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Designing the program trajectory for steering a spacecraft under arbitrary boundary conditions

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Abstract. A problem is considered of designing the program trajectory of a spacecraft turning from an arbitrary initial orientation to an arbitrary final orientation, with the orientations being defined with unit quaternions. A projection of a group of unit quaternions $Sp(1)$ on a sphere with the radius of 2π is used to represent rotation of a body as a motion of a point inside the given sphere. Polynomials of the fifth degree are considered as a class of functions to define the program trajectories in the sphere. The suggested class of trajectories is demonstrated to be effective to provide the possibility of meeting boundary conditions at arbitrary values of velocities and accelerations.

1. Introduction

The problem of steering a spacecraft has been one of the major issues to be resolved in space navigation. This problem has been addressed in a great number of publications [1, 2, 3]. However, as mentioned in [4], there has been no analytical solution found for the position and angular velocities, which would meet arbitrary boundary conditions. One of the methods used for solving the problem of constructing program trajectories is the application of kinematic approach followed by the inverse solution of dynamics to find the program control [5, 6]. The cited works cover a kinematic trajectory being constructed through setting the coordinates of a current orientation quaternion with the polynomials of the fifth degree. Our suggested approach, however, differs in that there is a projection of a group of unit quaternions $Sp(1)$ onto a sphere of the radius 2π, with the motion being defined with a polynomial of the fifth degree. The possibility of projecting unit quaternions onto a sphere of the radius π is reported in [7, 8]. To reach completeness, the class of functions adequate for describing the turning trajectories requires a change in the configuration space of rotations, and the sphere of the radius 2π becomes such configuration space. Section 2 covers the statement of the problem of finding an optimum program trajectory of a turning spacecraft. In Section 3, a solution of the stated problem is considered for the case that the program trajectory is represented as a polynomial of the fifth degree. Section 4 contains examples for defining the program trajectories. Section 5 summarizes the major results of the work presented.

2. Statement of problem

When solving the tasks of controlling the orientation of a spacecraft, a quaternion formalism is used actively. Using the quaternion parametrization, the kinematic equations, which define the

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relationship between the projections of angular velocity on the movable axes and the position of a body, take the following form

$$\begin{aligned} \dot{q}_0 &= -\tfrac{1}{2}(q_1\Omega_1 + q_2\Omega_2 + q_3\Omega_3), \\ \dot{q}_1 &= \tfrac{1}{2}(q_0\Omega_1 - q_3\Omega_2 + q_2\Omega_3), \\ \dot{q}_2 &= \tfrac{1}{2}(q_0\Omega_2 - q_1\Omega_3 + q_3\Omega_1), \\ \dot{q}_3 &= \tfrac{1}{2}(q_0\Omega_3 - q_2\Omega_1 + q_1\Omega_2), \end{aligned} \tag{1}$$

here $q_1,\ q_2,\ q_3$ are the Rodrig-Hamilton parameters; $\Omega_1,\ \Omega_2,\ \Omega_3$ are hence and henceforth the projections of the angular velocity vector on the axes related to a rotating body.

Let us consider the problem of finding an optimum program trajectory of a turning spacecraft, which motion is defined with the equations (1) over the time T. Let the following boundary conditions be imposed on the orientation, the angular velocity and its derivative:

$$q(0) = q^0,\ q(T) = q^T,\ \Omega(0) = \Omega^0,\ \Omega(T) = \Omega^T,\ \dot{\Omega}(0) = \dot{\Omega}^0,\ \dot{\Omega}(T) = \dot{\Omega}^T \tag{2}$$

The program control realizing the trajectory can be found using the Euler's dynamic equations:

$$\begin{cases} A\dot{\Omega}_1 + (C - B)\Omega_2\Omega_3 = M_1, \\ B\dot{\Omega}_2 + (A - C)\Omega_1\Omega_3 = M_2, \\ C\dot{\Omega}_3 + (B - A)\Omega_1\Omega_2 = M_3, \end{cases}$$

where $M_1,\ M_2,\ M_3$ are the projections of the controlling moment M on the moving coordinate axes.

As a quality criterion, a certain functional can be selected depending on the program motion and program control:

$$I_1(q(t), \Omega(t), M(t)) \tag{3}$$

which reaches the minimum with the optimum program motion.

