_2^{(m)}(t, Y, \varepsilon)$ the value and the optimal minimax strategy in this game.

For initial system (1), let us consider the following control scheme. Let us fix a value $\varepsilon > 0$ and a partition Δ_δ (4). Let realizations $u[t_0[\cdot]\vartheta)$ and $\widetilde{v}[t_0[\cdot]\vartheta)$ be constructed by aiming procedure (20) on the basis of the partition Δ_δ and a realization $\widetilde{u}[t_0[\cdot]\vartheta)$ be formed according to the law $\{u_2^{(m)}(\cdot), \varepsilon, \Delta_\delta\}$:

$$\widetilde{u}(t) = u_2^{(m)}\big(\tau_j, Y(\tau_j), \varepsilon\big),\quad t \in [\tau_j, \tau_{j+1}),\quad j = \overline{1,k}. \tag{23}$$

Theorem 3 allows us to establish the following result (see also Lukoyanov and Plaksin (2015)).

Theorem 4. For any number $\zeta > 0$, there exists a number $M > 0$ such that, for any natural number $m \geq M$, the following statements are valid:

i) The inequality $|\Gamma^0 - \rho_2^{(m)}| \leq \zeta$ holds.

ii) There exist a number $\varepsilon_2^{(m)} > 0$ and a function $\delta_2^{(m)}(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_2^{(m)}]$, such that, for any value $\varepsilon \in (0, \varepsilon_2^{(m)}]$ and any partition Δ_δ (4) with $\delta \leq \delta_2^{(m)}(\varepsilon)$, the control scheme (20), (23) ensures inequality (6) for any admissible disturbance realization $v[t_0[\cdot]\vartheta)$.

Thus, for the control problem (1)–(3), in this Section the following solution method is obtained. Let us fix a sufficiently large natural number m and construct the value $\rho_2^{(m)}$ and the optimal minimax strategy $u_2^{(m)}(\cdot)$ in the auxiliary differential game (18), (19), (22) according to (10). Then, the value $\rho_2^{(m)}$ approximates the value of the optimal guaranteed result Γ^0 and, for an appropriate value ε and partition Δ_δ, the control scheme (20), (23) ensures inequality (6).

6. EXAMPLE

In this Section the following control problem is considered. A dynamical system is described by the delay differential equations

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -2x_1(t) - 0.4x_2(t) + 0.02x_3(t) - x_1(t-1) \\ \qquad - 0.4x_2(t-1) + 0.4x_3(t-1) - x_4(t-1) \\ \qquad + (5-t)u_1(t) + 2v_1(t), \\ \dot{x}_3(t) = x_4(t), \\ \dot{x}_4(t) = 0.01x_1(t) - x_3(t) - 0.1x_4(t) - 0.3x_1(t-1) \\ \qquad + 0.7x_2(t-1) - 0.4x_3(t-1) + 0.5x_4(t-1) \\ \qquad + (4-0.5t)u_2(t) + 3v_2(t), \end{cases}$$

$$0 \leq t < 4,\quad x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$
$$u = (u_1, u_2) \in \mathbb{R}^2,\quad v = (v_1, v_2) \in \mathbb{R}^2,$$

with the initial condition

$$x_{t_0}(\xi) = \big(\sin(\xi), \cos(\xi), \cos(\xi), -\sin(\xi)\big),\quad \xi \in [-1, 0].$$

The quality of a control process realization is evaluated by the index

$$\begin{aligned} \gamma = &\left(\int_{3}^{4}\Big[x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t)\Big]\mathrm{d}t\right)^{1/2} \\ &+ \int_{0}^{4}\Big[u_1^2(t) + u_2^2(t) - v_1^2(t) - v_2^2(t)\Big]\mathrm{d}t. \end{aligned}$$

For this control problem, the solution methods proposed in Sections 4 and 5 are tested for different values m, ε and the uniform partitions Δ_δ (4) with the diameter δ.

For the value of the optimal guaranteed result Γ^0, the following approximating values are obtained:

$$\rho_1^{(10)} = 1.769, \quad \rho_1^{(50)} = 1.754, \quad \rho_1^{(100)} = 1.753,$$
$$\rho_2^{(10)} = 1.678, \quad \rho_2^{(50)} = 1.736, \quad \rho_2^{(100)} = 1.743.$$

In order to illustrate the workability of the proposed control schemes disturbance actions are simulated in the following way. Within the game-theoretical approach, for system (1), initial condition (2) and quality index (3), similarly to the initial control problem let us consider the problem of forming the worst-case disturbances aimed to maximize index (3). For this problem, results which are similar to Theorems 1–4 can be established and the corresponding two schemes of forming disturbance actions based on solution (10) of the auxiliary differential games (14)–(16) or (18), (19), (22) can be applied.

The following two control process realizations are simulated. In the first case, control actions are formed according to the first method and disturbance actions are formed according to the second method. In the second case, conversely, control actions are formed according to the second method and disturbance actions are formed according to the first method. The corresponding simulation results are shown in Figures 1 and 2 respectively.

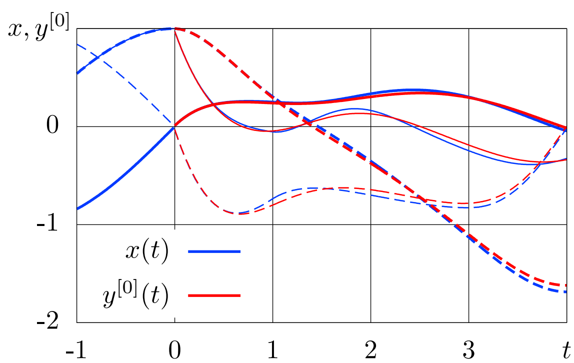

Fig. 1. Simulation results for $m = 50$, $\varepsilon = 0.02$, $\delta = 0.002$. The realized value of the quality index is $\gamma = 1.771$.

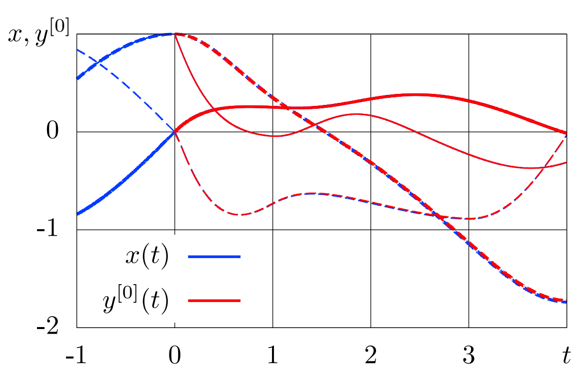

Fig. 2. Simulation results for $m = 100$, $\varepsilon = 0.01$, $\delta = 0.001$. The realized value of the quality index is $\gamma = 1.756$.

7. CONCLUSION

In the paper two effective methods for solving the control problem (1)–(3) are proposed. Both methods are based on the idea of reducing the problem to simple high-dimensional auxiliary differential games of type (7)–(9) that admit effective solution (10).

The key difference between these methods is the information needed for the construction of a current control action. In the first control scheme (17) this information is only a position of initial system (1). While in the second scheme (20), (23) this information includes values of the state vectors of initial (1) and approximating (18) systems and also a value of auxiliary variable $\alpha(t)$ (21). An important point here is that the value $\alpha(t)$ depends on the unknown disturbance actions realized in the initial system on $[t_0, t]$. This drawback certainly narrows the applicability of the second method.

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