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Stochastic Control in the Problem of Preventing Ecological Catastrophes

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Abstract: A problem of the analysis and prevention of catastrophic shifts in stochastically forced ecosystems is considered. For the solution of this problem, a new mathematical approach based on the analysis and synthesis of the stochastic sensitivity of dynamic regimes in population models is suggested. Technical details of this approach are discussed for the conceptual stochastically forced Bazykin-Berezovskaya predator-prey model with the Allee effect. For this population model, a phenomenon of the noise-induced extinction is analysed by the method of confidence domains. By reducing these domains we provide a stabilization of the persistence regime for both interacting species.

Keywords: Population dynamics, ecological shifts, random disturbances, stochastic sensitivity, stabilization.

1. INTRODUCTION

Even small inevitable deterministic or stochastic disturbances can cause the abrupt catastrophic shifts in ecosystems (see e.g. Ridolfi et al. (2011); Rietkerk et al. (2004); Scheffer et al. (2001)). Mathematically, such shifts can be explained by the coexistence of alternative stable states, nonuniformity of phase portraits, and a high sensitivity of attractors of the corresponding dynamic models (Bashkirtseva et al. (2018)).

Under the random disturbances, the solution of the multistable system can leave a basin of attraction of one attractor, cross the separatrix and continue to operate near another attractor. An analysis of the interplay between nonlinearity, multistability and stochasticity is an attractive problem of the modern theoretical ecology (Lande et al. (2003); Schreiber et al. (2003); Bashkirtseva et al. (2017b)).

Currently, for mathematical modeling of dynamics in randomly forced ecological systems, the stochastic differential equations are widely used (Spagnolo et al. (2004); Valenti et al. (2006); Bashkirtseva et al. (2017c)).

The detailed probabilistic description of the stochastic dynamics in these models is given by Kolmogorov-Fokker-Planck equation (Freidlin et al. (1984)). However, it is hard to use this equation directly, even in 2D case. In these circumstances, approximations and asymptotics based on stochastic sensitivity analysis are widely used (Bashkirtseva et al. (2016, 2017a)).

Along with the analysis of unwanted shifts in live systems, problems of the prevention of such shifts by controlling ecosystems with the help of additional artificial feedbacks are also highly relevant (Folke et al. (2005)).

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In the present paper, we show how the stochastic sensitivity analysis can be used for the study and prevention of undesirable shifts in ecosystems. To demonstrate the mathematical technique of our approach, we consider the conceptual predator-prey population model (Bazykin (1998)) suggested by Bazykin and Berezovskaya (BB-model for short).

2. DETERMINISTIC MODEL

Consider the BB-model (Bazykin (1998)) of the interacting prey-predator population system

$$\begin{cases} \dot{x} = rx(x-l)(k-x) - xy, \\ \dot{y} = y(x-m), \end{cases} \quad (1)$$

where x and y are densities of the prey and predator. The parameter r characterizes an intrinsic growth, l is defined a prey survival threshold corresponding to Allee effect, k is a carrying capacity, and m describes a mortality of the predator. All parameters are positive, and $l < k$.

The deterministic system (1) possesses four equilibria

$$M_0(0,0),\ M_1(m, r(m-l)(k-m)),\ M_2(l,0),\ M_3(k,0).$$

The nontrivial equilibrium M_1 has a biological sense for $l < m < k$.

For any values of parameters, the trivial equilibrium M_0 is stable, and the equilibrium M_2 is unstable. Stability of equilibria M_1 and M_3 depends on the mortality parameter m: M_3 is stable for $m > m_3 = k$, and unstable for $m < m_3$; M_1 is stable for $m_2 = (l+k)/2 < m < m_3$, and unstable otherwise.

