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Application of optimal control and stabilization to an infinite time horizon problem under constraints ⋆

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Abstract: In modeling the dynamics of capital, the Ramsey equation coupled with the Cobb-Douglas production function is reduced to a linear differential equation by means of the Bernoulli substitution. This equation is used in the optimal growth problem with logarithmic preferences. We consider a vector field of the Hamiltonian system in the Pontryagin maximum principle, taking into account control constraints. We prove the existence of two alternative steady states, depending on the constraints. A proposed algorithm for constructing growth trajectories combines methods of open-loop control and closed-loop stabilizing control. Results are supported by modeling examples.

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1. INTRODUCTION

An ordinary differential equation proposed in Ramsey (1928) and describing capital dynamics, is used in modern economic growth and welfare theory (Acemoglu, 2008). The optimal control problem formalized in the neoclassical growth model, using this equation, provides the basis for various applied studies. The optimization functional in the problem is given by the discounted consumption index on infinite time horizon. This leads to a singularity in the control problem, which stimulates studies, devoted to formulations of the Pontryagin maximum principle (Pontryagin et al., 1962) for problems with infinite horizon (Aseev and Kryazhimskii, 2007; Aseev and Veliov, 2014). These studies deal with characteristics of the adjoint variable in the necessary optimality conditions, as well as with the elaboration of transversality conditions. Let us stress the importance of the stability analysis with respect to the derived optimal solutions; particularly, in the applied studies. At the same time the infinite time horizon is standardly considered in the optimal stabilization problems (see, e.g. Al'brekht (1961)). Here we propose an algorithm combining the optimal control and stabilization approaches.

While the Ramsey equation is nonlinear in general case (Grass et al., 2008), in the standard problem statement, ascending to Solow (1957), the nonlinear equation can be transformed to linear differential equation using the Bernoulli substitution. This transformation is considered in several papers for special cases in the economic framework (Smith, 2006). Here we study the problem in the control framework (Shell, 1969; Krasovskii and Tarasyev, 2008; Tarasyev and Usova, 2011), specially focusing on the impacts of control constraints. We analyze the vector field of the Hamiltonian system under admissible control regimes. This analysis allows us to identify the steady states, at which the necessary optimality conditions are satisfied, and to characterize the behavior of the adjoint variable. Besides we find an additional steady state, which is usually not considered in models. However, we show that it admits the economic interpretation. Analytical results are illustrated by modeling examples.

2. OPTIMAL GROWTH PROBLEM

We consider an optimal growth problem with the Cobb-Douglas production function, technological change, and logarithmic preferences (Acemoglu, 2008). The output Y at time $t \geq 0$ is determined by the production factors:

$$Y(t) = A(t)K^{\alpha}(t)L^{1-\alpha}(t), \quad \alpha \in (0,1). \qquad (1)$$

Here the Cobb-Douglas production function is applied; $A > 0$ is the exogenous factor of technological change, $L > 0$ is labor, $K > 0$ is capital. Constant α stands for the capital elasticity. Capital dynamics is subject to equation:

$$\dot{K} = sY(A,K,L) - \mu K, \quad K(t_0) = K^0 > 0, \qquad (2)$$

where $\mu > 0$ is coefficient of capital depreciation, $s(t)$ is the fraction, satisfying constraints $s \in [0, g]$, $0 < g < 1$, of current production that is saved and invested in the capital growth. Here g is a given investments restriction; it plays an important role in our analysis. The investment process, subject to dynamics (2), starts at time t_0 from the initial capital K^0. In the paper we follow a standard assumption concerning the labor growth:

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$$\dot{L} = nL, \quad L(t_0) = L^0, \tag{3}$$

with the constant rate $n \geq 0$ and initial level $L^0 > 0$.