It should be noted that the choice of limitations imposed on the edge values of $\dot{\Omega}_0,\ \dot{\Omega}_T$ can be made based on a variety of considerations. For example, a requirement can be that these edge values would satisfy the Euler's dynamic equations containing zero right parts, i.e., the controlling moment be turned to zero at the moments that the turning maneuver starts or ends. Either, for a smooth turn to be realized, the choice of $\dot{\Omega}_0 = 0,\ \dot{\Omega}_T = 0$ can be made.

The task is to find an optimum, according to quality criterion (3), program trajectory for the dynamic system (1) that would satisfy the boundary conditions (2) at a given maneuver time T.

Any doubly differentiable function $q = q(t)$ can be interpreted as a certain trajectory in the turning configuration space. As follows from the trigonometric formula to define a quaternion, the orientation of movable axes can be identified with the axis-angle parameters as a plurality of turns at every possible angle $0 \le \chi \le \pi$ around every possible axis defined with the unit vectors (Fig.1) .

$$\bar{e}(\alpha, \beta) = \sin\alpha\cos\beta\ \bar{i}_1 + \sin\alpha\sin\beta\ \bar{i}_2 + \cos\alpha\ \bar{i}_3,\ 0 \le \alpha \le \pi,\ 0 \le \beta \le 2\pi,$$

in the form

$$q = \cos\frac{\chi}{2} + \sin\frac{\chi}{2}\bar{e}.$$

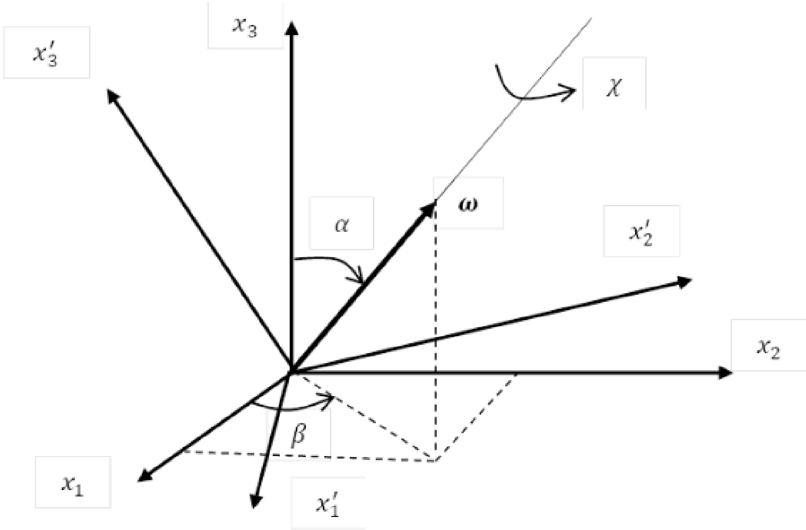

Figure 1. Defining a movable coordinate system using axis-angle parameters

It is reported in [9,10] that to a plurality of unit quaternions defining an orientation can correspond the points in a sphere with the radius π, with the point positions being determined with a radius-vector as follows:

$$\bar{r} = \chi\bar{e}$$

where $0 \leq \chi \leq \pi$. However, the motions corresponding to rotations of a spacecraft can extend beyond the sphere of the radius π, but, as a rule, they are confined within the sphere of the radius 2π. Then the vector-function defining a trajectory of a turning spacecraft can be written in the form.

$$\bar{r}(t) = \chi(t)\bar{e}(t)$$

where $0 \leq \chi \leq 2\pi$, or in the coordinate form

$$x_1(t) = \chi(t)\sin\alpha(t)\cos\beta(t),\ x_2(t) = \chi(t)\sin\alpha(t)\sin\beta(t),\ x_3(t) = \chi(t)\cos\alpha(t).$$

To solve the applied problems of controlling the motion, it may be convenient to switch from the totally nonintuitive representation of a trajectory on a hypersphere $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ in the four-dimensional space to its representation in a sphere with the radius 2π in the three-dimensional Euclidian space.