When the parameter m crosses the point m_2 of Andronov-Hopf bifurcation from right to left, the equilibrium M_1 loses its stability, and a stable limit cycles appears. As parameter m decreases from m_2, this limit cycle enlarges,

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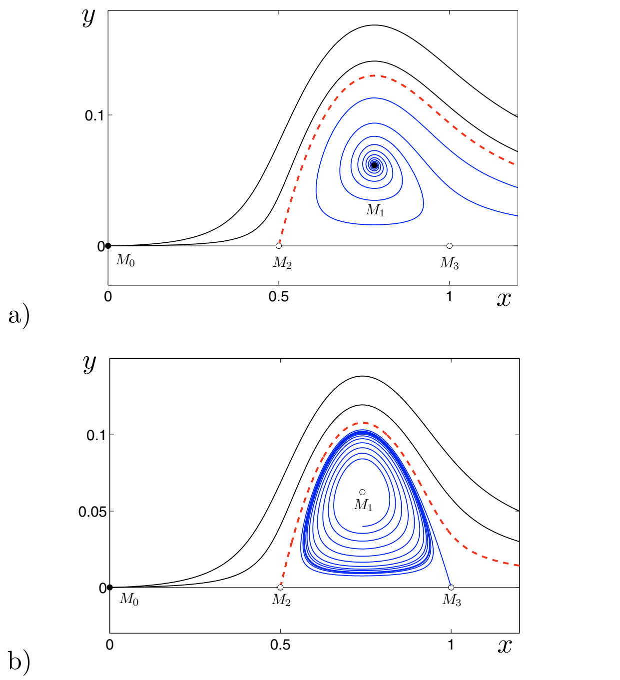

Fig. 1. Phase portraits of the deterministic model for a) $m = 0.78$, b) $m = 0.74$. Separatrices are shown by red dashed lines, stable equilibria by black circles, unstable equilibria by unfilled circles.

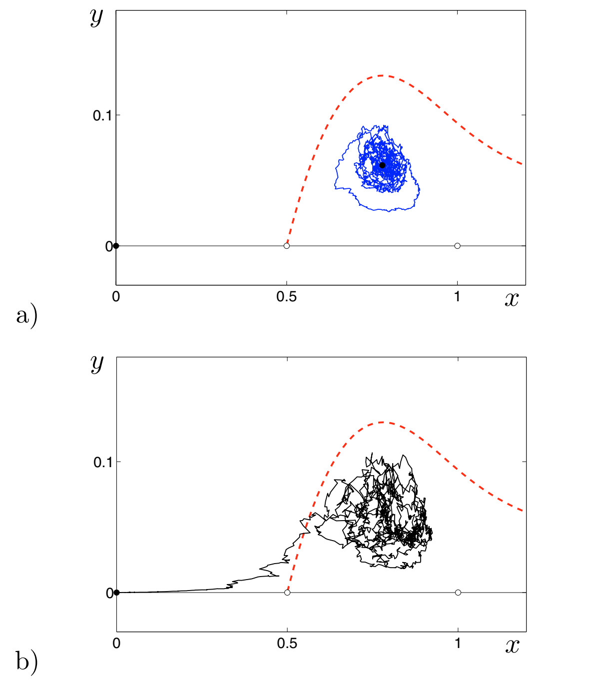

Fig. 2. Stochastic trajectories of the stochastic system with $m = 0.78$ for a) $\varepsilon = 0.05$, b) $\varepsilon = 0.15$.

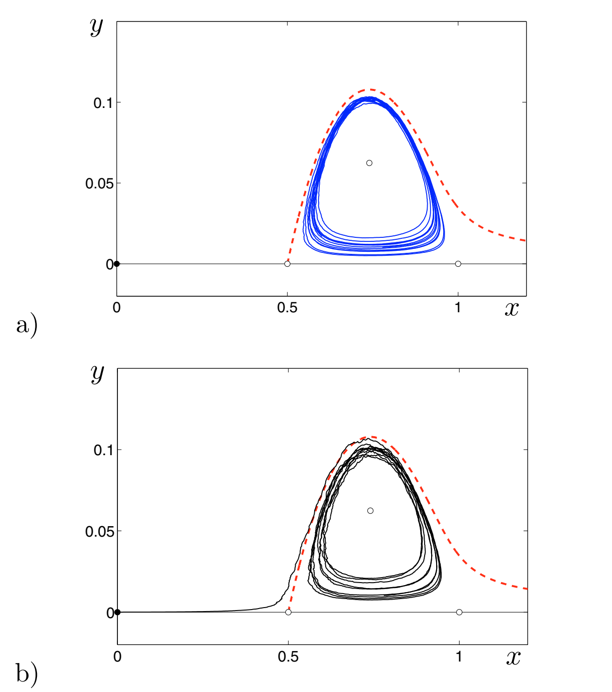

Fig. 3. Stochastic trajectories of the stochastic system with $m = 0.74$ for a) $\varepsilon = 0.005$, b) $\varepsilon = 0.015$.

and for $m = m_1$, the cycle is destroyed: a lower part of the cycle coalesces with the line $y = 0$ in the interval between equilibria M_2 and M_3, and upper part adheres to the separatrix. In the present paper, we fix $r = k = 1$, $l = 0.5$. Corresponding bifurcation points are as follows: $m_1 = 0.73544$, $m_2 = 0.75$, and $m_3 = 1$.