2.1 Bernoulli substitution in the Ramsey equation

The Bernoulli substitution:

$$x = \frac{1}{A}\Big(\frac{K}{L}\Big)^{1-\alpha} \tag{4}$$

transforms equation (2) to the linear form:

$$\dot{x} = ax + bu, \tag{5}$$

where $u(t) = s(t)$ is control; coefficients a and b are calculated as follows:

$$a(t) = -(1-\alpha)\lambda - \frac{\dot{A}}{A}, \; b = (1-\alpha) > 0, \; \lambda = \mu + n. \tag{6}$$

In the economic growth model the utility is defined as a function of consumption, which is calculated as follows:

$$C(t) = (1 - s(t))Y(t). \tag{7}$$

We consider the logarithmic utility:

$$\mathrm{U}(C(t)) = \beta(t) + \gamma \ln x(t) + \ln(1 - s(t)), \tag{8}$$

where

$$\beta(t) = \frac{\ln A(t)}{(1-\alpha)}, \quad \gamma = \frac{\alpha}{(1-\alpha)}. \tag{9}$$

In the dynamic investments optimization problem the task is to maximize the discounted consumption index on the infinite time horizon:

$$J = \int_{t_0}^{+\infty} \mathrm{U}(C(t))e^{-\delta t}dt, \tag{10}$$

where $\delta > 0$ is a discount factor.

Let us consider the case when the exogenous dynamics of technical change is given by the linear equation:

$$\dot{A} = rA, \quad A(t_0) = A^0. \tag{11}$$

Here $r \geq 0$ is constant growth rate, and $A^0 > 0$, meaning that coefficient a (6) is constant:

$$a(t) = a = -(1-\alpha)\lambda - r < 0. \tag{12}$$

The exogenous growth $A(t)$ enters parameter $\beta(t)$ in (9).

2.2 Optimal control problem

Problem 1. The controlled object x starts the motion from the initial condition $x(t_0) = x_0$. Its state is subject to the linear differential equation:

$$\dot{x} = ax + bu, \tag{13}$$

where $a < 0$ and $b > 0$ are given numbers. The problem is to find among admissible controls $u \in [0, g]$ the optimal control u^0, maximizing the functional:

$$\underset{u\in[0,g]}{\text{maximize}} \int_{t_0}^{+\infty} \big(\gamma \ln x(t) + \ln(1 - u(t))\big)e^{-\delta t}dt, \tag{14}$$

where $\gamma > 0$, $\delta > 0$, and $0 < g < 1$ are given numbers.

3. VECTOR FIELD ANALYSIS

We consider the vector field of velocities of the state variable x and the adjoint variable ψ in the maximum principle (Aseev and Kryazhimskii, 2007). Using the usual transformations (Krasovskii and Tarasyev, 2008; Tarasyev and Usova, 2011), the Hamiltonian in the Pontryagin maximum principle takes the form:

$$H(x, \psi, u, t) = \gamma \ln x + \ln(1 - u) + \psi(ax + bu). \tag{15}$$

Its maximum with respect to u, determined by equation:

$$\frac{-1}{1-u} + \psi b = 0, \tag{16}$$

is achieved at the control:

$$u^0 = 1 - \frac{1}{b\psi}. \tag{17}$$

Due to concavity of the Hamiltonian with respect to u, we determine the maximizing control under constraints $u \in [0, g]$ as follows:

$$u^0(x,\psi) = \begin{cases} 0, & \text{if} \quad (x,\psi) \in \mathcal{D}_1, \\ 1 - \dfrac{1}{b\psi}, & \text{if} \quad (x,\psi) \in \mathcal{D}_2, \\ g, & \text{if} \quad (x,\psi) \in \mathcal{D}_3, \end{cases} \tag{18}$$

where domains $\mathcal{D}_i, i = 1, 2, 3$ are determined as:

$$\mathcal{D}_1 = \big\{(x,\psi)\colon 0 < \psi \leq 1/b, \; x > 0\big\}, \tag{19}$$

$$\mathcal{D}_2 = \big\{(x,\psi)\colon 1/b \leq \psi \leq \frac{1}{b(1-g)}, \; x > 0\big\}, \tag{20}$$

$$\mathcal{D}_3 = \big\{(x,\psi)\colon \psi \geq \frac{1}{b(1-g)}, \; x > 0\big\}. \tag{21}$$

