Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-25 (2015) 178–182

On approximations of time-delay control systems

N. Lukoyanov * **A. Plaksin** *

* *Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, S.Kovalevskaya str. 16, Ekaterinburg, 620990, Russia;*

Institute of Mathematics and Computer Sciences, Ural Federal University, Mira str. 19, Ekaterinburg, 620002, Russia,

(*e-mail: nyul@imm.uran.ru, a.r.plaksin@gmail.com*)

Abstract: The paper deals with a dynamical system described by delay differential equations which is controlled under conditions of unknown disturbances. This system is approximated by a system of ordinary differential equations which is used as a leader. An aiming procedure between the delay and approximating systems is elaborated. The procedure is organized in such a way that the necessary proximity of motions of these systems is guaranteed by means of the delay system control and a certain part of the approximating system controls. Another part of the approximating system controls can be used to compensate disturbances in the initial delay system and to ensure the required quality of the control process. Thus, by using the elaborated aiming procedure, results of control theory for ordinary differential systems can be applied to delay differential systems.

Keywords: control schemes, delay elements, feedback methods, differential games

1. INTRODUCTION

The research of approximations of delay differential equations by ordinary differential equations have an extensive history. The convergence of such approximations for linear systems with constant delays was proved in Krasovskii (1964). In Repin (1965), this result was extended to nonlinear systems, and in Kurjanskii (1967) – to the case of variable delays. Later, similar approximations, their generalizations and applications to different kinds of problems were considered, for example, by Kryajimskii (1978), Banks and Burns (1978), Banks and Kappel (1979), Kunisch (1980), Fabiano (2013). Within the game-theoretical approach (see, e.g., Krasovskii (1985), Krasovskii and Subbotin (1988)), it was proposed in Krasovskii and Kotelnikova (2011) to use the approximating system of ordinary differential equations as a leader for the initial time-delay dynamical system controlled under conditions of disturbances or counteractions. Results concerning the validity of such control procedure for a quite general class of delay differential systems are given in the present paper.

2. TIME-DELAY CONTROL SYSTEM

Consider a dynamical system described by the following nonlinear delay differential equation

$$\dot{x}[t] = f\big(t, x_t[\cdot], u[t], v[t]\big), \tag{1}$$

$$t \in [t_0, \vartheta], \quad x[t] \in \mathbb{R}^n, \quad u[t] \in U, \quad v[t] \in V,$$

with the initial condition

$$x_{t_0}[\zeta] = x[t_0 + \zeta] = z[\zeta], \quad \zeta \in [-h, 0]. \tag{2}$$

Here t is the time variable; $x[t]$ is the value of the state vector at the time t; $\dot{x}[t] = \mathrm{d}x[t]/\mathrm{d}t$ is the rate of its variation at this time; $h = \mathrm{const} > 0$; $x_t[\cdot]$ is the motion history on the interval $[t-h, t]$, defined by $x_t[\zeta] = x[t + \zeta]$, $\zeta \in [-h, 0]$; $u[t]$ and $v[t]$ are respectively the current control and disturbance actions. It is assumed that times t_0, ϑ are given, and U, V are known compact sets of finite-dimensional spaces.

Admissible control and disturbance realizations are Borel measurable functions

$$u[t_0[\cdot]\vartheta) = \big\{u[t] \in U, t_0 \le t < \vartheta\big\}$$

and

$$v[t_0[\cdot]\vartheta) = \big\{v[t] \in V, t_0 \le t < \vartheta\big\}.$$

Below, angle brackets $\langle \cdot, \cdot \rangle$ are used for the scalar product of vectors, and double brackets $\|\cdot\|$ denote the Euclidean norm. The space of all continuous functions from $[-h, 0]$ to $\mathbb{R}^n$ is denoted by $C = C\big([-h, 0], \mathbb{R}^n\big)$ and is endowed with the supremum norm $\|\cdot\|_C$.

The following conditions are assumed:

(C1) The mapping $f\colon [t_0, \vartheta] \times C \times U \times V \mapsto \mathbb{R}^n$ is continuous.

(C2) There exists a constant $\alpha > 0$ such that

$$\big\|f\big(t, w[\cdot], u, v\big)\big\| \le \alpha\big(1 + \big\|w[\cdot]\big\|_C\big)$$

for all $t \in [t_0, \vartheta]$, $w[\cdot] \in C$, $u \in U$, and $v \in V$.

