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Towards an efficient approximability for the Euclidean Capacitated Vehicle Routing Problem with Time Windows and multiple depots

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Abstract: We consider the Euclidean Capacitated Vehicle Routing Problem with Time Windows (CVRPTW). For the long time, approximability of this well-known problem in the class of algorithms with theoretical guarantees was poorly studied. This year, for the case of a single depot, we proposed two approximation algorithms, which are the Efficient Polynomial Time Approximation Schemes (EPTAS) for any fixed given capacity q and the number p of mutually disjunctive time windows. The former scheme extends the celebrated approach proposed by M. Haimovich and A. Rinnooy Kan and allows the evident parallelization, while the latter one has an improved time complexity bound and the enlarged domain in terms $q = q(n)$ and $p = p(n)$, where it retains polynomial time complexity. In this paper, we announce an extension of these results to the case of multiple depots. So, the first scheme is also EPTAS for any fixed parameters q, p, and m, where m is the number of depots, and remains PTAS for $q = o(\log\log n)$ and $mp = o(\log\log n)$. In other turn, the second one is a PTAS for $p^3q^4 = O(\log n)$ and $(pq)^2 \log m = O(\log n)$.

Keywords: Capacitated Vehicle Routing Problem, Time Windows, Multiple Depots, Polynomial Time Approximation Schemes

1. INTRODUCTION

The Capacitated Vehicle Routing Problem (CVRP) is the famous combinatorial optimization problem introduced by G. Dantzig and J. Ramser in their seminal paper (Dantzig and Ramser, 1959). The problem has a wide range of practical applications in operations research (see survey in (Toth and Vigo, 2014) and references within, see also the recent paper (Mugayskikh et al., 2018)).

Besides that, several models of unsupervised learning can be successfully reduced to the appropriate CVRP settings (see, e.g. (Aggarwal and Reddy, 2013; Braekers et al., 2016)). In CVRP, we are given a number of *customer* locations and a fleet of *vehicles* of a specified *capacity* initially located at a single or multiple *depots*. The goal is to construct a collection of vehicle routes minimizing the total transportation cost, such that each route departs from and arrives to maybe different depots, each customer is serviced by a single route, and the capacity constraint is fulfilled.

In this paper, we consider the Capacitated Vehicle Routing Problem with Time Windows (CVRPTW) (Kumar and Panneerselvam, 2012; Toth and Vigo, 2014), which is an extension of the CVRP, where each customer should be serviced at a specified time interval, called a *time window*. There are two different types of time windows. For the former type, called *hard*, any feasible solution should visit each customer within its dedicated time window exactly, whilst, for the latter one, known as *soft*, this constraint can be violated barring some penalty cost. Along with traditional applications in postal or bank deliveries, industrial refuse collection, school bus scheduling, etc., CVRPTW with hard windows (or just CVRPTW) is a widely employed mathematical model in continent-scale distribution of building materials (Pace et al., 2015), in the low-carbon economy (Shen et al., 2018), in dial-ride company planning (Gschwind and Irnich, 2015) and other practical transportation problems (see, e.g. (Savelsbergh and van Woensel, 2016)).

In this paper, our goal is to propose for CVRPTW novel efficient algorithms with theoretical performance guarantees. Therefore, despite the obvious progress in solving practical instances of this problem by local-search heuristics (Hashimoto and Yagiura, 2008), genetic (Vidal et al., 2013), memetic (Blocho and Czech, 2013; Nalepa and Blocho, 2016), and ant colony algorithms (Necula et al., 2017), in the following short literature overview, we concentrate intensionally on results concerning the complexity of CVRPTW and its approximability with theoretically proven bounds.

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1.1 Related work

Containing the famous Traveling Salesman Problem (TSP) (Toth and Vigo, 2014), the Capacitated Vehicle Routing Problem is strongly NP-hard even in the Euclidean plane (Papadimitriou, 1977), if capacity q belongs to the instance. For $q = 2$, the metric CVRP can be efficiently reduced to the minimum weight perfect matching problem and solved to optimality in a polynomial time. For any fixed $q \geq 3$, the problem is strongly NP-hard. The proof follows from a polynomial time reduction of Partition into Isomorphic Subgraphs (Problem GT12 in (Garey and Johnson, 1979)) to CVRP with $(1, 2)$-metric.