Let us note, that the Bernoulli substitution simplifies the shapes of switching lines; here they are straight lines parallel to axis x (cf. Krasovskii and Tarasyev (2008)). The following dynamics of the adjoint variable:

$$\dot{\psi} = \delta\psi - \frac{\partial H}{\partial x} = (\delta - a)\psi - \frac{\gamma}{x}, \tag{22}$$

is valid for all domains $\mathcal{D}_i, i = 1, 2, 3$. The sign of the velocity of the adjoint variable ψ (22) is determined by:

$$\Psi(x) = \frac{\gamma}{(\delta - a)x}. \tag{23}$$

Due to $a < 0$, $\gamma > 0$, $\delta > 0$, $\Psi(x)$ is monotonically decreasing. The following conditions take place:

$$\begin{array}{lll} \dot{\psi} < 0, & \text{when} & \psi(x) < \Psi(x), \\ \dot{\psi} = 0, & \text{when} & \psi(x) = \Psi(x), \\ \dot{\psi} > 0, & \text{when} & \psi(x) > \Psi(x). \end{array} \tag{24}$$

Below we consider the Hamiltonian system:

$$\begin{cases} \dot{x} &= ax + bu^0, \\ \dot{\psi} &= (\delta - a)\psi - \dfrac{\gamma}{x}. \end{cases} \tag{25}$$

for control regimes u^0 (18). We are interested in directions of $\dot{x}$ in every domain (19)–(21). Our analysis is based on the fact (Aseev and Kryazhimskii, 2007), that the optimal trajectory (x^0, ψ^0) must satisfy the following transversality condition:

$$\lim_{t\to+\infty} x^0(t)\psi^0(t)e^{-\delta t} = 0. \tag{26}$$

3.1 Hamiltonian system in domain $\mathcal{D}_1$ (19)

is given by the relations:

$$\begin{cases} \dot{x} &= ax, \\ \dot{\psi} &= (\delta - a)\psi - \dfrac{\gamma}{x}. \end{cases} \tag{27}$$

In domain $\mathcal{D}_1$:

$$\dot{x} < 0. \tag{28}$$

3.2 Hamiltonian system in domain $\mathcal{D}_2$ (20)

takes the form:

$$\begin{cases} \dot{x} &= ax + b - \dfrac{1}{\psi}, \\ \dot{\psi} &= (\delta - a)\psi - \dfrac{\gamma}{x}. \end{cases} \tag{29}$$

Defining the function:

$$X(\psi) = \frac{1 - b\psi}{a\psi}, \quad \psi \in \mathcal{D}_2, \tag{30}$$

we derive the following relations:

$$\begin{array}{ll} \dot{x} > 0, & \text{when} \quad x(\psi) < X(\psi), \\ \dot{x} = 0, & \text{when} \quad x(\psi) = X(\psi), \\ \dot{x} < 0, & \text{when} \quad x(\psi) > X(\psi). \end{array} \tag{31}$$

3.3 Hamiltonian system in domain $\mathcal{D}_3$ (21)

takes the form:

$$\begin{cases} \dot{x} &= ax + bg, \\ \dot{\psi} &= (\delta - a)\psi - \dfrac{\gamma}{x}. \end{cases} \tag{32}$$

Defining the function:

$$Z = \big\{(x, \psi) \in \mathcal{D}_3\colon\ x = -ga/b\big\}, \tag{33}$$

we derive the following relations:

$$\begin{array}{ll} \dot{x} > 0, & \text{when} \quad x < Z, \\ \dot{x} = 0, & \text{when} \quad x = Z, \\ \dot{x} < 0, & \text{when} \quad x > Z. \end{array} \tag{34}$$

3.4 Steady states of the Hamiltonian system

Conditions (24), (28), (31), (34) characterize the vector field of the Hamiltonian system under admissible controls. **Definition.** The steady state of the Hamiltonian system (25) is the solution to the system of equations:

$$\begin{cases} ax + bu^0(x, \psi) &= 0, \\ (\delta - a)\psi - \dfrac{\gamma}{x} &= 0, \end{cases} \tag{35}$$

where $u^0(x, \psi)$ is subject to (18).

Geometrically the equations in the system (35) are satisfied on the following lines: Ψ (23), X (30) in domain $\mathcal{D}_2$ (19), as well as on Z (33) in domain $\mathcal{D}_3$ (21).

Theorem 1. There exist two alternative steady states of the Hamiltonian system (25), determined by constraints $u \in [0, g]$ in Problem 1.

