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Towards a relaxation of the pursuit-evasion differential game

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Abstract: We consider a special case of the non-linear zero-sum pursuit-evasion differential game. The instance of this game is defined by two closed sets - target set and one specifying state constraints. We find an optimal non-anticipating strategy for player I (the pursuer). Namely, we construct his successful solvability set specified by limit function of the iterative procedure in space of positions. For positions located outside the successful solvability set, we provide a relaxation of our game by determining the smallest size of a neighborhoods of two mentioned sets, for which the pursuer can solve his problem successfully. Then, we construct his successful solvability set in terms of those neighborhoods.

Keywords: Pursuit-Evasion differential game, non-anticipating strategies, program iterations method.

1. INTRODUCTION AND RELATED WORK

During the last decades, differential game theory has been an actively developing field of operations research and control theory. The first mention of such a differential game goes back to R. Isaacs. In the well-known study (Isaacs, 1969), Isaacs overviewed a number of applications that can be reduced to a model of differential game. These applications are of great practical importance. Later, this theory was significantly developed by Soviet mathematicians L.S. Pontryagin (Pontrjagin, 1981), N.N. Krasovskii (Krasovskii, 1970) and improved by A.B. Kurzhanski, Yu.S. Osipov and A.N. Subbotin.

In this paper, we consider the non-linear zero-sum pursuit-evasion differential game, defined by two closed sets in space of positions. This setting was introduced in (Krasovskii and Subbotin, 1970, 1987). For differential game considered in (Krasovskii and Subbotin, 1970), the fundamental Theorem on alternative was established. The important generalization of this result was obtained by (Kryazhimskii, 1978). According to theorem of alternative, the set defining state constraints, can be split into two disjoint subsets, specifying successful solvability sets for each player. If differential game satisfies Isaacs condition, then the alternative can be implemented in terms of pure positional strategies and, of course, in terms of non-anticipating strategies (Roxin, 1969; Elliott and Kalton, 1972). In literature those strategies are also known as Elliott-Kalton strategies (Elliott and Kalton, 1972) and quasi-strategies (Subbotin and Chentsov, 1981). Multivalued cases of non-anticipating strategies were considered in (Chentsov, 1976; Subbotin and Chentsov, 1981). The problem of construction of an alternative partition can be reduced to problem of finding the set of successful solvability of a pursuer (player I) (that is, the maximum stable bridge by N.N. Krasovskii), who is interested in the guaranteed feasibility of approach. Various methods were used to build this set. In (Ushakov et al., 2015), special kind of procedures constructed by dynamic programming was considered. Besides that, the program constructions can be used to find a solutions for differential game (see (Krasovskii and Subbotin, 1987; Krasovskii, 1988a,b) and others).

For the general case of the differential game in question, the program iterations method was introduced in (Chentsov, 1976) (see also (Chentsov, 2017a)). In our case, we use adaptation of the program iterations method, also known as stability iterations technique (Chentsov, 2017b). This approach provides a way to solve pursuit-evasion games with additional constraint on the number of control switchings (Chentsov, 2017b). More precisely, at each stage of the iterative procedure, we construct the successful solvability set.

In this study, we relax the initial setting of the considered pursuit-evasion differential game by analyzing possibilities of player I in terms of reachability of closed neighborhoods of target set within corresponding neighborhoods of state constraints set. Moreover, for each fixed position, our goal is to find the minimal neighborhood, guaranteeing the solvability of the pursuit problem. Also, such a neighborhood estimates possibilities of player II (evader) in following way: for smaller neighborhood, one can perform evasion procedure with finite number of switchings. Based on this approach, we can define minimax function, which values can be represented as guaranteed result for special payoff. To define this function, we construct special iterative procedure in space of position functions; the function itself is a fixed point of special conversion operator.