In the interval $m_1 < m < m_2$, system (1) exhibits a coexistence of the stable equilibrium M_0 and stable limit cycle. In the interval $m_2 < m < m_3$, this system has the stable equilibrium M_0 and stable equilibrium M_1. Their basins of attraction are separated by the stable manifold of the saddle M_2. So, the system behavior depends on the choice of the initial point: if the starting point belongs to the basin of attraction of M_0 then both populations go to extinction, otherwise both species coexist in the equilibrium or oscillatory regime. These deterministic regimes are illustrated in Fig. 1. Here, the separatrix is plotted by red dashed line.

Under the random disturbances, this population system is subject to the ecological shifts. Inevitable noise can transit the system from the persistence to extinction regime. Underlying reasons and condition of such ecological catastrophe are considered below.

3. NOISE-INDUCED EXTINCTION

Consider the BB-model in presence of parametric random disturbances:

$$\begin{cases} \dot{x} = rx\,(x - [l + \varepsilon\xi_1(t)])\,(k - x) - xy, \\ \dot{y} = y(x - [m + \varepsilon\xi_2(t)]). \end{cases} \tag{2}$$

Here, $\xi_i(t)\,(i = 1, 2)$ are white uncorrelated Gaussian noises with parameters $\langle \xi_i(t) \rangle = 0$, $\langle \xi_i(t)\xi_i(\tau) \rangle = \delta(t - \tau)$, and ε is the noise intensity. These noises model random fluctuations of the prey Allee parameter l and predator mortality m.

Under the random disturbances, trajectories leave the deterministic attractor. For small noise, the stochastic trajectories are concentrated near the initial attractor. As noise intensity increases, a dispersion of random trajectories increases too. In bistable systems, for the quite large noise, stochastic trajectories can exit from the basin of attraction of the initial attractor, cross the separatrix, and fall into the basin of another attractor. Such scenario is demonstrated in Figs. 2,3.

The Fig. 2 illustrates these two variants of the stochastic dynamics for $m = 0.78$. For small noise $\varepsilon = 0.05$, random trajectories are concentrated near the stable non-trivial equilibrium M_1. So, populations of prey and predator exhibit small-amplitude stochastic oscillations (see Fig. 2a). For larger noise ($\varepsilon = 0.05$), the stochastic trajectories exit from the basin of attraction of M_1, cross the separatrix, and tend to the trivial equilibrium M_0 (see Fig. 2b).

A similar scenario can be seen in Fig. 3 for $m = 0.74$ when non-trivial attractor is the stable limit cycle. Note that here the noise-induced extinction occurs for smaller noise intensities.

A parametric study of these phenomena can be carried out on the basis of the stochastic sensitivity function technique and confidence domains method. In the present paper, we consider in detail a case of stochastically forced equilibria.

4. ANALYSIS OF THE STOCHASTIC SENSITIVITY

Consider a general nonlinear stochastic system

$$\dot{x} = f(x) + \varepsilon\sigma(x)\xi, \quad (3)$$

where x is an n-dimensional vector, ξ is an m-dimensional Gaussian white noise satisfying $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(\tau)\rangle = \delta(t-\tau)I$, I is an identity matrix, $\sigma(x)$ is $n \times m$-matrix function of random disturbances with scalar intensity ε.