(1) If

$$0 < g \le \frac{-a\gamma}{\delta - a - a\gamma}, \tag{36}$$

then there is a unique steady state in domain $\mathcal{D}_3$ (21):

$$x^*_S = -g\frac{b}{a}, \quad \psi^*_S = \frac{-\gamma a}{gb(\delta - a)}. \tag{37}$$

(2) If

$$\frac{-a\gamma}{\delta - a - a\gamma} \le g < 1, \tag{38}$$

then there is a unique steady state in domain $\mathcal{D}_2$ (20):

$$x^* = \frac{b\gamma}{\delta - a - a\gamma}, \quad \psi^* = \frac{\delta - a - a\gamma}{b(\delta - a)}. \tag{39}$$

Each of these steady states possesses the saddle property.

Proof. By conditions of Problem 1: $a < 0$, $b > 0$, $\gamma > 0$, $\delta > 0$, $g \in (0, 1)$, meaning that:

$$0 < \frac{-a\gamma}{\delta - a - a\gamma} < 1. \tag{40}$$

The steady state (39) in domain $\mathcal{D}_2$, corresponds to the unique intersection of lines X (30) and Ψ (23). The definition of domain $\mathcal{D}_2$ (20) includes parameter g. Hence the existence of steady state (x^*, ψ^*) in domain $\mathcal{D}_2$ is subject to inequalities:

$$\frac{1}{b} \le \psi^* \le \frac{1}{b(1 - g)}. \tag{41}$$

The left-hand side inequality is always satisfied:

$$\psi^* = \frac{\delta - a - a\gamma}{b(\delta - a)} > \frac{\delta - a}{b(\delta - a)} = \frac{1}{b}. \tag{42}$$

Substituting in the right-hand side of (41) the expression for ψ^* (39), we get (38).

The steady state (37) is determined by the intersection of lines Z (33) and Ψ (23) in domain $\mathcal{D}_3$ (21). The existance of this intersection is subject to the condition:

$$\psi^*_S \ge \frac{1}{b(1 - g)}. \tag{43}$$

Substituting (37) to (43), we get inequality (36).

In the case of equality in (38), (36):

$$g = \frac{-a\gamma}{\delta - a - a\gamma}, \tag{44}$$

steady states coincide: $x^* = x^*_S$, $\psi^* = \psi^*_S$.

As there is no steady state of the Hamiltonian system (25) in domain $\mathcal{D}_1$ (19) (according to (27), (28)), we have proven the existance of two alternative steady states.

The Jacobi matrix of the Hamiltonian system (29), linearized in the neighborhood of steady state (39), is:

$$\mathcal{J}_{\mathcal{D}_2}(x^*, \psi^*) = \begin{pmatrix} a & \dfrac{1}{\psi^{*2}} \\ \dfrac{\gamma}{x^{*2}} & (\delta - a) \end{pmatrix}. \tag{45}$$

Eigenvalues are calculated as follows:

$$\xi_{\text{n,p}} = 0.5\left(\delta \mp \sqrt{\delta^2 - 4a(\delta - a) + \frac{(\delta - a)^2}{\gamma^2}}\right). \tag{46}$$

Using $a < 0$ and $\delta > 0$, we get:

$$\xi_{\text{n}} < 0, \quad \xi_{\text{p}} > 0, \tag{47}$$

which proves the saddle character of steady state (x^*, ψ^*). The direction of eigenvector, corresponding to ξ_{n} is stable (see, e.g., Hartman (1964)).

Similarly, for steady state (37), we have:

$$\mathcal{J}_{\mathcal{D}_3}(x^*_S, \psi^*_S) = \begin{pmatrix} a & 0 \\ \dfrac{\gamma a^2}{g^2 b^2} & (\delta - a) \end{pmatrix}, \tag{48}$$

Here eigenvalues are calculated as follows:

$$\sigma_{\text{n,p}} = 0.5\big(\delta \mp \sqrt{\delta^2 - 4a(\delta - a)}\big),\ \sigma_{\text{n}} < 0,\ \sigma_{\text{p}} > 0, \tag{49}$$

proving the saddle character of steady state (x^*_S, ψ^*_S). □

Corollary 2. The following inequalities imply from Theorem 1:

$$0 < x^*_S \le x^*, \quad 0 < \psi^* \le \psi^*_S. \tag{50}$$

4. CONSTRUCTION OF TRAJECTORIES

The analysis above identifies two qualitative portraits, reflecting the vector field of the Hamiltonian system. In this section we propose algorithms for constructing trajectories, satisfying optimality conditions.

