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Selection Principles and Games in Bitopological Function Spaces

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Abstract. For a Tychonoff space X, we denote by $(C(X), \tau_k, \tau_p)$ the bitopological space of all real-valued continuous functions on X, where τ_k is the compact-open topology and τ_p is the topology of pointwise convergence. In the papers [6, 7, 13] variations of selective separability and tightness in $(C(X), \tau_k, \tau_p)$ were investigated. In this paper we continue to study the selective properties and the corresponding topological games in the space $(C(X), \tau_k, \tau_p)$.

1. Introduction

In the papers [1, 3, 4, 9–12, 14] the authors investigated the selectors of dense subsets of the space $C(X)$ of all real-valued continuous functions on a Tychonoff space X with the topology τ_p of pointwise convergence and with the compact-open topology τ_k. For a Tychonoff space X, we denote by $(C(X), \tau_k, \tau_p)$ the bitopological space. In the articles [6, 7, 13] variations of selective separability and tightness in $(C(X), \tau_k, \tau_p)$ were investigated. In this paper, we continue to study the selective properties and the corresponding topological games in the space $(C(X), \tau_k, \tau_p)$. The following selection properties for $(C(X), \tau_k, \tau_p)$ are considered.

$$S_1(\mathcal{D}^k, \mathcal{S}^p) = S_{fin}(\mathcal{D}^k, \mathcal{S}^p) \Rightarrow S_1(\mathcal{D}^k, \mathcal{D}^p) \Rightarrow S_{fin}(\mathcal{D}^k, \mathcal{D}^p)$$

For example, a space $(C(X), \tau_k, \tau_p)$ satisfies $S_1(\mathcal{D}^k, \mathcal{S}^p)$ (resp., $S_{fin}(\mathcal{D}^k, \mathcal{S}^p)$) if whenever $(D_n : n \in \mathbb{N})$ is a sequence of dense subsets of $C_k(X)$, one can take points $f_n \in D_n$ (resp., finite $F_n \subset D_n$) such that $\{f_n : n \in \mathbb{N}\}$ (resp., $\bigcup\{F_n : n \in \mathbb{N}\}$) is sequentially dense in $C_p(X)$. There is a topological game, denoted by $G_*(\mathcal{A}, \mathcal{B})$, corresponding to $S_*(\mathcal{A}, \mathcal{B})$.

In this paper, we give characterizations for the bitopological space $(C(X), \tau_k, \tau_p)$ to satisfy the selection properties and the corresponding games.

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2. Main Definitions and Notation

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then many topological properties are characterized in terms of the following classical selection principles:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [15].

$\binom{\mathcal{A}}{\mathcal{B}}$: For each $\mathcal{U} \in \mathcal{A}$ there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$. In this paper we accept that $|\mathcal{V}| = \aleph_0$.

Then $S_{fin}(\mathcal{A},\mathcal{B})$ implies $\binom{\mathcal{A}}{\mathcal{B}}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is called:

• an ω-cover (a k-cover) if each finite (compact) subset C of X is contained in an element of $\mathcal{U}$.

• a γ-cover (a γ_k-cover) if $\mathcal{U}$ is infinite and for each finite (compact) subset C of X the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

For a topological space X we denote:

• $\mathcal{O}$ — the family of all open covers of X;
• Γ — the family of all open γ-covers of X;
• Γ_k — the family of all open γ_k-covers of X;
• Ω — the family of all open ω-covers of X;
• $\mathcal{K}$ — the family of all open k-covers of X;
• $\mathcal{D}^k$ — the family of all dense subsets of $C_k(X)$;
• $\mathcal{D}^p$ — the family of all dense subsets of $C_p(X)$;
• $\mathcal{S}^k$ — the family of all sequentially dense subsets of $C_k(X)$;
• $\mathcal{S}^p$ — the family of all sequentially dense subsets of $C_p(X)$;
• $\mathbb{K}(X)$ — the family of all non-empty compact subsets of X;
• $\mathbb{F}(X)$ — the family of all non-empty finite subsets of X.

A space X is said to be a γ_k-set if each k-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover of X [5].

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set. Clearly, every sequentially separable space is separable.

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collections:

• $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
• $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

We write $\Pi(\mathcal{A}_x,\mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x,\mathcal{B}_x)$.