The coordinates of the function $q = q(t)$ are defined with the equations as follows:

$$\begin{cases} q_0(t) = \cos\dfrac{\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}}{2}, \\ q_k(t) = \dfrac{x_k(t)}{\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}}\sin\dfrac{\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}}{2},\ k = 1,\ 2,\ 3. \end{cases} \quad (4)$$

Hence,

$$x_k(t) = \frac{2q_k(t)\arccos q_0}{\sqrt{1 - q_0^2}} = 2q_k(t)Q(q_0),\ k = 1,\ 2,\ 3,\ Q(q_0) = \frac{\arccos q_0}{\sqrt{1 - q_0^2}}. \quad (5)$$

From Eq. 5, we obtain

$$\dot{x}_k = 2\left(q_k \frac{\partial Q}{\partial q_0}\dot{q}_0 + Q\dot{q}_k\right), \ k = 1, \ 2, \ 3,$$

where $\dfrac{\partial Q}{\partial q_0} = -\dfrac{1}{1-q_0^2} + \dfrac{q_0 \arccos q_0}{(1-q_0^2)^{3/2}}$.

Upon successive substitution of Eq.(1) and Eq.(4) into the equations, we then obtain the kinematic equations for the new parameters in the form

$$\dot{x}_k = f_k(x_1, x_2, x_3, \Omega_1, \Omega_2, \Omega_3), \ k = 1, \ 2, \ 3. \tag{6}$$

From Eq.(6) we can find

$$\ddot{x}_k = 2\left(\dot{q}_k \frac{\partial Q}{\partial q_0}\dot{q}_0 + q_k \frac{\partial^2 Q}{\partial q_0^2}\dot{q}_0^2 + q_k \frac{\partial Q}{\partial q_0}\ddot{q}_0 + \frac{\partial Q}{\partial q_0}\dot{q}_0\dot{q}_k + Q\ddot{q}_k\right) \tag{7}$$

where $\dfrac{\partial^2 Q}{\partial q_0^2} = -\dfrac{3q_0}{(1-q_0^2)^2} + \dfrac{3q_0^2 \arccos q_0}{(1-q_0^2)^{5/2}} + \dfrac{\arccos q_0}{(1-q_0^2)^{3/2}}$.

From Eq.(1) we find:

$$\begin{aligned}
\ddot{q}_0 &= -\tfrac{1}{2}(\dot{q}_1\Omega_1 + \dot{q}_2\Omega_2 + \dot{q}_3\Omega_3 + q_1\dot{\Omega}_1 + q_2\dot{\Omega}_2 + q_3\dot{\Omega}_3), \\
\ddot{q}_1 &= \tfrac{1}{2}(\dot{q}_0\Omega_1 - \dot{q}_3\Omega_2 + \dot{q}_2\Omega_3 + q_0\dot{\Omega}_1 - q_3\dot{\Omega}_2 + q_2\dot{\Omega}_3), \\
\ddot{q}_2 &= \tfrac{1}{2}(\dot{q}_0\Omega_2 - \dot{q}_1\Omega_3 + \dot{q}_2\Omega_1 + q_0\dot{\Omega}_2 - q_1\dot{\Omega}_3 + q_3\dot{\Omega}_1), \\
\ddot{q}_3 &= \tfrac{1}{2}(\dot{q}_0\Omega_3 - \dot{q}_2\Omega_1 + \dot{q}_1\Omega_2 + q_0\dot{\Omega}_3 - q_2\dot{\Omega}_1 + q_1\dot{\Omega}_2),
\end{aligned} \tag{8}$$

Upon successive substitution of Eq. (8), (1) and (4) into Eq. (7), we obtain equations in the form

$$\ddot{x}_k = g_k(x_1, x_2, x_3, \Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_1, \dot{\Omega}_2, \dot{\Omega}_3).$$