4.1 Steady state in domain $\mathcal{D}_2$

Figure 1 depicts the vector field in the case of steady state belonging to domain $\mathcal{D}_2$ (20). It corresponds to the case, when parameter g satisfies (38); this case is standard.

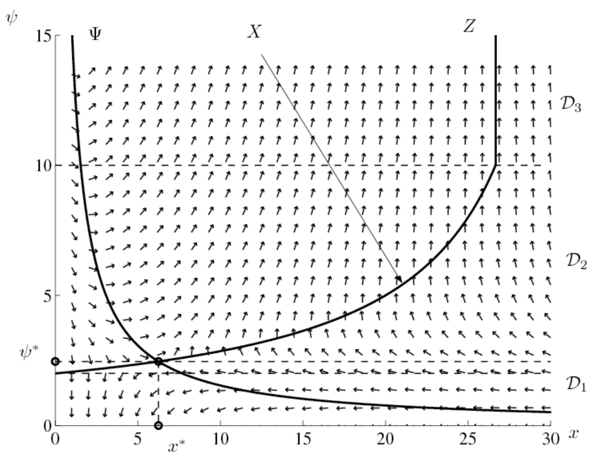

Fig. 1. Vector field of the Hamiltonian system (25) in the case of standard contraints (38). Bold lines: Z (33), X (30), Ψ (23). Steady state $(x^*, \psi^*) = (6.25, 2.46)$ (39) belongs to domain $\mathcal{D}_2$ (20). Parameter values: $a = -0.015$, $b = 0.5$, $\gamma = 1$, $\delta = 0.05$, $g = 0.8$.

The figure shows that transversality condition (26) is valid only in steady state (x^*, ψ^*) (39). Thus, for initial condition $x(t_0)$, on needs to find $\psi(t_0)$, such that the solution of the Hamiltonian system with this initial condition converges to the steady state. Using $\Psi(x)$ (23) and ψ^* (39), we obtain the rule for searching $\psi(t_0)$, when $x(t_0)$ is fixed:

(1) if $x(t_0) < x^*$, then $\psi^* < \psi(t_0) < \Psi(x(t_0))$.
(2) if $x(t_0) = x^*$, then $\psi(t_0) = \psi^*$.
(3) if $x(t_0) > x^*$, then $\Psi(x(t_0)) < \psi(t_0) < \psi^*$.

For every initial position $(x(t_0), \psi(t_0))$ one can find the solution to the Hamiltonian system (25) by integrating it in the direct time, taking into account switching between domains $\mathcal{D}_i, i = 1, 2, 3$ (see previous section). Among these trajectories, there is the one that converges to (x^*, ψ^*), and, hence, satisfies transversality condition (26); this trajectory is optimal.

Due to the saddle character of the steady state, it is extremely difficult to find the unique trajectory that ideally (theoretically) converges to (x^*, ψ^*). Therefore, we propose to choose the trajectory reaching the neighborhood of (x^*, ψ^*) in time T and afterwards, stabilize it. Figure 2 shows trajectories, constructed for two initial values $x(t_0) = x(0)$. In each case the initial value of $\psi(0)$ is chosen, such that the trajectory reaches the neighborhood of (x^*, ψ^*) in time T.

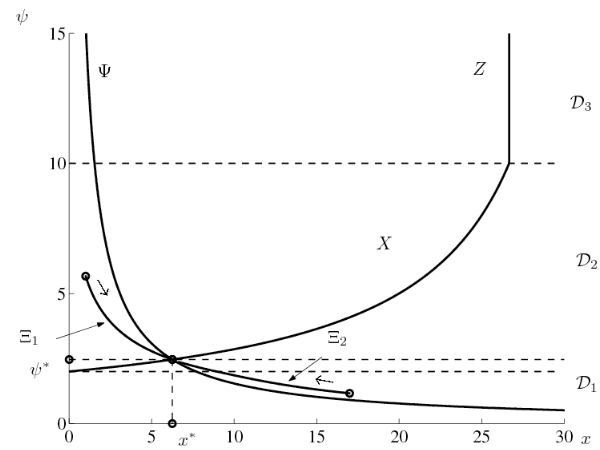

Fig. 2. Trajectories $\Xi_1 = \{(x_1, \psi_1)\colon x_1 = x_1(t), \psi_1 = \psi_1(t), t \in [0, T]\}$ for $x_1(0) = 1$, $\psi_1(0) = 5.667$, and $\Xi_2 = \{(x_2, \psi_2)\colon x_2 = x_2(t), \psi_2 = \psi_2(t), t \in [0, T]\}$ for $x_2(0) = 17$, $\psi_2(0) = 1.162$. In time $T = 80$ trajectories reach the neighborhood of the steady state in domain $\mathcal{D}_2$: $x_1(T) = 6.186$, $\psi_1(T) = 2.49$; $x_2(T) = 6.638$, $\psi_2(T) = 2.362$.