⋆ This work is supported by the Russian Science Foundation, project no. 15-11-10018.

2405-8963
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.11.080

(C3) For any compact set $D \subset C$ there exists a number $\lambda(D) > 0$ such that

$$\big\|f\big(t, w_1[\cdot], u, v\big) - f\big(t, w_2[\cdot], u, v\big)\big\| \leq \lambda(D)\,\big\|w_1[\cdot] - w_2[\cdot]\big\|_C$$

for all $t \in [t_0, \vartheta]$, $u \in U$, $v \in V$, and $w_1[\cdot], w_2[\cdot] \in D$.

(C4) The initial function $z[\cdot]$ is absolutely continuous and satisfies inequalities

$$\big\|z[\zeta]\big\| \leq R_0,\ \zeta \in [-h, 0],$$

$$\big\|\dot{z}[\zeta]\big\| \leq R_0 \text{ for almost all } \zeta \in [-h, 0], \quad R_0 = \text{const} > 0.$$

(C5) For all $t \in [t_0, \vartheta]$, $w[\cdot] \in C$, and $s \in \mathbb{R}^n$ the following equality holds

$$\min_{u\in U}\max_{v\in V}\big\langle f\big(t, w[\cdot], u, v\big), s\big\rangle = \max_{v\in V}\min_{u\in U}\big\langle f\big(t, w[\cdot], u, v\big), s\big\rangle.$$

In differential games theory condition (C4) is called "Isaacs condition" (see, e.g., Isaacs (1965)) or "the saddle point condition in a small game" (see, e.g., Krasovskii and Subbotin (1988)). This assumption does not play a crucial role and can be omitted (see below, Remark 4).

If conditions (C1)–(C4) hold, then, for any admissible realizations $u[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$, and any initial function $z[\cdot]$ the problem (1), (2) has a unique solution that is the absolutely continuous function

$$x[t_0 - h[\cdot]\vartheta] = \{x[t] \in \mathbb{R}^n,\ t_0 - h \leq t \leq \vartheta\}$$

which satisfies (2) and almost everywhere satisfies (1) together with $u[t]$ and $v[t]$. Moreover, there exists a number $R_x > 0$, such that for any solution $x[t_0 - h[\cdot]\vartheta]$ the following inequalities hold

$$\big\|x[t]\big\| \leq R_x,\ t \in [t_0 - h, \vartheta], \tag{3}$$

$$\big\|\dot{x}[t]\big\| \leq R_x \text{ for almost all } t \in [t_0 - h, \vartheta].$$

3. APPROXIMATING ORDINARY DIFFERENTIAL SYSTEM

Let $m \in \mathbb{N}$, $\Delta h = h/m$. Let $x[t_0 - h[\cdot]\vartheta]$ be a solution of the problem (1), (2). We define functions

$$x^{[i]}[t] = x[t - i\Delta h], \quad t \in [t_0, \vartheta], \quad i = \overline{0, m},$$

$$X[t] = \big(x^{[0]}[t], x^{[1]}[t], \ldots, x^{[m]}[t]\big).$$

By $S\big(X[t]\big)[\cdot]$ we denote the linear spline on the interval $[-h, 0]$ with notes $-i\Delta h$, $i = \overline{0, m}$, such that

$$S\big(X[t]\big)[-i\Delta h] = x^{[i]}[t], \quad i = \overline{0, m}. \tag{4}$$

In accordance with system (1), by using the divided differences formula and the approximation $S\big(X[t]\big)[\cdot] \approx x_t[\cdot]$ we can assume

$$\dot{x}^{[0]}[t] \approx f\big(t, S\big(X[t]\big)[\cdot], u[t], v[t]\big),$$

$$\dot{x}^{[i]}[t] \approx \big(x^{[i-1]}[t] - x^{[i]}[t]\big)/\Delta h,\ i = \overline{1, m},$$