So we have three types of topological properties of $(C(X),\tau_k,\tau_p)$ described through the selection principles of X where the index k means the compact-open topology and the index p - the topology of pointwise convergence:

• local properties of the form $S_*(\Phi^k_x,\Psi^p_x)$;
• global properties of the form $S_*(\Phi^k,\Psi^p)$;
• semi-local properties of the form $S_*(\Phi^k,\Psi^p_x)$.

There is a game, denoted by $G_{fin}(\mathcal{A},\mathcal{B})$, corresponding to $S_{fin}(\mathcal{A},\mathcal{B})$; two players, ONE and TWO, play a round for each natural number n. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$ and TWO responds with a finite subset B_n of A_n. A play $A_1, B_1; ...; A_n, B_n; ...$ is won by TWO if $\bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

A strategy of a player is a function σ from the set of all finite sequences of moves of the opponent into the set of (legal) moves of the strategy owner.

If ONE does not have a winning strategy in the game $G_*(\mathcal{A},\mathcal{B})$, then the selection hypothesis $S_*(\mathcal{A},\mathcal{B})$ is true; it is easy to prove. The converse implication is not always true.

Similarly, one defines the game $G_1(\mathcal{A},\mathcal{B})$, associated with $S_1(\mathcal{A},\mathcal{B})$.

So we have three types of topological games on $(C(X),\tau_k,\tau_p)$ described through the selection principles (or topological games) of X:

- local games of the form $G_*(\Phi^k_x,\Psi^p_x)$;
- global games of the form $G_*(\Phi^k,\Psi^p)$;
- semi-local games of the form $G_*(\Phi^k,\Psi^p_x)$.

The symbol $\mathbf{0}$ denotes the constantly zero function in $C(X)$. Since the compact-open topology coincides with the topology of uniform convergence on compact subsets of X, we can represent a basic neighborhood of the point $f\in C_k(X)$ as $\langle f,A,\epsilon\rangle$ where $\langle f,A,\epsilon\rangle := \{g\in C(X) : |f(x)-g(x)|<\epsilon\ \forall x\in A\}$, A is a compact subset of X and $\epsilon>0$.

3. $S_1(\mathcal{D}^k,\mathcal{D}^p)$ and $G_1(\mathcal{D}^k,\mathcal{D}^p)$

Theorem 3.1. ([13, Theorem 3.7] for $\lambda=k$ and $\mu=p$) *For a space X the following are equivalent:*

1. *$(C(X),\tau_k,\tau_p)$ has the property $S_1(\Omega^k_{\mathbf{0}},\Omega^p_{\mathbf{0}})$;*
2. *X has the property $S_1(\mathcal{K},\Omega)$.*

Recall that the *i*-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ. Note that a space X has a coarser second countable topology iff $iw(X)=\aleph_0$.

Theorem 3.2. (Noble [8]) *A space $C_k(X)$ is separable if and only if $iw(X)=\aleph_0$.*

Recall that a subset A of a bitopological space (X,τ_1,τ_2) is bidense (double dense or short *d*-dense) in X if A is dense in both (X,τ_1) and (X,τ_2) ([2]). (X,τ_1,τ_2) is *d-separable* if there is a countable set A which is *d*-dense in X. Note that if $iw(X)=\aleph_0$, then $(C(X),\tau_k,\tau_p)$ is *d*-separable.

Theorem 3.3. *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. *$(C(X),\tau_k,\tau_p)$ has the property $S_1(\mathcal{D}^k,\mathcal{D}^p)$;*
2. *$(C(X),\tau_k,\tau_p)$ has the property $S_1(\mathcal{D}^k,\Omega^p_0)$;*
3. *$(C(X),\tau_k,\tau_p)$ has the property $S_1(\Omega^k_0,\Omega^p_0)$;*
4. *X has the property $S_1(\mathcal{K},\Omega)$;*
5. *$(C(X),\tau_k,\tau_p)$ has the property $\binom{\mathcal{D}^k}{\mathcal{D}^p}$;*
6. *ONE has no winning strategy in the game $G_1(\mathcal{K},\Omega)$;*
7. *ONE has no winning strategy in the game $G_1(\mathcal{D}^k,\mathcal{D}^p)$;*
8. *ONE has no winning strategy in the game $G_1(\Omega^k_{\mathbf{0}},\Omega^p_0)$;*
9. *ONE has no winning strategy in the game $G_1(\mathcal{D}^k,\Omega^p_0)$.*