Using eigenvalue ξ_{n} (46) and u^0 (18), we construct the feedback (closed-loop) stabilizer in domain $\mathcal{D}_2$:

$$u^0_{D_2}(x) = 1 - \frac{1}{b(\psi^* + \omega(x - x^*))}, \tag{51}$$

where ω is determined by the slope of eigenvector corresponding to ξ_{n}:

$$\omega = (\xi_{\mathrm{n}} - a)\psi^{*2}. \tag{52}$$

The feedback rule (51) stabilizers the Hamiltonian system in domain $\mathcal{D}_2$ (see, e.g. Ayres et al. (2009)). Trajectories, constructed by applying stabilizer (51) in the neighborhood of steady state, are depicted in Figure 3.

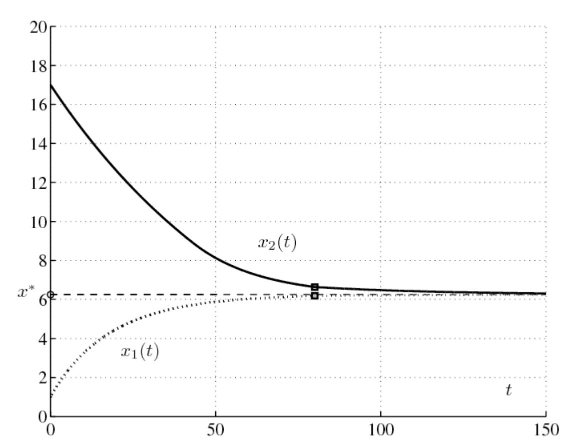

Fig. 3. Plots of $x_1(t)$ and $x_2(t)$ (see Figure 2). In time $T = 80$ trajectories reach $x_1(T)$, $x_2(T)$. Afterwards they are stabilized by feedbacks $u^0_{D_2}(x)$ (51).

Remark 1. Another method deals with constucting trajectories in the inverse time, i.e. by integrating the Hamiltonian system backwards starting from the initial condition in the neighborhood of the steady state (see, e.g. Krasovskii and Tarasyev (2008); Tarasyev and Usova (2011)).

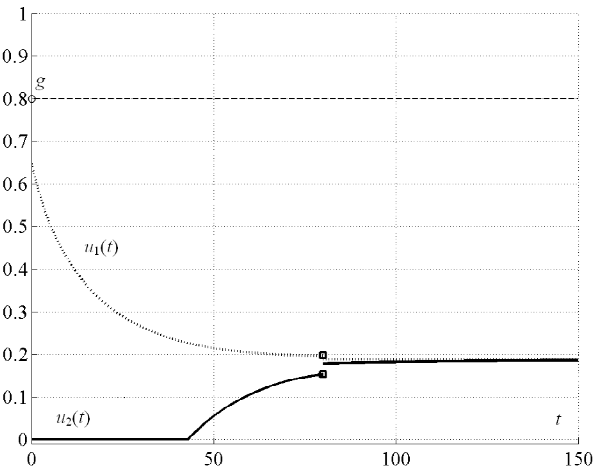

Fig. 4. Graphs of controls generating corresponding trajectories in Figure 3.

Plots of controls u_1 and u_2 corresponding to trajectories x_1 and x_2 are shown in Figure 4.

4.2 Steady state in domain $\mathcal{D}_3$

Figure 5 depicts the vector field in the case of steady state belonging to domain $\mathcal{D}_3$ (21). Let us call this case, when parameter g satisfies (36), the case of hard constraints. Here the optimal trajectory must converge to (x_S^*, ψ_S^*).