Setting the problem of steering a spacecraft in terms of the parameters $x_1, \ x_2, \ x_3$, the boundary conditions (2) can be substituted with the following:

$$\begin{gathered}
x_k(0) = 2q_k^0 Q(q_0^0), \ x_k(T) = 2q_k^T Q(q_0^T), \ \dot{x}_k(0) = f_k(x_1^0, x_2^0, x_3^0, \Omega_1^0, \Omega_2^0, \Omega_3^0) \\
\dot{x}_k(T) = f_k(x_1^T, x_2^T, x_3^T, \Omega_1^T, \Omega_2^T, \Omega_3^T), \ \ddot{x}_k(0) = g_k(x_1^0, x_2^0, x_3^0, \Omega_1^0, \Omega_2^0, \Omega_3^0, \dot{\Omega}_1^0, \dot{\Omega}_2^0, \dot{\Omega}_3^0), \\
\ddot{x}_k(T) = g_k(x_1^T, x_2^T, x_3^T, \Omega_1^T, \Omega_2^T, \Omega_3^T, \dot{\Omega}_1^T, \dot{\Omega}_2^T, \dot{\Omega}_3^T).
\end{gathered} \tag{9}$$

The quality criterion can be rewritten as

$$I_2(x_1(t), x_2(t), x_3(t), \Omega(t), M(t)) = I_1(q(x_1(t), x_2(t), x_3(t)), \Omega(t), M(t)) \tag{10}$$

It is required to find an optimum program trajectory of the system (6) that would satisfy the conditions (6), at a given maneuver time T .

Therefore, the problem of controlling the motion (1)-(3) formulated in terms of the Rodrig-Hamilton parameters is transformed into the problem of controlling the motion of a point within a sphere of the radius 2π (6), (9)-(10). In this problem, the motion of a point is described with Eq.(6) with the conditions imposed by Eq. (9). That makes it possible to get an intuitive interpretation of the rotating body as a point moving in the three-dimensional Euclidian space.

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3. Finding an optimum trajectory in the form of a polynomial of the fifth degree

Let us consider finding an optimum trajectory as it is defined with a vector polynomial of the fifth degree

$$\bar{r}(t) = \sum_{k=1}^{5} \bar{a}_k t^k, \;\; 0 \le t \le T. \tag{11}$$

As will be shown below, with such limitation the trajectory is defined unambiguously from the conditions of Eq. (9), which makes a quality criterion unnecessary, because the trajectory becomes optimal automatically.

The coefficients (11) have to satisfy the conditions

$$\begin{cases} \bar{a}_0 = \bar{r}_0, \\ \bar{a}_1 = \dot{\bar{r}}_0 \\ \bar{a}_2 = \frac{1}{2}\ddot{\bar{r}}_0 \end{cases} \tag{12}$$

$$\begin{cases} \bar{a}_5 T^5 + \bar{a}_4 T^4 + \bar{a}_3 T^3 = \bar{r}_T - \bar{r}_0 - \dot{\bar{r}}_0 T - \frac{1}{2}\ddot{\bar{r}}_0 T^2, \\ 5\bar{a}_5 T^4 + 4\bar{a}_4 T^3 + 3\bar{a}_3 T^2 = \dot{\bar{r}}_T - \dot{\bar{r}}_0 - \ddot{\bar{r}}_0 T, \\ 20\bar{a}_5 T^3 + 12\bar{a}_4 T^2 + 6\bar{a}_3 T = \ddot{\bar{r}}_T - \ddot{\bar{r}}_0. \end{cases}$$

The latter system of equations can be written out in the matrix form as follows

$$\begin{pmatrix} T^5 & T^4 & T^3 \\ 5T^4 & 4T^3 & 3T^2 \\ 20T^3 & 12T^2 & 6T \end{pmatrix} \begin{pmatrix} \bar{a}_5 \\ \bar{a}_4 \\ \bar{a}_3 \end{pmatrix} = \begin{pmatrix} \bar{r}_T - \bar{r}_0 - \dot{\bar{r}}_0 T - \frac{1}{2}\ddot{\bar{r}}_0 T^2 \\ \dot{\bar{r}}_T - \dot{\bar{r}}_0 - \ddot{\bar{r}}_0 T \\ \ddot{\bar{r}}_T - \ddot{\bar{r}}_0 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \bar{a}_5 \\ \bar{a}_4 \\ \bar{a}_3 \end{pmatrix} = \begin{pmatrix} \dfrac{6}{T^5} & \dfrac{-3}{T^4} & \dfrac{1}{2T^3} \\ \dfrac{-15}{T^4} & \dfrac{7}{T^3} & \dfrac{-1}{T^2} \\ \dfrac{10}{T^3} & \dfrac{-4}{T^2} & \dfrac{1}{2T} \end{pmatrix} \begin{pmatrix} \bar{r}_T - \bar{r}_0 - \dot{\bar{r}}_0 T - \frac{1}{2}\ddot{\bar{r}}_0 T^2 \\ \dot{\bar{r}}_T - \dot{\bar{r}}_0 - \ddot{\bar{r}}_0 T \\ \ddot{\bar{r}}_T - \ddot{\bar{r}}_0 \end{pmatrix}$$