As it follows from the seminal paper by M. Haimovich and A. Rinnooy Kan (Haimovich and Rinnooy Kan, 1985), any ρ-approximation algorithm for TSP induces $(2 + \rho)$-approximation algorithm for the metric CVRP in general case and asymptotically $(1 + \rho)$-approximation algorithm, if $q = o(n)$. On the other hand, for any fixed $q \geq 3$, the problem is APX-hard as it was shown in (Asano et al., 1996) by cost-preserving reduction from the known Max-SNP-hard Maximum H-matching problem (Kann, 1994).

The Euclidean version of CVRP can be approximated much more better. The first Polynomial Time Approximation Scheme (PTAS) for the case of planar Euclidean CVRP for $q = o(\log\log n)$ was proposed in (Haimovich and Rinnooy Kan, 1985). In (Asano et al., 1997), this approach was extended to the case of $q = O(\log n/\log\log n)$. Also, they observed that Arora's (Arora, 1998) PTAS for the two-dimensional Euclidean TSP implies a PTAS for the corresponding CVRP where $q = \Omega(n)$. Then, in (Das and Mathieu, 2015), the first Quasi-Polynomial Time Approximation Scheme (QPTAS) for the two-dimensional Euclidean CVRP, where q belongs to the instance, was proposed. Their algorithm combines the approach developed by Arora (Arora, 1998) for Euclidean TSP with some new ideas to deal with CVRP and gives a $(1 + \varepsilon)$-approximation for the two-dimensional Euclidean CVRP in time $n^{(\log n)^{O(1/\varepsilon)}}$. Further, in (Adamaszek et al., 2010), it was showed that $(1 + \varepsilon)$-approximate solution can be found in polynomial time for $q \leq 2^{\log^\delta n}$, where $\delta = \delta(\varepsilon)$. Recently, some of the aforementioned results were extended to the case of Euclidean d-dimensional space for an arbitrary fixed $d > 2$ (Khachay and Zaytseva, 2015; Khachay and Dubinin, 2016; Khachai and Dubinin, 2017) and some special graphs (Becker et al., 2017, 2018).

Unlike CVRP, approximability of CVRPTW is investigated significantly less. To the best of our knowledge, up to 2018, all known approximability results for this problem were exhausted by the QPTAS scheme extending the results from (Das and Mathieu, 2015) for the case of CVRPTW with finite amount of mutually disjoint time windows introduced in (Song and Huang, 2017; Song et al., 2016).

The rest of the paper is structured as follows. In Section 2, we introduce problem statement of the CVRPTW for the case of multiple depots. We distinguish two different settings of this problem. In the former one, we call in CVRPTW1, any route can depart from and arrive to the different depots, while in the latter setting CVRPTW2, each route should return to its departure depot. In Section 3, we present the first PTAS for the CVRPTW with multiple depots that extends the famous scheme by M. Haimovich and A. Rinnooy Kan, whilst Section 4 presents the further improvement inspired by the scheme proposed in (Asano et al., 1997). Finally, in Section 5, we summarize the results obtained and overview the future work.

2. PROBLEM STATEMENT

We are given by a set $X = \{x_1, \ldots, x_n\}$ of *customers*, a set $Y = \{y_1, \ldots, y_m\}$ of *depots*, and a set $T = \{t_1, \ldots, t_p\}$ of consecutive *time windows*. The set T is assumed to be ordered with respect to the natural precedence, i.e.

$$t_{j_1} \preceq t_{j_2} \text{ for any } 1 \leq j_1 \leq j_2 \leq p.$$

Each customer x_i has a unit non-splittable demand that should be serviced in a given time window $t(x_i) \in T$ by a fleet of identical *vehicles* having the same *capacity* q. Initially, each vechicle is located at some depot $y \in Y$. Any customer should be visited by some vehicle *route* starting and finishing at some depots. The goal is, to propose a collection of capacitated vehicle routes visiting each customer exactly once, during its time window, and minimizing the total transportation cost.