Basing on this idea of approximation, we construct the following system of ordinary differential equation

$$\begin{cases} \dot{y}^{[0]}[t] = f\big(t, S\big(Y[t]\big)[\cdot], \tilde{u}[t], \tilde{v}[t]\big), \\ \dot{y}^{[i]}[t] = \big(y^{[i-1]}[t] - y^{[i]}[t]\big)/\Delta h,\ i = \overline{1, m}, \end{cases} \tag{5}$$

$$t \in [t_0, \vartheta], \quad y^{[i]}[t] \in \mathbb{R}^n,\ i = \overline{0, m}, \quad \tilde{u}[t] \in U, \quad \tilde{v}[t] \in V,$$

$$Y[t] = \big(y^{[0]}[t], y^{[1]}[t], \ldots, y^{[m]}[t]\big),$$

with the initial condition

$$y^{[i]}[t_0] = z[-i\Delta h], \quad i = \overline{0, m}. \tag{6}$$

If conditions (C1)–(C4) hold, then, for any admissible realizations $\tilde{u}[t_0[\cdot]\vartheta)$ and $\tilde{v}[t_0[\cdot]\vartheta)$, the problem (5), (6) has a unique absolutely continuous on $[t_0, \vartheta]$ solution

$$Y[t_0[\cdot]\vartheta] = \{Y[t],\ t_0 \leq t \leq \vartheta\}.$$

Lemma 1. Let conditions (C1)–(C4) be valid. Then, there exists a number $R_y > 0$ such that, for any integer $m > 0$, any initial function $z[\cdot]$, and any admissible realizations $\tilde{u}[t_0[\cdot]\vartheta)$ and $\tilde{v}[t_0[\cdot]\vartheta)$ the solution $Y[t_0[\cdot]\vartheta]$ of the problem (5), (6) satisfies inequalities

$$\big\|y^{[i]}[t]\big\| \leq R_y,\ t \in [t_0, \vartheta], \tag{7}$$

$$\big\|\dot{y}^{[i]}[t]\big\| \leq R_y \text{ for almost all } t \in [t_0, \vartheta], \quad i = \overline{0, m}.$$

Proof. From (5), for any $t \in [t_0, \vartheta]$ and $i = \overline{1, m}$, we have

$$y^{[i]}[t] = y^{[i]}[t_0]e^{-(t-t_0)/\Delta h} + \int_{t_0}^{t} y^{[i-1]}[\xi]e^{-(t-\xi)/\Delta h}/\Delta h\, \mathrm{d}\xi.$$

Hence, using (C4) and (6) we deduce

$$\big\|y^{[i]}[t]\big\| \leq \big\|y^{[i]}[t_0]\big\|e^{-\frac{t-t_0}{\Delta h}} + \max_{\tau\in[t_0,t]}\big\|y^{[i-1]}[\tau]\big\|\int_{t_0}^{t}\frac{e^{-\frac{t-\xi}{\Delta h}}}{\Delta h}\,\mathrm{d}\xi$$

$$\leq \max\Big\{R_0, \max_{\xi\in[t_0,t]}\big\|y^{[i-1]}[\xi]\big\|\Big\}\bigg(e^{-\frac{t-t_0}{\Delta h}} + \int_{t_0}^{t}\frac{e^{-\frac{t-\xi}{\Delta h}}}{\Delta h}\,\mathrm{d}\tau\bigg)$$

$$= \max\Big\{R_0, \max_{\xi\in[t_0,t]}\big\|y^{[i-1]}[\xi]\big\|\Big\}. \tag{8}$$

By definition (4) of the spline $S(Y[t])$ the following equality holds

$$\big\|S\big(Y[t]\big)[\cdot]\big\|_C = \max_{i=\overline{0,m}}\big\|y^{[i]}[t]\big\|, \quad t \in [t_0, \vartheta]. \tag{9}$$

From (C2), (5), (8) and (9), for almost all $t \in [t_0, \vartheta]$, we obtain

$$\big\|\dot{y}^{[0]}[t]\big\| = \big\|f\big(t, S\big(Y[t]\big)[\cdot], \tilde{u}[t], \tilde{v}[t]\big)\big\| \tag{10}$$

$$\leq \alpha\Big(1 + \max\Big\{R_0, \max_{\xi\in[t_0,t]}\big\|y^{[0]}[\xi]\big\|\Big\}\Big).$$

Then, using (C4) and (6) we conclude

$$\big\|y^{[0]}[t]\big\| \leq \big\|y^{[0]}[t_0]\big\| + \int_{t_0}^{t}\big\|\dot{y}^{[0]}[\tau]\big\|\,d\tau \tag{11}$$

$$\leq R_0 + \int\limits_{t_0}^{t} \alpha\Big(1 + \max\Big\{R_0, \max_{\xi\in[t_0,\tau]} \big\|y^{[0]}[\xi]\big\|\Big\}\Big)\, d\tau.$$