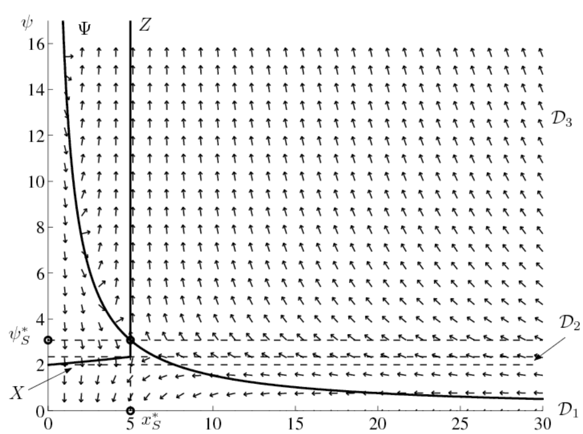

Fig. 5. Vector field in the case of hard contraints (36). Steady state $(x_S^*, \psi_S^*) = (5, 3.077)$ belongs to domain $\mathcal{D}_3$ (21). Parameter values: $a = -0.015$, $b = 0.5$, $\gamma = 1$, $\delta = 0.05$, $g = 0.15$.

Similarly, we obtain the rule for searching $\psi(t_0)$, based on $\Psi(x)$ (23) and ψ_S^* (37):

(1) if $x(t_0) < x_S^*$, then $\psi_S^* < \psi(t_0) < \Psi(x(t_0))$,
(2) if $x(t_0) = x_S^*$, then $\psi(t_0) = \psi_S^*$,
(3) if $x(t_0) > x_S^*$, then $\Psi(x(t_0)) < \psi(t_0) < \psi_S^*$.

In the case, when $x(t_0) \leq x_S^*$, the optimal trajectory always belongs to domain $\mathcal{D}_3$, meaning that the optimal control is constant: $u^0 = g$. In this case the solution can be derived, applying the Cauchy formula to (32):

$$x^0(t) = x(t_0)e^{a(t-t_0)} + \int_{t_o}^{t} bge^{a(t-\tau)}d\tau$$
$$= (x(t_0) + \frac{bg}{a})e^{a(t-t_0)} - \frac{bg}{a}. \quad (53)$$

Due to $a < 0$ and (37), we have:

$$\lim_{t\to+\infty} x^0(t) = -\frac{bg}{a} = x_S^*. \quad (54)$$

Thus, regulator $u^0_{\mathcal{D}_3}(x) = g$ stabilizes the Hamiltonian system in the neighborhood of steady state (x_S^*, ψ_S^*) (37) in domain $\mathcal{D}_3$ (21).

Figure 6 depicts optimal trajectories, converging to the neighborhood of the steady state in domain $\mathcal{D}_3$. Their plots are given in Figure 7, where trajectory $x_1(t)$ is calculated analyticaly using (53).

Let us note that both cases: steady state in domain $\mathcal{D}_3$, and steady state in domain $\mathcal{D}_2$ are modeled for exactly the same parameters, except for g. Figures 1 and 5 show that $x_S^* < x^*$ and $\psi_S^* > \psi^*$, which illustrates Corollary 2. Figure 3 and Figure 7 show that in the case of hard constraints the time of approaching the neighborhood of steady state is longer.

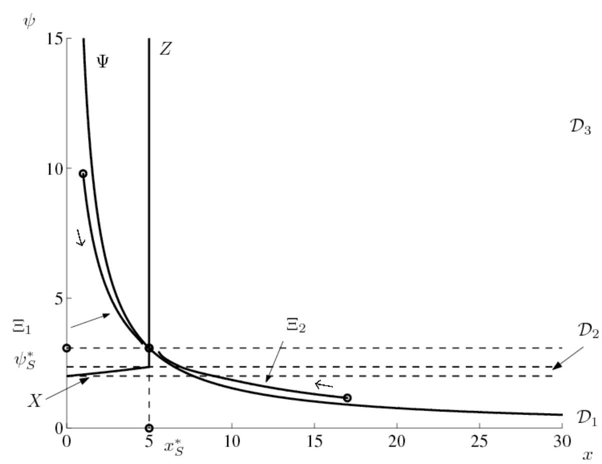

Fig. 6. Trajectories $\Xi_1 = \{(x_1, \psi_1)\colon x_1 = x_1(t), \psi_1 = \psi_1(t), t \in [0, T]\}$ for $x(0) = 1, \psi(0) = 9.788$, and $\Xi_2 = \{(x_2, \psi_2)\colon x_2 = x_2(t), \psi_2 = \psi_2(t), t \in [0, T]\}$ for $x(0) = 17$, $\psi(0) = 1.162$. In time $T = 150$ trajectories reach the neighborhood of the steady state in domain $\mathcal{D}_3$: $x_1(T) = 4.579$, $\psi_1(T) = 3.25$; $x_2(T) = 5.59$, $\psi_2(T) = 2.928$.