Proof. (1) ⇒ (4). Let $(\mathcal{U}^k_i : i\in\mathbb{N})$ be a sequence of *k*-covers of X and let $D=\{f_s : s\in\mathbb{N}\}$ be a countable dense set in $C_k(X)$. Consider $P_i := \{h^i_{L,W,f_s}\in C(X) : h^i_{L,W,f_s}\upharpoonright L = f_s\upharpoonright L, L\in\mathbb{K}(X), L\subset W, W\in\mathcal{U}^k_i, h^i_{L,W,f_s}\upharpoonright(X\setminus W)=1, f_s\in D\}$. Note P_i is a dense subset of $C_k(X)$ for each $i\in\mathbb{N}$. Indeed fix $f\in C(X), K\in\mathbb{K}(X), \epsilon>0$. For $\langle f,K,\epsilon\rangle$ there exists $W_k\in\mathcal{U}^k_i$ and $f_s\in D$ such that $K\subset W_k$ and $f_s\in\langle f,K,\epsilon\rangle$. Take $h^i_{K,W_k,f_s}\in\langle f,K,\epsilon\rangle$.

Since $\{P_i : i\in\mathbb{N}\}$ is a countable set of dense sets of $C_k(X)$, by (1), there exists $\{p_i : i\in\mathbb{N}\}$ such that $p_i\in P_i$ and $\{p_i : i\in\mathbb{N}\}$ is a dense subset of $C_p(X)$. For $\{p_i = h^i_{L_i,W_i,f_{s_i}} : i\in\mathbb{N}\}$, we have that $\{W_i : i\in\mathbb{N}\}$ is an ω-cover of X. Indeed, let $M=\{x_1,x_2,...,x_k\}\in\mathbb{F}(X)$. Consider $U=\langle\mathbf{0},M,(-\frac{1}{2};\frac{1}{2})\rangle$, then there exists i' such that $p_{i'}\in U$. It follows that $M\subset W_{i'}$.

(3) $\Rightarrow$ (2) is immediate.

(4) $\Rightarrow$ (3). By Theorem 3.1.

(2) $\Rightarrow$ (1). Let $\{D_{i,j} : i, j \in \mathbb{N}\}$ be a countable set of dense sets in $C_k(X)$. Let $D = \{d_i : i \in \mathbb{N}\}$ be a countable dense set in $C_k(X)$. By $S_1(D^k, \Omega^p_{d_i})$ there exists $\{d_{i,j} : j \in \mathbb{N}\}$ such that $d_{i,j} \in D_{i,j}$ and $\{d_{i,j} : j \in \mathbb{N}\} \in \Omega^p_{d_i}$. Consider $M = \{d_{i,j} : i, j, \in \mathbb{N}\}$. The set M is dense in $C_p(X)$. Fix $f \in C(X)$. Let $L = \{x_1, x_2, ..., x_n\} \in \mathbb{F}(X)$ and $\epsilon > 0$. The set $\langle f, L, \epsilon\rangle$ is a neighborhood of f, then there is $d_{i'} \in D$ such that $d_{i'} \in \langle f, L, \epsilon\rangle$, than there is j' such that $d_{i',j'} \in \langle f, L, \epsilon\rangle$, hence $M \in D^p$

(6) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (6). Let σ be a strategy for ONE in $G_1(\mathcal{K}, \Omega)$ and let the first move of ONE be a k-cover $\sigma(\emptyset) = \{U_{(\alpha^1)} : \alpha^1 \in \Lambda^1\}$. Suppose that for each finite sequence s of numbers $\alpha^i \in \Lambda^i$ of length at most m, U_s has been already defined. Then define $\{U_{(\alpha^1,...,\alpha^m,\alpha^k)} : \alpha^k \in \Lambda^k\}$ to be the set $\sigma(U_{(\alpha^1)}, U_{(\alpha^1,\alpha^2)}, ..., U_{(\alpha^1,...,\alpha^m)}) \setminus \{U_{(\alpha^1)}, U_{(\alpha^1,\alpha^2)}, ..., U_{(\alpha^1,...,\alpha^m)}\}$. Because each compact subset of X belongs to infinitely many elements of a k-cover, we have that, for each s, a finite sequence of numbers $\alpha^i \in \Lambda^i$, the set $\{U_{s\frown(\alpha^n)} : \alpha^n \in \Lambda^n\}$ is a k-cover. Apply (4) and, for each s, choose $\alpha^s \in \Lambda^s$ such that $\{U_{s\frown(\alpha^s)} : s$ a finite sequence of numbers $\alpha^i \in \Lambda^i, i \in \mathbb{N}\}$ is a ω-cover of X. Then inductively define a sequence $\alpha^1 = \alpha^{\emptyset}, \alpha^{k+1} = \alpha^{(\alpha^1,...,\alpha^k)}$ for $k \geq 1$. Then $U_{\alpha^1}, U_{\alpha^1,\alpha^2}, ..., U_{\alpha^1,...,\alpha^k}, ...$ is a ω-cover, and because it is, in fact, a sequence of moves TWO in a play of game $G_1(\mathcal{K}, \Omega)$, σ is not a winning strategy for ONE.