From this estimation, according to Gronwall's lemma (see., e. g. Bellman and Cooke (1963)) there exists a number $R_1 > 0$ such that, the following inequality holds

$$\big\|y^{[0]}[t]\big\| \leq R_1, \quad t \in [t_0, \vartheta]. \tag{12}$$

Let us set $R_y = (\alpha+1)(1+\max\{R_0, R_1\})$. Then, from (8), (10) and (12) we obtain

$$\big\|y^{[i]}[t]\big\| \leq R_y,\ t \in [t_0, \vartheta],\ i = \overline{0, m}, \tag{13}$$

$$\big\|\dot{y}^{[0]}[t]\big\| \leq R_y \text{ for almost all } t \in [t_0, \vartheta].$$

Now, we show that $\big\|\dot{y}^{[i]}[t]\big\| \leq R_y$, $i = \overline{1, m}$. Let us denote

$$q^{[0]}[t] = f\big(t, S\big(Y[t]\big)[\cdot], \tilde{u}[t], \tilde{v}[t]\big), \tag{14}$$

$$q^{[i]}[t] = \big(y^{[i]}[t] - y^{[i-1]}[t]\big)/\Delta h,\ i = \overline{1, m}, \quad t \in [t_0, \vartheta].$$

Taking into account (5) we have

$$\dot{y}^{[i]}[t] = q^{[i]}[t],\ \text{for almost all } t \in [t_0, \vartheta], \quad i = \overline{0, m}. \tag{15}$$

Therefore, the following relations hold

$$\dot{q}^{[i]}[t] = \big(q^{[i-1]}[t] - q^{[i]}[t]\big)/\Delta h \tag{16}$$

$$\text{for almost all } t \in [t_0, \vartheta], \quad i = \overline{1, m}.$$

For system (16), similarly to (8), we have

$$\big\|q^{[i]}[t]\big\| \leq \max\Big\{R_0, \max_{\zeta\in[t_0,t]} \big\|q^{[0]}[\zeta]\big\|\Big\}, \tag{17}$$

$$t \in [t_0, \vartheta], \quad i = \overline{1, m}.$$

Using the second estimation in (13) and relations (15), (17) we obtain

$$\big\|\dot{y}^{[i]}[t]\big\| \leq R_y \text{ for almost all } t \in [t_0, \vartheta], \quad i = \overline{1, m}.$$

Lemma is proved.

Let us define the function $y^{[0]}[t]$ on the interval $[t_0 - h, t_0]$ as follows:

$$y^{[0]}[t_0 + \zeta] = y^{[0]}_{t_0}[\zeta] = z[\zeta], \quad \zeta \in [-h, 0]. \tag{18}$$

Theorem 2. There exists a number $K > 0$ such that, for any integer $m > 0$, any initial function $z[\cdot]$, and any admissible realizations $\tilde{u}[t_0[\cdot]\vartheta)$ and $\tilde{v}[t_0[\cdot]\vartheta)$, under conditions (C1)–(C4) the solution $Y[t_0[\cdot]\vartheta]$ of the problem (5), (6) satisfies the inequality

$$\big\|y^{[0]}_t[\cdot] - S\big(Y[t]\big)[\cdot]\big\|_C \leq K m^{-1/2},\ t \in [t_0, \vartheta]. \tag{19}$$

Proof. By definition (4) of the spline $S(Y[t])[\cdot]$ taking into account equations (5) and Lemma 1 we have

$$\big\|S(Y[t])[\zeta'] - S(Y[t])[\zeta'']\big\| \leq \max_{i=\overline{1,m}} \big\|\dot{y}^{[i]}[t]\big\| |\zeta' - \zeta''| \tag{20}$$

$$\leq R_y|\zeta' - \zeta''|, \quad \zeta', \zeta'' \in [-h, 0], \quad t \in [t_0, \vartheta].$$

From Lemma 1 and results in Repin (1965) we obtain

$$\big\|y^{[0]}[t - i\Delta h] - y^{[i]}[t]\big\| \leq 4R_y h m^{-1/2}, \quad i = \overline{1, m}.$$

Therefore, we conclude

$$\big\|y^{[0]}_t[\zeta] - S\big(Y[t]\big)[\zeta]\big\| \leq 2R_y\Delta h + 4R_y h m^{-1/2}, \tag{21}$$

$$\zeta \in [-i\Delta h, -(i-1)\Delta h], \quad i = \overline{1, m}.$$

Let $K_1 = 6R_y h$. Then, the required estimation (19) follows from (21). Theorem is proved.