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2. PROBLEM STATEMENT

Consider following control system on finite time interval $T \triangleq [t_0, \vartheta_0]$:

$$\dot{x} = f(t, x, u, v), \quad u \in P, \ v \in Q, \tag{1}$$

where P and Q are non-empty compact sets in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, $p, q \in \mathbb{N}$. As $u(\cdot)$ we define control of player I and as $v(\cdot)$ - control of player II. Also we assume that system 1 satisfies the conditions of generalized uniqueness and uniform boundedness of solutions, similar to one in (Chentsov, 2017b). We are given by two following sets:

$$(\mathbf{M} \in \mathcal{F}) \ \& \ (\mathbf{N} \in \mathcal{F}),$$

where $\mathcal{F}$ is the family of all subsets of $T \times \mathbb{R}^n$, closed in topological space, $\mathbf{M} \subset \mathbf{N}$, $\mathbf{M} \neq \emptyset$. First one is a target set for first player and second one defines state constraints in terms of set cross-sections: $\mathbf{N}\langle t\rangle \triangleq \{x \in \mathbb{R}^n | \ (t, x) \in \mathbf{N}\}$, $\forall t \in T$. We need to construct an approximation to $\mathbf{M}$ under state constraints $\mathbf{N}\langle t\rangle, t \in T$. Most of the time $(\mathbf{M}, \mathbf{N})$-approximation cannot be contructed, thus problem is intractable. However, we can relax this problem by defining some quality guarantee. Namely, let us consider $(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))$-approximation with guarantee $\varepsilon \in [0, \infty[$ (see (Chentsov and Khachay, 2018)). For fixated position (t_*, x_*), we find the smallest number $\varepsilon_* \in [0, \infty[$, for which $(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))$-approximation can be constructed in terms of admissible control procedures. Later, we show that this number exists.

Thus, according to such admissible procedure, for every trajectory $x(\cdot) = x(t)$, $t_* \leq t \leq \vartheta_0$, generating by given procedure, the following property

$$((\theta, x(\theta)) \in S_0(\mathbf{M}, \varepsilon_*)) \ \& \ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon_*) \ \forall t \in [t_*, \theta]) \tag{2}$$

where $\theta \in [t_*, \vartheta_0]$, holds. Moreover, for trajectory $x(\cdot)$ in (2), the equality $x(t_*) = x_*$ is fulfilled. Therefore, we can estimate ε_* by two following values $\rho((t_*, x_*), \mathbf{M})$ and $\rho((t_*, x_*), \mathbf{N})$. In addition, by choice of $\mathbf{M}$ and $\mathbf{N}$

$$\rho((t_*, x_*), \mathbf{N}) \leq \rho((t_*, x_*), \mathbf{M}),$$

where $\rho(\cdot, H)$ was designated in (Chentsov and Khachay, 2018). We suppose that $\varepsilon_*^{(1)} \triangleq \rho((t_*, x_*), \mathbf{N})$ and $\varepsilon_*^{(2)} \triangleq \rho((t_*, x_*), \mathbf{M})$. Then, by (2), $\varepsilon_*^{(1)} \leq \varepsilon_*^{(2)}$. Therefore, it is obvious that $(S_0(\mathbf{M}, \varepsilon_*^{(2)}), S_0(\mathbf{N}, \varepsilon_*^{(2)}))$-approximation can be constructed. By the choice of ε_*, we obtain $\varepsilon_* \leq \varepsilon_*^{(2)}$.

On the other hand, from (2), for admissible procedure, which constructs $(S_0(\mathbf{M}, \varepsilon_*), S_0(\mathbf{N}, \varepsilon_*))$-approximation, we obtain $\varepsilon_*^{(1)} \leq \varepsilon_*$. This follows from (2) and definition of $\varepsilon_*^{(1)}$. Therefore, we obtain the chain of inequalities $\varepsilon_*^{(1)} \leq \varepsilon_* \leq \varepsilon_*^{(2)}$. Using definition of $\varepsilon_*^{(1)}$ and $\varepsilon_*^{(2)}$, we have

$$\rho((t_*, x_*), \mathbf{N}) \leq \varepsilon_* \leq \rho((t_*, x_*), \mathbf{M}). \tag{3}$$

Thus, by (3), we defined the range for optimal value of ε_* for every position from $T \times \mathbb{R}^n$. Therefore, $\rho((\cdot), \mathbf{N})$ and $\rho((\cdot), \mathbf{M})$ - are two boundary functions which define corresponding range in functional space.