Similarly to (4) $\Leftrightarrow$ (6) we have that (1) $\Leftrightarrow$ (7), (2) $\Leftrightarrow$ (9) and (3) $\Leftrightarrow$ (8). □

4. $S_{fin}(\mathcal{D}^k, \mathcal{D}^p)$ and $G_{fin}(\mathcal{D}^k, \mathcal{D}^p)$

Theorem 4.1. ([13, Theorem 3.9] for $\lambda = k$ and $\mu = p$) *For a space X the following are equivalent:*

1. $(C(X), \tau_k, \tau_p)$ *has the property* $S_{fin}(\Omega^k_{\mathbf{0}}, \Omega^p_{\mathbf{0}})$*;*
2. *X has the property* $S_{fin}(\mathcal{K}, \Omega)$*.*

Theorem 4.2. *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. $(C(X), \tau_k, \tau_p)$ *has the property* $S_{fin}(\mathcal{D}^k, \mathcal{D}^p)$*;*
2. $(C(X), \tau_k, \tau_p)$ *has the property* $S_{fin}(\mathcal{D}^k, \Omega^p_0)$*;*
3. $(C(X), \tau_k, \tau_p)$ *has the property* $S_{fin}(\Omega^k_0, \Omega^p_0)$*;*
4. *X satisfies the selection principle* $S_{fin}(\mathcal{K}, \Omega)$*;*
5. *ONE has no winning strategy in the game* $G_{fin}(\mathcal{K}, \Omega)$*;*
6. *ONE has no winning strategy in the game* $G_{fin}(\mathcal{D}^k, \mathcal{D}^p)$*;*
7. *ONE has no winning strategy in the game* $G_{fin}(\Omega^k_{\mathbf{0}}, \Omega^p_0)$*;*
8. *ONE has no winning strategy in the game* $G_{fin}(\mathcal{D}^k, \Omega^p_0)$*.*

Proof. The implications are proved similarly to the proof of Theorem 3.3. □

5. $S_1(\mathcal{D}^k, \mathcal{S}^p)$ and $G_1(\mathcal{D}^k, \mathcal{S}^p)$

Theorem 5.1. ([3, Theorem 15]) *For a space X the following are equivalent:*

1. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\Omega^k_{\mathbf{0}}, \Gamma^p_{\mathbf{0}})$*;*
2. *X has the property* $S_1(\mathcal{K}, \Gamma)$*.*

Theorem 5.2. ([4, Theorem 10]) *For a space X the following are equivalent:*

1. *X has the property* $S_{fin}(\mathcal{K}, \Gamma)$*;*
2. *X has the property* $S_1(\mathcal{K}, \Gamma)$*;*
3. *ONE has no winning strategy in the game* $G_1(\mathcal{K}, \Gamma)$*.*

Theorem 5.3. *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\mathcal{D}^k, \mathcal{S}^p)$;
2. $(C(X), \tau_k, \tau_p)$ *has the property* $\binom{\mathcal{D}^k}{\mathcal{S}^p}$;
3. X *has the property* $S_1(\mathcal{K}, \Gamma)$;
4. $(C(X), \tau_k, \tau_p)$ *has the property* $S_{fin}(\mathcal{D}^k, \mathcal{S}^p)$;
5. X *has the property* $S_{fin}(\mathcal{K}, \Gamma)$;
6. *Each finite power of X has the property* $S_1(\mathcal{K}, \Gamma)$;
7. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\Omega^k_{\mathbf{0}}, \Gamma^p_{\mathbf{0}})$;
8. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\mathcal{D}^k, \Gamma^p_{\mathbf{0}})$;
9. X *has the property* $\binom{\mathcal{K}}{\Gamma}$;
10. *ONE has no winning strategy in the game* $G_1(\mathcal{K}, \Gamma)$;
11. *ONE has no winning strategy in the game* $G_1(\mathcal{D}^k, \mathcal{S}^p)$;
12. *ONE has no winning strategy in the game* $G_1(\Omega^k_{\mathbf{0}}, \Gamma^p_{\mathbf{0}})$;
13. *ONE has no winning strategy in the game* $G_1(\mathcal{D}^k, \Gamma^p_{\mathbf{0}})$.