4. AIMING PROCEDURE

Let us describe an aiming procedure between systems (1) and (5). This procedure is based on a partition of the control interval $[t_0, \vartheta]$:

$$\Delta_\delta = \big\{t_j\colon 0 < t_{j+1} - t_j < \delta,\ j = \overline{0, k-1},\ t_k = \vartheta\big\}. \tag{22}$$

Realizations $u[t_0[\cdot]\vartheta)$ and $\tilde{v}[t_0[\cdot]\vartheta)$ are formed in this procedure according to the following feedback rule:

$$u[t] = u^\circ_j, \quad \tilde{v}[t] = \tilde{v}^\circ_j, \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, k-1}, \tag{23}$$

where

$$u^\circ_j \in \arg\min_{u\in U}\max_{v\in V} \big\langle f\big(t_j, x_{t_j}[\cdot], u, v\big),\ x[t_j] - y^{[0]}[t_j]\big\rangle, \tag{24}$$

$$\tilde{v}^\circ_j \in \arg\max_{v\in V}\min_{u\in U} \big\langle f\big(t_j, S\big(Y[t_j]\big)[\cdot], u, v\big),\ x[t_j] - y^{[0]}[t_j]\big\rangle.$$

Theorem 3. Let conditions (C1)–(C5) be valid. Then, for any $\varepsilon > 0$, there exist numbers $M > 0$ and $\delta > 0$ such that, for any integer $m > M$, any initial function $z[\cdot]$, and any admissible realizations $\tilde{u}[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$, if realizations $u[t_0[\cdot]\vartheta)$ and $\tilde{v}[t_0[\cdot]\vartheta)$ are formed according to the aiming procedure (22)–(24), then, for solutions $x[t_0 - h[\cdot]\vartheta]$ of the problem (1), (2) and $Y[t_0[\cdot]\vartheta]$ of the problem (5), (6), the following inequality holds

$$\big\|x[t] - y^{[0]}[t]\big\| \leq \varepsilon, \quad t \in [t_0, \vartheta].$$

Proof. We denote

$$s[t] = x[t] - y^{[0]}[t], \quad t \in [t_0 - h, \vartheta], \tag{25}$$

and $R_s = R_x + R_y$. Then, from (3) and Lemma 1 we have

$$\big\|s[t]\big\| \leq R_s,\ t \in [t_0 - h, \vartheta], \tag{26}$$

$$\big\|\dot{s}[t]\big\| \leq R_s \text{ for almost all } t \in [t_0, \vartheta].$$

Let us define a compact set $D_* \subset C$ consisting of absolutely continuous functions $\tilde{x}\colon [-h, 0] \mapsto \mathbb{R}^n$, which satisfy inequalities

$$\big\|\tilde{x}[t]\big\| \leq R_s,\ t \in [-h, 0], \tag{27}$$

$$\big\|\dot{\tilde{x}}[t]\big\| \leq R_s \text{ for almost all } t \in [-h, 0].$$

Then, taking into account relations (3), (9), (20) and Lemma 1 we obtain

$$x_t[\cdot] \in D_*, \quad S\big(Y[t]\big)[\cdot] \in D_*. \tag{28}$$

In accordance with (C3) let us define a number $l_* = l(D_*)$ and a function

$$V[t] = \max_{\zeta\in[t_0,t]} \|s[\zeta]\|^2 e^{-2l_*(t-t_0)}, \quad t \in [t_0,\vartheta]. \qquad (29)$$

The function $V[t]$ is absolutely continuous, and almost everywhere on $[t_0,\vartheta]$ we have

$$\dot V[t] = 2\max\{0,\ \langle \dot s[t],\ s[t]\rangle\} e^{-2l_*(t-t_0)} \qquad (30)$$

$$-2l_* \max_{\zeta\in[t_0,t]} \|s[\zeta]\|^2 e^{-2l_*(t-t_0)}.$$

Let us first obtain the estimation for $\langle \dot s[t],\ s[t]\rangle$. From (1), (5) and (29), for almost all $t \in [t_0,\vartheta]$, we have

$$\langle \dot s[t],\ s[t]\rangle = \langle f(t, x_t[\cdot], u[t], v[t]),\ s[t]\rangle \qquad (31)$$