Mathematically an instance CVRPTW(X, Y, T, q) of the CVRPTW is defined by the edge-weighted complete digraph $G = (X \cup Y, E, w)$, capacity bound $q \in \mathbb{N}$, and the partition

$$X_1 \cup \ldots \cup X_p = X \tag{1}$$

induced by the set T of time windows, where [1]

$$X_j = \{x \in X : t(x) = t_j \in T\}.$$

Hereinafter,

(i) for any mutually distinct customers $x_{i_1}, \ldots, x_{i_s} \in X$ and some depots $y_s, y_d \in Y$, the following simple directed path

$$R = y_s, x_{i_1}, \ldots, x_{i_s}, y_d \tag{2}$$

is called *a feasible route* [2], if R satisfies the following constraints

$$s \leq q \quad \text{(capacity)} \tag{3}$$

$$t(x_{i_j}) \preceq t(x_{i_{j+1}}) \quad \text{(time windows)} \tag{4}$$

(ii) if $|Y| = 1$, then the problem in question is called Single Depot CVRPTW. In this case, every feasible route is a simple circuit. Otherwise, the problem is called Multiple Depot CVRPTW. In this paper, we distinguish two different settings of this kind. In the former setting, we will call it CVRPTW1, any route is free to depart from and arrive to separate depots, whilst in the latter one, CVRPTW2, every route is constrained to start and finish at the same depot.

(iii) by construction, the non-negative weighting function w defines direct transportation costs $w(v_1, v_2)$ for any arc $(v_1, v_2) \in E$ in the given graph G, such that each feasible route (2) has the cost

$$w(R) = w(y_s, x_{i_1}) + \sum_{j=1}^{s-1} w(x_{i_j}, x_{i_{j+1}}) + w(x_{i_s}, y_d).$$

[1] tights can be broken arbitrarily.

[2] visiting the customers $x_{i_1}, \ldots, x_{i_s}$

For each aforementioned setting, it is required to find a collection of feasible routes $S = \{R_1, \ldots, R_l\}$ that visits each customer $x \in X$ exactly once and has the minimum total transportation cost

$$w(S) = \sum_{i=1}^{l} w(R_i). \tag{5}$$

If the graph G is undirected, the weighting function w is symmetric and satisfies the triangle inequality, then CVRPTW is called *metric* and the transportation costs $w(v_i, v_j)$ in turn are called distances between the locations v_i and v_j.

In this paper, we consider the Euclidean CVRPTW, where $X \cup Y \subset \mathbb{R}^2$ and $w(v_i, v_j) = \|v_i - v_j\|_2$, although most results remain valid for an arbitrary metric w. For any considered combinatorial optimization problem, we use $*$-notation for its optimum value. For instance, by $\mathrm{TSP}^*(X)$ and $\mathrm{CVRPTW}^*(X, T, Y, q)$ we denote optimum values of the corresponding instances of TSP and CVRPTW, respectively.

3. HAIMOVICH AND RINNOOY KAN SCHEME

Our first scheme extends the framework proposed for the Euclidean Single Depot CVRP in the seminal paper (Haimovich and Rinnooy Kan, 1985). Although their approach cannot be applied to the case of CVRP with time windows and multiple depots directly, our scheme have a lot of common with it. Therefore, we name this scheme after M. Haimovich and A. Rinnoy Kan.

The main idea of the proposed scheme is quite simple and is based on the following points. Consider an arbitrary instance $\mathrm{CVRPTW}(X, Y, T, q)$ of the metric CVRPTW defined by some edge-weighted complete graph $G = (X \cup Y, E, w)$, partition $X_1 \cup \ldots \cup X_p = X$, and a capacity bound q.

(i) To any customer $x_i \in X$, we assign the minimum distance $r_i = r(x_i) = \min\{w(x_i, y) \colon y \in Y\}$ to the set of depots and the closest depot $y(x_i)$, for which $w(x_i, y(x_i)) = r_i$. Then, we relabel the customers such that $r_1 \geq \ldots \geq r_n$.