Let $\bar{x}$ be an exponentially stable equilibrium of the corresponding deterministic system with $\varepsilon = 0$. For stochastic system (3) with small noise, random trajectories are concentrated near $\bar{x}$ with the stationary probabilistic distribution $\rho(x,\varepsilon)$. For $\rho(x,\varepsilon)$, one can write (Freidlin et al. (1984)) the following Gaussian approximation:

$$\rho(x,\varepsilon) \approx K \cdot \exp\left(-\frac{(x-\bar{x}, W^{-1}(x-\bar{x}))}{2\varepsilon^2}\right).$$

Here, the stochastic sensitivity matrix W is a unique solution of the equation

$$FW + WF^\top + S = 0, \quad F = \frac{\partial f}{\partial x}(\bar{x}), \quad S = \sigma(\bar{x})\sigma^\top(\bar{x}). \quad (4)$$

A dispersion of random states around $\bar{x}$ can be approximated by the following formula

$$\mathrm{E}(x-\bar{x})(x-\bar{x})^\top \approx \varepsilon^2 W.$$

Eigenvalues of the matrix W can be considered as scalar characteristics of the stochastic sensitivity of the equilibrium.

In two-dimensional case, this matrix defines the corresponding confidence ellipse

$$(x-\bar{x}, W^{-1}(x-\bar{x})) = 2\varepsilon^2 q.$$

Here, $q = -\ln(1-P)$, and P is a fiducial probability.

The confidence ellipse allows us to describe a spatial arrangement of random states near the stable equilibrium, and can be used in parametric analysis of noise-induced transitions.

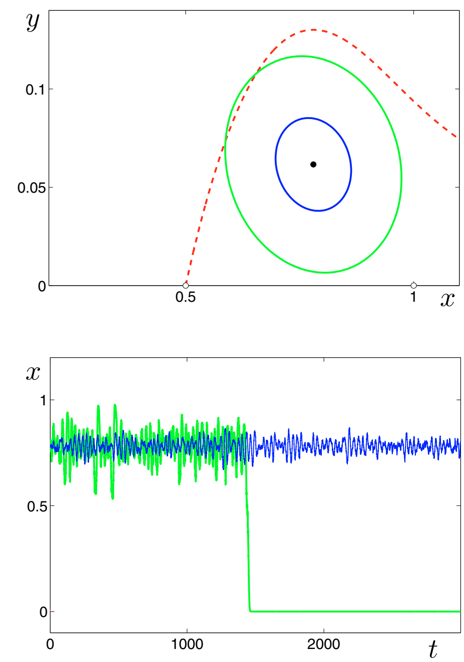

Fig. 4. Stochastic system with $m = 0.78$: confidence ellipses and time series for $\varepsilon = 0.03$ (blue), $\varepsilon = 0.07$ (green). Separatrix is shown by red dashed line.

Consider how this theory can be applied to the analysis of the noise-induced extinction in the BB-model. In Fig. 3a, we show confidence ellipses for system (2) with $m = 0.78$ and two values of the noise intensity. For weak noise $\varepsilon = 0.03$, the confidence ellipse plotted by blue color entirely belongs to the basin of the attraction of the equilibrium M_1, and corresponding random trajectories (see Fig. 3b) demonstrate small-amplitude oscillations near M_1.

With increasing noise, confidence ellipses expand, cross the separatrix (dashed line), and begin to occupy a basin of attraction of trivial equilibrium M_0. For $\varepsilon = 0.07$, the ellipse is plotted by green in Fig. 3a. This means that random trajectories with a high probability can leave the basin of attraction of M_1 and tend to M_0 (see time series in Fig. 3b). Note that a size of the ellipse is defined by the noise intensity and stochastic sensitivity of the equilibrium M_1. For considered here $m = 0.78$, eigenvalues of the stochastic sensitivity matrix W are following: $\lambda_1 = 0.83$, $\lambda_2 = 0.066$.

In controlled systems we can change the stochastic sensitivity by the choice of the appropriate regulator. To prevent the undesired noise-induced extinction, it is required to reduce the sensitivity of the equilibrium to noise. A mathematical background of the corresponding control theory is presented below.

5. PREVENTING ECOLOGICAL CATASTROPHES BY THE STOCHASTIC SENSITIVITY SYNTHESIS

Consider a nonlinear stochastic system with the control:

$$\dot{x} = f(x,u) + \varepsilon\sigma(x)\xi. \quad (5)$$

Here, $f(x,u)$ is a continuously differentiable vector-function, and u is a control input. Let $\bar{x}$ be an equilibrium