$$-\langle f(t, S(Y[t])[\cdot], \tilde u[t], \tilde v[t]),\ s[t]\rangle.$$

Let

$$\sigma = \frac{e^{-\lambda(\vartheta-t_0)}}{10(\vartheta-t_0)}. \qquad (32)$$

By condition (C1) the mapping f is continuous. Hence, since the set D_* is compact set in C, there exists a number $\nu = \nu(\varepsilon) > 0$ such that, for any $t', t'' \in [t_0,\vartheta]$ and any $w'[\cdot], w''[\cdot] \in D_*$, satisfying inequalities $|t'-t''| \le \nu$ and $\|w'[\cdot] - w''[\cdot]\|_C \le \nu$ the following inequality holds

$$\|f(t', w'[\cdot], u, v) - f(t'', w''[\cdot], u, v)\| \le \sigma\varepsilon^2/R_s, \qquad (33)$$

for any $u \in U$ and $v \in V$. Let $\delta_1 = \min\{\nu/R_x,\ \nu/R_y\}$. Then, from (3) and Lemma 1, for any $t', t'' \in [t_0,\vartheta]$, $|t'-t''| \le \delta_1$, we have

$$\|x_{t'}[\cdot] - x_{t''}[\cdot]\|_C = \max_{\xi\in[-h,0]} \|x[\xi+t'] - x[\xi+t'']\| \qquad (34)$$

$$\le R_x|t'-t''| \le \nu,$$

$$\|S(Y[t'])[\cdot] - S(Y[t''])[\cdot]\|_C = \max_{i=\overline{0,m}} \|y^{[i]}[t'] - y^{[i]}[t'']\|$$

$$\le R_y|t'-t''| \le \nu.$$

Let $R_f > 0$ be such that

$$\|f(t, w[\cdot], u, v)\| \le R_f, \qquad (35)$$

$$t \in [t_0,\vartheta],\ w[\cdot] \in D_*,\ u \in U,\ v \in V.$$

Let $\delta_2 = \sigma\varepsilon/(R_f R_s)$. Then, from (26), for any $t', t'' \in [t_0,\vartheta]$, $|t'-t''| \le \delta_2$, we deduce

$$\|s[t'] - s[t'']\| \le R_s|t'-t''| \le \sigma\varepsilon^2/R_f. \qquad (36)$$

Put $\delta = \min\{\nu, \delta_1, \delta_2\}$. Then, by adding and subtracting $\langle f(t_j, x_{t_j}[\cdot], u[t], v[t]) - f(t_j, S(Y[t_j])[\cdot], \tilde u[t], \tilde v[t]), r[t] - r[t_j]\rangle$ in the right side of (31), and taking into account (22), (26), and (33)–(36), we obtain

$$\langle \dot s[t],\ s[t]\rangle \le \langle f(t_j, x_{t_j}[\cdot], u[t], v[t]),\ s[t_j]\rangle \qquad (37)$$

$$-\langle f(t_j, S(Y[t_j])[\cdot], \tilde u[t], \tilde v[t]),\ s[t_j]\rangle + 4\sigma\varepsilon^2.$$

Further, in accordance with (23) and (C5), for $t \in [t_j, t_{j+1})$, we have

$$\langle f(t_j, x_{t_j}[\cdot], u[t], v[t]),\ s[t_j]\rangle \qquad (38)$$

$$-\langle f(t_j, S(Y[t_j])[\cdot], \tilde u[t], \tilde v[t]),\ s[t_j]\rangle$$

$$\le \max_{v\in V} \langle f(t_j, x_{t_j}[\cdot], u_j^\circ, v), s[t_j]\rangle$$

$$-\min_{u\in U} \langle f(t_j, S(Y[t_j])[\cdot], u, \tilde v_j^\circ), s[t_j]\rangle$$

$$= \min_{u\in U}\max_{v\in V} \langle f(t_j, x_{t_j}[\cdot], u, v), s[t_j]\rangle$$

$$-\max_{v\in V}\min_{u\in U} \langle f(t_j, S(Y[t_j])[\cdot], u, v), s[t_j]\rangle$$

$$= \min_{u\in U}\max_{v\in V} \langle f(t_j, x_{t_j}[\cdot], u, v), s[t_j]\rangle$$

$$-\min_{u\in U}\max_{v\in V} \langle f(t_j, S(Y[t_j])[\cdot], u, v), s[t_j]\rangle.$$