(ii) For a given $\varepsilon > 0$, we partition the customer set X onto subsets $X_{out} = \{x_1, \ldots, x_{k-1}\}$ and $X_{in} = X \setminus X_{out}$ of the *outer* and *inner* customers respectively, where $k = k(\varepsilon, p, q, m)$ does not depend on the number of customers. Hence, the initial instance is decomposed to two independent subinstances $\mathrm{CVRPTW}(X_{out}, Y, T, q)$ and $\mathrm{CVRPTW}(X_{in}, Y, T, q)$, which can be solved in parallel. The main point is that such a decomposition really exists for any given ε ensuring that the subsequent constructions do provide us with an $(1 + \varepsilon)$-approximate solution. For the case $m = 1$, we show it in (Khachay and Ogorodnikov, 2018). The common case will be proven in the forthcoming paper.

(iii) The resulting approximate solution is combined from an arbitrary optimal solution of the subinstance $\mathrm{CVRPTW}(X_{out}, Y, T, q)$ obtained by the dynamic programming and an approximate solution of $\mathrm{CVRPTW}(X_{in}, Y, T, q)$ provided by our modification (Khachay and Ogorodnikov, 2018) of the famous Iterated Tour Partition (ITP) heuristic (Haimovich and Rinnooy Kan, 1985) adapted to address the additional time windows constraint.

This approach can be successfully applied to the case of CVRPTW2, where each route should return to the starting depot. For the CVRPTW1, we propose a small modification that leads to further improvement in time complexity bounds. We introduce a fictitious depot y_0 and, for any customer x_i, assign the distance $w(x_i, y_0) = r_i$. Thus, we reduce the initial instance $\mathrm{CVRPTW1}(X, Y, T, q)$ to the appropriate instance $\mathrm{CVRPTW}(X, \{y_0\}, T, q)$ of the single depot CVRPTW, which is solved using the approximation scheme proposed in our recent paper (Khachay and Ogorodnikov, 2018). Evidently, the obtained solution can be easily adapted to solution of the initial instance with the same approximation guarantee. To do it, in any route

$$R = y_0, x_{i_1}, \ldots, x_{i_s}, y_0$$

of the solution obtained, we replace the fictitious depot y_0 with the closest depots $y(x_{i_1})$ and $y(x_{i_s})$.

In the sequel, we give time complexity bounds both for the CVRPTW1 and CVRPTW2 problems. For the CVRPTW1, assigning customers to the nearest depots takes time $O(nm)$. An exact solution for the outer instance $\mathrm{CVRPTW}(X_{out}, \{y_0\}, T, q)$ can by found by the dynamic programming in time $O\left(qk^2 2^k\right)$. Then, to employ for the inner subinstance $\mathrm{CVRPTW}(X_{in}, \{y_0\}, T, q)$ the ITP heuristic, we need to found an approximate solution for the auxiliary instance of the TSP, which can be obtained, e.g. by the famous Christofides algorithm (Christofides, 1975) in time $O(n^3)$. Since, the running time of the ITP itself and the final solution decoding can be performed in time $O(n^2)$, the overall complexity of our scheme for the CVRPTW1 is $O\left(mn + n^3 + qk^2 2^k\right)$.

In the case of CVRPTW2 an exact solution for the subinstance $\mathrm{CVRPTW2}(X_{out}, Y, T, q)$ can be found by dynamic programming in $O(m \cdot k^q 2^k)$ (see, e.g. (Cardon et al., 2008)). Since, in this case, all other steps of the scheme have the same complexity as for the CVRPTW1, we obtain the the following overall time complexity $O\left(mn + n^3 + mk^q 2^k\right)$.

Summarizing, we obtain the following theorem.

Theorem 1. For any $\varepsilon > 0$ an $(1+\varepsilon)$-approximate solution for the Euclidean CVRPTW1 and CVRPTW2 can be obtained in time $O\left(mn + n^3 + qk^2 2^k\right)$ and $O\left(mn + n^3 + mk^q 2^k\right)$, respectively, where $k = k(\varepsilon, p, q, m) = O\left(mp \exp\left(O(q/\varepsilon)\right) \exp\left(O(\sqrt{mpq/\varepsilon})\right)\right)$.