or

$$\begin{aligned} \bar{a}_3 &= \frac{10}{T^3}(\bar{r}_T - \bar{r}_0 - \dot{\bar{r}}_0 T - \frac{1}{2}\ddot{\bar{r}}_0 T^2) - \frac{4}{T^2}(\dot{\bar{r}}_T - \dot{\bar{r}}_0 - \ddot{\bar{r}}_0 T) + \frac{1}{2T}(\ddot{\bar{r}}_T - \ddot{\bar{r}}_0), \\ \bar{a}_4 &= -\frac{15}{T^4}(\bar{r}_T - \bar{r}_0 - \dot{\bar{r}}_0 T - \frac{1}{2}\ddot{\bar{r}}_0 T^2) + \frac{7}{T^3}(\dot{\bar{r}}_T - \dot{\bar{r}}_0 - \ddot{\bar{r}}_0 T) - \frac{1}{T^2}(\ddot{\bar{r}}_T - \ddot{\bar{r}}_0), \\ \bar{a}_5 &= \frac{6}{T^5}(\bar{r}_T - \bar{r}_0 - \dot{\bar{r}}_0 T - \frac{1}{2}\ddot{\bar{r}}_0 T^2) - \frac{3}{T^4}(\dot{\bar{r}}_T - \dot{\bar{r}}_0 - \ddot{\bar{r}}_0 T) + \frac{1}{2T^3}(\ddot{\bar{r}}_T - \ddot{\bar{r}}_0). \end{aligned} \tag{13}$$

Therefore, the coefficients of the vector polynomial (11) are defined unambiguously from Eq. (12) and (13) upon substitution of the values from Eq. (9).

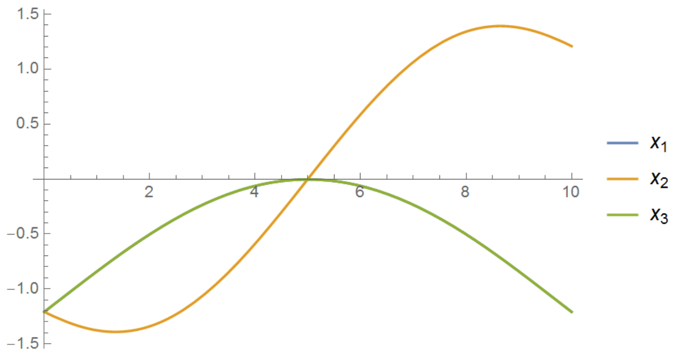

Figure 2. Plotted functions of $x_1(t),\ x_2(t),\ x_3(t)$

4. Example

As an example, let us consider the problem of finding a program trajectory of a body turning from the position $q^0 = (0.5, -0.5, -0.5, -0.5)$ into the position $q^T = (0.5, -0.5, 0.5, -0.5)$ over the time $T = 10$ sec, with the following initial conditions:

$$\Omega^0 = (0.5, 0, 0),\ \Omega^T = (0, 0, -0.5),\ \dot{\Omega}^0 = (0, 0, 0),\ \dot{\Omega}^T = (0, 0, 0).$$

Upon calculation of the polynomial coefficients (3.1), rounding them off to five decimal digits, we obtain

$$x_1(t) = -1.2092 + 0.3682t + 0.00872t^2 - 0.00911t^3 + 0.00046t^4,$$
$$x_2(t) = -1.2092 - 0.2364t + 0.041667t^2 + 0.031157t^3 - 0.00509t^4 + 0.0002t^5,$$
$$x_3(t) = -1.2092 + 0.3682t + 0.00872t^2 - 0.00911t^3 + 0.00046t^4.$$

Figure 2 displays the graphs of functions: $x_1(t),\ x_2(t),\ x_3(t)$. Figure 3 shows a turning trajectory inside a sphere of the radius 2π .