Proof. By Theorem 5.1 ([3, Theorem 15]), (3) ⇔ (7).

By Theorem 5.2 (Theorem 10 in [4]), (3) ⇔ (5) ⇔ (10).

By Theorem 14 in [3], (3) ⇔ (9).

(3) ⇔ (6) (Proposition 13 and Theorem 10 in [4]).

(1) ⇒ (4) is immediate.

(7) ⇒ (8) is immediate.

Similarly to (3) ⇔ (10) (the implication (2) ⇒ (3) in Theorem 10 in [4]) we have that (1) ⇔ (11), (7) ⇔ (12) and (8) ⇔ (13).

(4) ⇒ (2). Let D be a dense subset of $C_k(X)$. By the property $S_{fin}(\mathcal{D}^k, \mathcal{S}^p)$, for sequence $(D_i : D_i = D$ and $i \in \mathbb{N})$ there is a sequence $(K_i : i \in \mathbb{N})$ such that for each i, K_i is finite, $K_i \subset D_i$, and $\bigcup_{i\in\mathbb{N}} K_i$ is a countable sequentially dense subset of $C_p(X)$.

(2) ⇒ (9). Let $\mathcal{U}$ be an open k-cover of X. Note that the set $\mathcal{D} := \{f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1$ for some $U \in \mathcal{U}\}$ is dense in $C_k(X)$ and, hence, $\mathcal{D}$ contains a countable sequentially dense set A in $C_p(X)$. Take $\{f_n : n \in \mathbb{N}\} \subset A$ such that $f_n \mapsto \mathbf{0}$ $(n \mapsto \infty)$ in $C_p(X)$. Let $f_n \upharpoonright (X \setminus U_n) \equiv 1$ for some $U_n \in \mathcal{U}$. Then $\{U_n : n \in \mathbb{N}\}$ is a γ-subcover of $\mathcal{U}$, because of $f_n \mapsto \mathbf{0}$. Hence, X satisfies $\binom{\mathcal{K}}{\Gamma}$.

(3) ⇒ (1). Let $(D_{i,j} : i, j \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ and let $D = \{f_i : i \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$.

For every $f_i \in D$ and $j \in \mathbb{N}$ consider $\mathcal{U}_{i,j} = \{U_{h,i,j} : U_{h,i,j} = (f_i - h)^{-1}(-\frac{1}{j}, \frac{1}{j}) \wedge (U_{h,i,j} \neq \emptyset)$ for $h \in D_{i,j}\}$. Note that $\mathcal{U}_{i,j}$ is a k-cover of X for every $i, j \in \mathbb{N}$. Since X satisfies $S_1(\mathcal{K}, \Gamma)$, there is a sequence $(U_{h(i,j),i,j} : i, j \in \mathbb{N})$ such that $U_{h(i,j),i,j} \in \mathcal{U}_{i,j}$, and $\phi := \{U_{h(i,j),i,j} : i, j \in \mathbb{N}\}$ is an element of Γ.

We claim that $\{h(i, j) : i, j \in \mathbb{N}\}$ is a sequentially dense subset of $C_p(X)$.

Fix $g \in C(X)$. There exists $(f_{i_k} : k \in \mathbb{N})$ such that $f_{i_k} \to g$ $(k \to \infty)$ in τ_p. Then $(g - f_{i_k}) \to \mathbf{0}$ in τ_p. Show that $h(i_k, j) \to g$ in τ_p. Let $W = \langle g, A, \epsilon\rangle$ be a base neighborhood of g in $C_p(X)$, where $A \in \mathbb{F}(X)$ and $\epsilon > 0$. Since ϕ is a γ-cover of X, then $\{U_{h(i_k,j),i_k,j} : k, j \in \mathbb{N}\}$ is a γ-cover of X, too. There exists k', j' such that $\frac{1}{j'} < \frac{\epsilon}{2}$ and for every $k > k', j > j'$ the following statements are true: $(g - f_{i_k})(A) \subset (-\frac{\epsilon}{2}; \frac{\epsilon}{2})$ and $(f_{i_k} - h(i_k, j))(A) \subset (-\frac{1}{j'}; \frac{1}{j'}) \subset (-\frac{\epsilon}{2}; \frac{\epsilon}{2})$. Notice, that $((g - f_{i_k}) + (f_{i_k} - h(i_k, j)))(A) = (g - h(i_k, j))(A) \subset (-\epsilon; \epsilon)$. Then $h(i_k, j) \subset W$ for every $k > k', j > j'$.