From (C3) and (25) we deduce

$$\min_{u\in U}\max_{v\in V} \langle f(t, x_t[\cdot], u, v), s[t_j]\rangle \qquad (39)$$

$$-\min_{u\in U}\max_{v\in V} \langle f(t, S(Y[t])[\cdot], u, v), s[t_j]\rangle$$

$$\le l_*\|s[t_j]\| \|x_t[\cdot] - S(Y[t])[\cdot]\|_C \le l_*\|s[t_j]\| \|s_t[\cdot]\|_C$$

$$+ l_* R_s \|y_t^{[0]}[\cdot] - S(Y[t])[\cdot]\|_C.$$

By Theorem 1 there exists a number $M > 0$ such that, for any integer $m > M$, the following inequality holds

$$\|y^{[0]}[\zeta - h] - y^{[m]}[\zeta]\| \le \sigma\varepsilon^2/(l_* R_s). \qquad (40)$$

Combining relations (30), (37)–(40), we obtain

$$\dot V[t] \le 10\sigma\varepsilon e^{-\lambda(t-t_0)} \le 10\sigma\varepsilon. \qquad (41)$$

From (29), (32) and (41) we conclude

$$\max_{\zeta\in[t_0,\vartheta]} \|s[\zeta]\|^2 \qquad (42)$$

$$= V[\vartheta] e^{\lambda(\vartheta-t_0)} \le 10\sigma\varepsilon^2(\vartheta-t_0) e^{\lambda(\vartheta-t_0)} = \varepsilon^2.$$

Theorem is proved.

Thus, by using the aiming procedure (22)–(24), we can apply results of the control theory of ordinary differential systems to delay differential systems.

Remark 4. Theorem 3 is valid without condition (C5) if we change the aiming procedure (22)–(24) as follows:

$$u[t] = u_j^\circ, \quad \tilde v[t] = \tilde v_j^\circ(\tilde u[t]), \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, k-1},$$

where

$$u_j^\circ \in \arg\min_{u\in U}\max_{v\in V} \langle f(t_j, x_{t_j}[\cdot], u, v),\ x[t_j] - y^{[0]}[t_j]\rangle,$$

$$\tilde v_j^\circ(u) \in \arg\max_{v\in V} \langle f(t_j, S(Y[t_j])[\cdot], u, v),\ x[t_j] - y^{[0]}[t_j]\rangle.$$

5. EXAMPLE

Consider a dynamical system described by the delay differential equation

$$\dot{x}[t] = 0.5x[t] - 7\int_{t-1}^{t} x[\tau]e^{-0.3|x[\tau]|}\,d\tau + u[t] + v[t]x[t-1],$$

$$t \in [0,9], \quad x[t] \in \mathbb{R}, \quad \big|u[t]\big| \le 1, \quad \big|v[t]\big| \le 1,$$

with the initial condition

$$x[\zeta] = \zeta\cos(12\zeta), \quad \zeta \in [-1,0].$$

For this system, according to (5), (6) we construct the approximating ordinary differential system, perform the aiming procedure (22)–(24) and simulate the situation with the worst-case realizations $\tilde{u}[t_0[\cdot]\vartheta)$ and $v[t_0[\cdot]\vartheta)$. The results of simulation are shown in Figures 1–3.

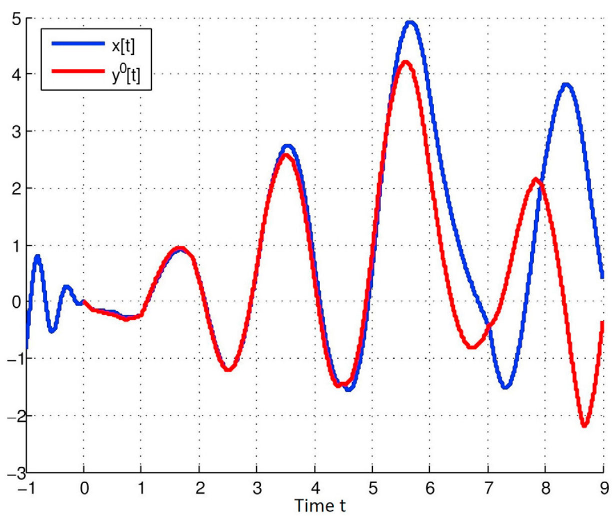

Fig. 1. Simulation results for $m = 100$, $\delta = 0,004$.