As it follows from Theorem 1, the scheme proposed is EPTAS for any fixed capacity q, number of time windows p, and amount of the depots m with running time

$$O\left(n^3 + \exp(\exp(O(1/\varepsilon)))\right) \tag{6}$$

and remains PTAS for $q = o(\log \log n)$ and $mp = o(\log \log n)$.

4. IMPROVED SCHEME

Although the approximation scheme proposed in the previous section is the first PTAS for the CVRPTW with multiple depots, its time complexity (6) still remains impractical due to its double exponent on ε. In this section, we obtain another approximation scheme as a trade-off between the improvement (up to exp) its time complexity and the opportunity to solve the *inner* and *outer* subinstances in parallel.

Following to (Asano et al., 1997), we proceed with the introduction of an auxiliary problem called the Partial Capacitated Vehicle Routing Problem (PCVRPTW). The settings of the PCVRPTW and CVRPTW seem to be very similar to each other. As for the CVRPTW problem, the PCVRPTW is given by a weighted complete digraph $G = (X \cup Y, E, w)$ specifying customer and depot locations in combination with the transportation costs, partition $X_1 \cup \ldots \cup X_p = X$ of the customer set X to subsets induced by time windows $\{t_1, \ldots, t_p\}$, and a common vehicle capacity q.

The main difference of PCVRPTW compared to the CVRPTW is that the feasible solutions should not visit all the customers but only so called *privileged* customers belonging a given subset $Q \subset X$. If a customer is omitted, a penalty term is included to the total cost function as follows. Let $S = \{R_1, \ldots R_l\}$ be an arbitrary feasible solution. By $X[S]$ and $\bar{X}[S]$ we denote the customer subsets visited and omitted by the solution S, respectively. The total cost of the solution S is defined by the equation

$$\operatorname{cost}(S) = w(S) + \frac{2}{q} \sum_{x \in \bar{X}[S]} r(x), \tag{7}$$

where $w(S)$ is determined by equation (5) and $r(x)$ denotes the distance the customer x and its closest depot $y(x)$. The goal is to find a feasible solution U minimizes (7).

As for the CVRPTW, for the instance of the PCVRRTW specified by a set of customers X, subset of privileged customers Q, set of depots Y, time windows collection Y, and capacity q, we use the notation $\mathrm{PCVRPTW}(X, Q, Y, T, q)$.

The main idea of our second scheme is also quite simple and presented as follows

(i) As for the first scheme, we start with the reordering a set of customers X by descending of the distances $r_i = \{w(x_i, y) \colon y \in Y\}$ from customers x_i to the set of depots Y. Then, for a given $\varepsilon > 0$, we find two numbers k and k' independent on the number of customers and decompose the set X to three subsets $X_{out} = \{x_1, \ldots x_{k-1}\}$, $X_{mid} = \{x_1, \ldots x_{k'-1}\}$ and $X_{in} = \{x_{k'}, \ldots x_n\}$ named *outer, middle and inner* customers, respectively. The main point is that we can find such fixed numbers k and k' for any given accuracy ε.

(ii) Then, applying the proposed smart search algorithm, we find an exact solution U for the $\mathrm{PCVRPTW}(X_{out} \cup X_{mid}, X_{out}, T, Y, q)$ instance.

(iii) The last stage of this scheme is similar to the first one. Employing our adaptation of the ITP heuristic, we find an approximate solution S_{ITP} for the *inner* subinstance $\mathrm{CVRPTW}CVRPTW(\bar{X}[U] \cup X_{in}, Y, T, q)$ and output the combined solution

$$S_{APP} = U \cup S_{ITP}.$$

In the sequel we propose a sketch of our smart search optimal for algorithm for the PCVRPTW. Consider an arbitrary instance $\mathrm{PCVRPTW}(Z, Q, Y, T, q)$. The main idea is based on the following observations.

(i) Since feasibility of any optimal solution $S = \{R_1, \ldots, R_l\}$ does not depend on the order of its routes, we can assume that the routes are ordered by decreasing of their individual capacities q_j, where q_j is equal to the number of customers visited by the route R_j.