Graphs of controls $u_1(t)$ and $u_2(t)$ generating trajectories $x_1(t)$ and $x_2(t)$, respectively, are presented in Figure 8.

Modeling examples, presented above, are coded in the MATLAB software using the packages developed for solving dynamic optimization problems (see, e.g., Lebedev et al. (2016)).

Remark 2. The case, when the trajectory always belongs to domain $\mathcal{D}_3$ and the solution is given by constant control $u(t) = g$, draws back to the origins of growth theory. Namely, the constant exogenously given control was considered by R. Solow in his model, using the Ramsey equation (see, e.g., Acemoglu (2008)).

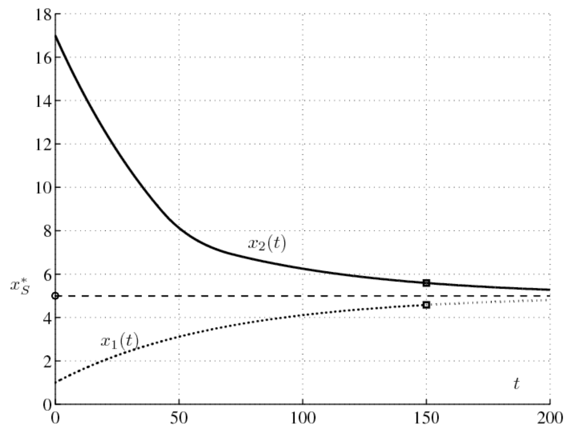

Fig. 7. Plots of $x_1(t)$ and $x_2(t)$ (see Figure 6). Starting from $T = 150$ the stabilizer $u^0_{D_3}(x) = g$ is applied in domain $\mathcal{D}_3$. Trajectory x_1 always belongs to domain $\mathcal{D}_3$.

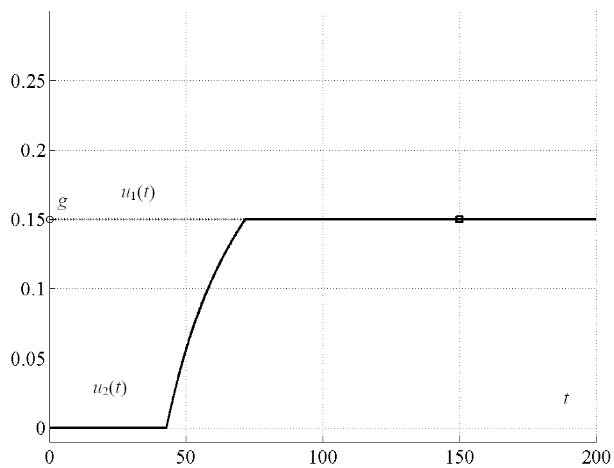

Fig. 8. Graphs of controls generating corresponding trajectories in Figure 7.

5. CONCLUSIONS

In this study we analyzed the impacts of the control constraints on solutions to the neoclassical growth model. The situation of restricted investment possibilities could happen, when investments in the economy are splited into several sectors, or due to corruption. Our analysis shows that hard constraints change the optimal solution. Namely, Theorem 1 determines two different steady states depending on the parameter values. We show that the steady state achieved by constant controls is lower compared to the standard one. This fact is reflected in Corollary 2.

Based on the vector field analysis, we proposed an algorithm for constructing optimal trajectories, combining the optimal control and stabilization approaches. Here we provided results, illustrating the construction of trajectories for any initial level of the state variable. The modeling results demonstrate the switching of controls between the admissible regimes, and the convergence of trajectories to the steady states.

The further analysis could deal with consideration of nonstationary coefficients in the dynamics of state variable. For example, coefficient a in (13) is time-dependent, when the exogenous technological change $A(t)$ is subject to the logistic growth (see, e.g. Krasovskii et al. (2010)). The approach, proposed in this paper, facilitates the analysis of such optimal control problems, as well as their application to real data.

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