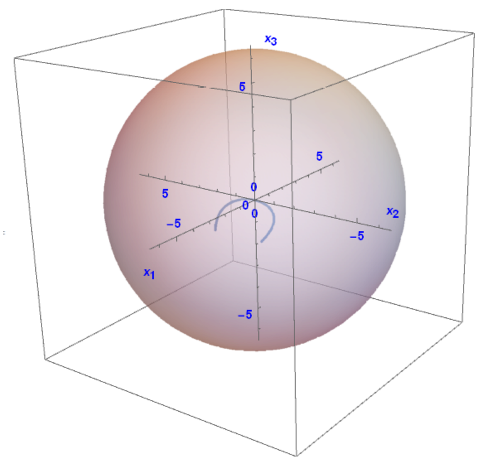

Figure 3. Turning trajectory inside a sphere of the radius 2π

Figures 4, 5 and 6 show the graphs of components of quaternion, angular velocity and angular velocity derivate, respectively.

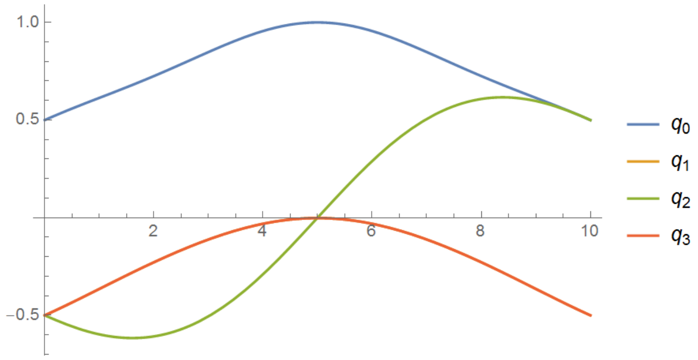

Figure 4. Graphs of the quaternion components $q_0(t),\ q_1(t),\ q_2(t),\ q_3(t)$

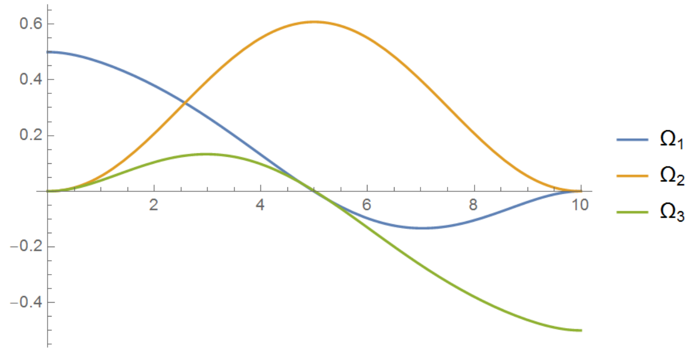

Figure 5. Graphs of angular velocity components $\Omega_1(t),\ \Omega_1(t),\ \Omega_1(t)$

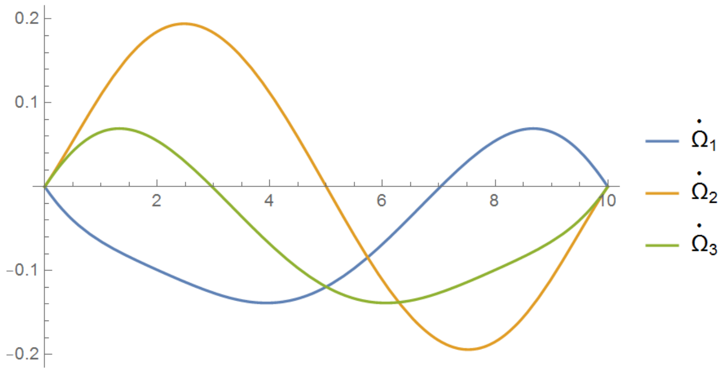

Figure 6. Graphs of angular velocity derivative components $\dot{\Omega}_1(t),\ \dot{\Omega}_2(t),\ \dot{\Omega}_3(t)$.

5. Conclusion

The analytical algorithm has been obtained for constructing program trajectories in the form of a polynomial of the fifth degree. Examples are given to demonstrate the effectiveness of the proposed approach. Further studies are envisioned to find program controls in the framework of the concept of inverse problems of dynamics.

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