(ii) Since any route $_j$ visits at least one customer from Q and each customer $x \in Q$ should be visited exactly once, the number l of routes can not exceed $b = |Q|$.

As it follows from these observations, to each solution S, we can assign the *shape* $\lambda = \lambda(S) = q_1, \ldots, q_l$, where $q \geq q_1 \geq q_2 \geq \ldots \geq q_l \geq 1$ and $l \leq b$. Further, any two solutions S_1 and S_2 we call *equivalent*, if $\lambda(S_1) = \lambda(S_2)$. Thus, the set of feasible solutions of the instance $\mathrm{PCVRPTW}(Z, Q, Y, T, q)$ is partitioned into equivalence classes. It is convenient to represent each class with corresponding *Young diagram* (see, e.g. (Andrews and Eriksson, 2004)).

Each j-th row of such a diagram corresponds to the j-th route excluding its starting and finishing node x_0. To obtain the one-to-one correspondence with a certain solution belonging to the considered equivalence class, we just fill each row of the diagram, left to right, cell by cell by customer locations in the order of visiting them by the corresponding route.

We start by preprocessing of the given instance that depends on its type. If we are given by a PCVRPTW1 instance, where each route is free to depart from and arrive to the different depots, we introduce the *fictitious* depot y_0 again. Otherwise, if we are faced to the instance of the PCVRPTW2 problem, where each route should return to the starting depot, the preliminary depot-to-route stage is included to the algorithm, which enlarges time complexity to the factor m^b. Then, we enumerate all possible shapes for the given q and b.

Further, given a certain shape, we enumerate all the possible injections of the set Q into the cell set of the corresponding Young diagram and filter out injections that induce infeasible solutions (violating time windows constraints).

At the last stage, for any partially filled diagram (by the points of Q), we enumerate all injections of the subset of its free cells into the set $Z \setminus Q$. For any such an injection, we check out whether the candidate solution obtained is feasible and (if so) compute a value of objective function (7). Finally, we output the least cost feasible solution.

Coming back to the approximation scheme, we remind that, in our case, $Q = X_{out}$ and $b = k - 1$. The final result is presented in the following theorem extending our recent results obtained for the case of $m = 1$ (Khachai and Ogorodnikov, 2018; Khachay and Ogorodnikov, 2019).

Theorem 2. For any $\varepsilon > 0$ an $(1+\varepsilon)$-approximate solution for the Euclidean CVRPTW can be obtained in time

$$TIME(\mathrm{TSP}, \rho, n) + O(n^2) + \\ + O\left(e^{O\left(q\left(\frac{q}{\varepsilon}\right)^3 (p\rho)^2 \log(p\rho)\right)}\right) O(m^k),$$

where

$$k = k(\varepsilon, q, p, \rho) = O\left((qp\rho/\varepsilon)^2\right)$$
and $TIME(\text{TSP}, \rho, n)$ denotes time complexity of finding ρ-approximate solution for the auxiliary metric TSP instance.

Note, that the scheme proposed is PTAS for $p^3q^4 = O(\log n)$ and $(pq)^2 \log m = O(\log n)$. Furthermore, for any fixed $p \geq 1$ and $q \geq 1$, the scheme is PTAS with time complexity
$$O\left(n^3 + m^{O(1/\varepsilon^2)} \exp(O(1/\varepsilon^3))\right)$$
provided the inner TSP instance is solved by the Christofides algorithm.

5. CONCLUSION

In this paper, we proposed sketches of two approximation algorithms for the Euclidean Capacitated Vehicle Routing Problem with Time Windows and multiple depots, both for the setting CVRPTW1, where we are free to start and finish any route at different depots, and for the setting CVRPTW2, where each route is constrained to depart from and arrive to the same depot. To the best of our knowledge, these algorithms are the first known Polynomial Time Approximation schemes (PTAS) for the problem in question.

For the sake of brevity, we skip the rigorous mathematical proofs of the presented results, which will be provided in the forthcoming paper. Also, we plan to extend our result for the case of Euclidean space of an arbitrary fixed dimension $d > 1$ and splittable demand soon.

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