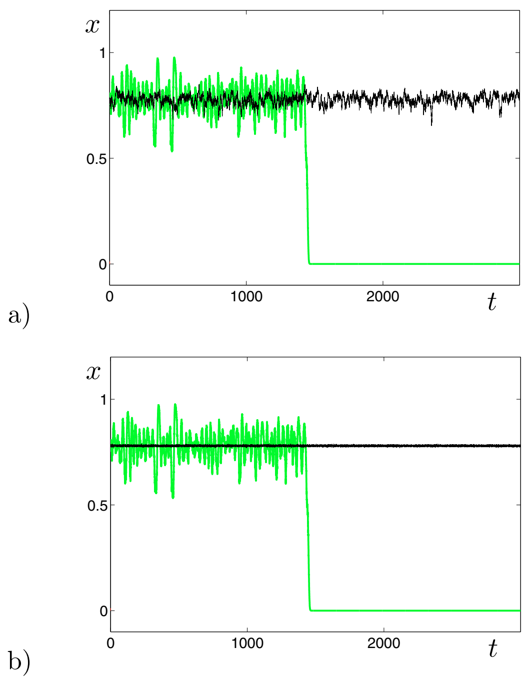

Fig. 5. Time series of the stochastic system with $m = 0.78$, $\varepsilon = 0.07$ without control (green) and with control (black) providing a) $w = 0.1$, b) $w = 0.001$.

of the corresponding deterministic system (5) (with $\varepsilon = 0$ and $u = 0$). A stability of $\bar{x}$ is not assumed.

Here, we will use a feedback regulator

$$u(x) = K(x - \bar{x}). \tag{6}$$

The equilibrium $\bar{x}$ of the closed-loop system (5), (6) is exponentially stable if the feedback matrix K belongs to the following set (see Wonham (1979)):

$$\mathbf{K} = \{K|\mathrm{Re}\lambda_i(F + BK) < 0\}.$$

Here, $F = \dfrac{\partial f}{\partial x}(\bar{x}, 0), \quad B = \dfrac{\partial f}{\partial u}(\bar{x}, 0).$

The aim of the control is to provide an assigned stochastic sensitivity matrix W of this stable equilibrium by the choice of the appropriate matrix K of the regulator (6).

The stochastic sensitivity matrix W for system (5), (6) is a solution of the following equation (Ryashko et al. (2008))

$$(F + BK)W + W(F + BK)^\top + S = 0. \tag{7}$$

So, the feedback matrix K providing the assigned stochastic sensitivity matrix W can be found from the equation:

$$BKW + WK^\top B^\top + S + FW + WF^\top = 0. \tag{8}$$

A detailed mathematical description of the solution of this equation can be found in (Ryashko et al. (2008)).

Here, we present the basic results for $\mathrm{rank}(B) = n$. In this case, there exists the matrix B^{-1}, and for any assigned positive defined matrix W it holds that

$$K = -B^{-1}\left(F + \frac{1}{2}SW^{-1}\right). \tag{9}$$

Consider how this control theory can be applied to the stabilization of the population dynamics in the stochastic BB-model.

To protect this population system from unwanted noise-induced ecological shifts, we will construct a feedback in such a way as to decrease the stochastic sensitivity of the equilibrium M_1. Remember that in uncontrolled system with $m = 0.78$, the stochastic sensitivity matrix W has eigenvalues $\lambda_1 = 0.83$, $\lambda_2 = 0.066$.

To decrease the stochastic sensitivity, let us assign the diagonal matrix W:

$$W = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$$

with the smaller value w. In Fig. 5, we plot time series of the uncontrolled system (2) with $m = 0.78$, $\varepsilon = 0.07$ by green color, and time series of the controlled system (5),(6) (by black color) with the regulators providing the stochastic sensitivity $w = 0.1$ and $w = 0.001$.

As one can see, a synthesis of the small stochastic sensitivity results in the localization of random trajectories near the equilibrium M_1. Note that the smaller the assigned stochastic sensitivity, the smaller the dispersion of random trajectories. So, using the constructed regulator, one can successfully solve the problem of prevention of undesired ecological shifts.

CONCLUSION

Our paper is devoted to the important problem how to prevent unwanted catastrophic shifts in stochastic ecosystems on the basis of the modern control theory. We have shown that the underlying reason of such shifts is in the high stochastic sensitivity of attractors of ecosystems. In our paper, a new mathematical approach based on the synthesis of the reduced stochastic sensitivity was developed. For general control systems, the necessary mathematical background was given. An efficiency of the proposed approach has been illustrated for the stochastically forced predator-prey model with the Allee effect.

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