(8) ⇒ (3). Let $\{\mathcal{U}_i : i \in \mathbb{N}\} \subset \mathcal{K}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. Consider $D_i = \{f_{K,U,i,j} \in C(X)$: such that $f_{K,U,i,j} \upharpoonright K \equiv d_j$, $f_{K,U,i,j} \upharpoonright (X \setminus U) \equiv 1$ where $K \in \mathbb{K}(X)$, $K \subset U \in \mathcal{U}_i\}$ for every $i \in \mathbb{N}$. Since D is a dense subset of $C_k(X)$, then D_i is a dense subset of $C_k(X)$ for every $i \in \mathbb{N}$. By (8), there is a set $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\}$ such that $f_{K(i),U(i),i,j(i)} \in D_i$ and $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma^p_{\mathbf{0}}$. Claim that the set $\{U(i) : i \in \mathbb{N}\} \in \Gamma$. Let $K \in \mathbb{F}(X)$ and let $W = \langle \mathbf{0}, K, \frac{1}{2}\rangle$ be a base neighborhood of $\mathbf{0}$. Since $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma^p_{\mathbf{0}}$, there is $i' \in \mathbb{N}$ such that $f_{K(i),U(i),i,j(i)} \in W$ for every $i > i'$. It follows that $K \subset U(i)$ for every $i > i'$ and hence $\{U(i) : i \in \mathbb{N}\} \in \Gamma$. □

We can summarize the relationships between considered notions in next diagrams.

$$\begin{array}{ccccccc}
G_1(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & G_{fin}(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Rightarrow & G_1(\mathcal{D}^k,\Omega^p_0) & \Rightarrow & G_{fin}(\mathcal{D}^k,\Omega^p_0)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
G_1(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & G_{fin}(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Rightarrow & G_1(\Omega^k_0,\Omega^p_0) & \Rightarrow & G_{fin}(\Omega^k_0,\Omega^p_0)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
G_1(\mathcal{D}^k,\mathcal{S}^p) & \Leftrightarrow & G_{fin}(\mathcal{D}^k,\mathcal{S}^p) & \Rightarrow & G_1(\mathcal{D}^k,\mathcal{D}^p) & \Rightarrow & G_{fin}(\mathcal{D}^k,\mathcal{D}^p)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\mathcal{D}^k,\mathcal{S}^p) & \Leftrightarrow & S_{fin}(\mathcal{D}^k,\mathcal{S}^p) & \Rightarrow & S_1(\mathcal{D}^k,\mathcal{D}^p) & \Rightarrow & S_{fin}(\mathcal{D}^k,\mathcal{D}^p)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & S_{fin}(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Rightarrow & S_1(\mathcal{D}^k,\Omega^p_0) & \Rightarrow & S_{fin}(\mathcal{D}^k,\Omega^p_0)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & S_{fin}(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Rightarrow & S_1(\Omega^k_0,\Omega^p_0) & \Rightarrow & S_{fin}(\Omega^k_0,\Omega^p_0)
\end{array}$$

Fig. 1. The Diagram of games and selectors of $(C(X), \tau_k, \tau_p)$.

$$\begin{array}{ccccccc}
G_1(\mathcal{K},\Gamma) & \Leftrightarrow & G_{fin}(\mathcal{K},\Gamma) & \Rightarrow & G_1(\mathcal{K},\Omega) & \Rightarrow & G_{fin}(\mathcal{K},\Omega)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\mathcal{K},\Gamma) & \Leftrightarrow & S_{fin}(\mathcal{K},\Gamma) & \Rightarrow & S_1(\mathcal{K},\Omega) & \Rightarrow & S_{fin}(\mathcal{K},\Omega)
\end{array}$$

Fig. 2. The Diagram of games and selection principles for a space X with $iw(X) = \aleph_0$ corresponding to selectors of $(C(X), \tau_k, \tau_p)$.

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