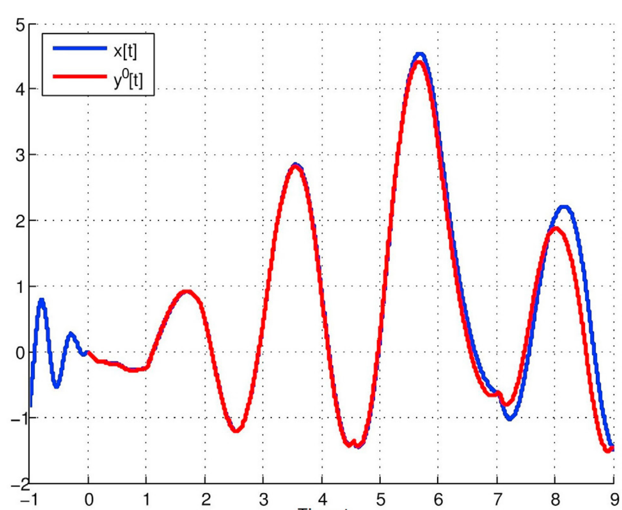

Fig. 2. Simulation results for $m = 500$, $\delta = 0,004$.

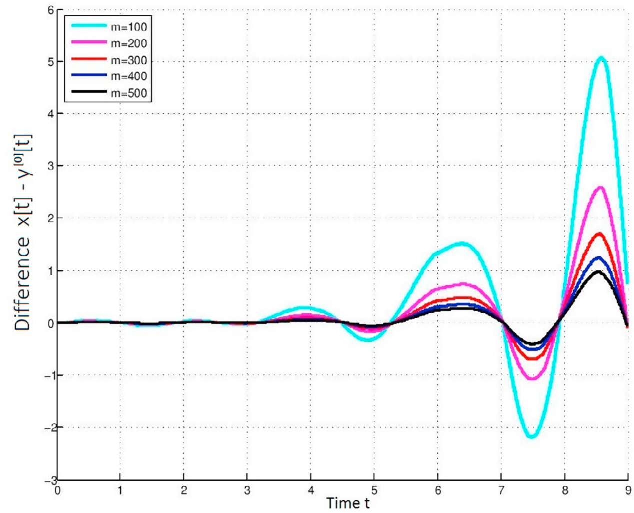

Fig. 3. Simulation results for different values of m, when $\delta = 0,004$.

REFERENCES

N.N. Krasovskii (1964). On an approximation of a problem of analytical constructing regulators in a delay system. *Prikl. Mat. Mekh.*, 28(4), 716–724.

U.M. Repin (1965). On an approximate replacement of delay systems by ordinary dynamic systems. *Prikl. Mat. Mekh.*, 29(2), 226–235.

A.B. Kurjanskii (1967). On an approximation of linear delay differential equations. *Differetial'niye Uravneniya*, 3, 2097–2107.

A.V. Kryajimskii and V.I. Maximov (1978). Approximation of linear differential-difference games. *Prikl. Mat. Mekh.*, 42(2), 202–209.

H.T. Banks and J.A. Burns (1978). Hereditary control problems: numerical methods based on averaging approximations. *SIAM J. Control Optim.*, 16, 169–208.

H.T. Banks and F. Kappel (1979). Spline approximations for functional differential equations. *J. Different. Equat.*, 34(3), 496–522.

K. Kunisch (1980). Approximation schemes for nonlinear neutral optimal control systems. *J. Math. Anal. Appl.*, 82, 112–143.

R.H. Fabiano (2013). A semidiscrete approximation scheme for neutral delay-differential equetions. *International Journal of Numerical Analysis and Modeling*, 10(3), 712–726.

R. Isaacs (1965). *Differential Games.* John Wiley, New York.

N.N. Krasovskii (1985). *Control of a Dynamic System.* Nauka, Moscow. (In Russian).

N.N. Krasovskii and A.I. Subbotin (1988). *Game-Theoretical Control Problems.* Springer-Verlag, New York.

N.U. Lukoyanov and T.N. Reshetova (1998). Problems of conflict control of high dimensionality functional systems. *J. Appl. Maths Mechs*, 62(4), 545–554.

N.N. Krasovskii and A.N. Kotelnikova (2011). Stochastic guide for a time-delay object in a positional differential game. *Proc. Steklov Inst. Math.*, 227(1), 145–151.

R. Bellman and K.L. Cooke (1963). *Differential-Difference Equations.* Academic Press, New York.