We should pay special attention to construction of function ε_0 (see (Chentsov and Khachay, 2018)). In particular, we use sequence of functions $(\varepsilon_0^{(s)})_{s \in \mathbb{N}_0}$ constructed on each iteration with number s. Later, we explain how those functions relate in terms of iterations. Let us consider special kind of number sets. Each set is generated on respective stage of our iteration procedure. We denote them as follows (see (Chentsov and Khachay, 2018) and (Chentsov, 2017b))

$$\Sigma_0^{(s)}(t, x) \triangleq \{\varepsilon \in [0, \infty[| (t, x) \in \mathcal{W}_s(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))\} \quad \forall s \in \mathbb{N}_0 \ \forall (t, x) \in T \times \mathbb{R}^n, \tag{4}$$

$$\Sigma_0(t, x) \triangleq \{\varepsilon \in [0, \infty[\ | \ (t, x) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon), S_0(\mathbf{N}, \varepsilon))\}$$

where according to (Chentsov, 2017b), to each sets $M \in \mathcal{F}$ and $N \in \mathcal{F}$ we assign the sequence of sets $(\mathcal{W}_k(M, N))_{k \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathcal{P}(T \times \mathbb{R}^n)$ and limit set

$$\mathcal{W}(M, N) = \bigcap_{k \in \mathbb{N}_0} \mathcal{W}_k(M, N) \in \mathcal{P}(T \times \mathbb{R}^n).$$

Following (Chentsov and Khachay, 2018), we obtain

$$\varepsilon_0(t, x) \triangleq \inf(\Sigma_0(t, x)) \in [0, \infty[.$$

Proposition 1. If $(t_*, x_*) \in T \times \mathbb{R}^n$, then $\varepsilon_0(t_*, x_*) \in \Sigma_0(t_*, x_*)$.

As a corollary, we note that

$$(t, x) \in \mathcal{W}(S_0(\mathbf{M}, \varepsilon_0(t, x)), S_0(\mathbf{N}, \varepsilon_0(t, x))) \quad \forall (t, x) \in T \times \mathbb{R}^n. \tag{5}$$

Let us introduce another option for functional on trajectories of the process. If $t_* \in T, x(\cdot) \in C_n([t_*, \vartheta_0])$ and $\theta \in [t_*, \vartheta_0]$, then we assume that

$$\omega(t_*, x(\cdot), \theta) = \sup\ (\{\rho((\theta, x(\theta)), \mathbf{M}); \max_{t_* \leq t \leq \theta} \rho((t, x(t)), \mathbf{N})\}), \tag{6}$$

$\omega(t_*, x(\cdot), \theta) \in [0, \infty[$. As a corollary, we define for $t_* \in T$ special payoff function

$$\gamma_{t_*} : C_n([t_*, \vartheta_0]) \to [0, \infty[, \tag{7}$$

by following conditions: $\forall \ x(\cdot) \in C_n([t_*, \vartheta_0])$.

$$\gamma_{t_*}(x(\cdot)) \triangleq \min_{\theta \in [t_*, \vartheta_0]} \omega(t_*, x(\cdot), \theta). \tag{8}$$

Minimum in (8) is attainable, since $\rho((\cdot, x(\cdot)), \mathbf{M})$ and $\rho((\cdot, x(\cdot)), \mathbf{N})$ - are uniformly continuous functions (see also (6)).

Proposition 2. Let $t_* \in T$, $x(\cdot) \in C_n([t_*, \vartheta_0])$ and $\varepsilon_* \in [0, \infty[$ be given. Then, following two conditions are equivalent:

((i)) $\exists \vartheta \in [t_*, \vartheta_0] : \ ((\vartheta, x(\vartheta)) \in S_0(\mathbf{M}, \varepsilon_*)) \ \& \ ((t, x(t)) \in S_0(\mathbf{N}, \varepsilon_*) \ \ \forall t \in [t_*, \vartheta])$;

((ii)) $\gamma_{t_*}(x(\cdot)) \leq \varepsilon_*$

We recall that $\mathbf{M} \subset \mathbf{N}$. If $\varepsilon \in [0, \infty[$, then $S_0(\mathbf{M}, \varepsilon) \subset S_0(\mathbf{N}, \varepsilon)$.

Hereinafter, we will consider special multivalued non-anticipating strategies as admissible control procedures. We define considered strategies in terms of non-anticipating operators on control-measure spaces. For this, it is helpful to recall some of the properties of measurable spaces, used in paragraph 2. We should improve some constructions given there. For arbitrary $t \in T$, let's fix compactum $Z_t \triangleq [t, \vartheta_0] \times Q$ and σ-algebra $\mathcal{D}_t$ of Borel subsets of Z_t.

Wherein, for moments of time $t_1 \in T$ and $t_2 \in [t_1, \vartheta_0]$ $\mathcal{D}_{t_2} = \mathcal{D}_{t_1}|_{Z_{t_2}} = \{D \in \mathcal{D}_{t_1} | D \subset Z_{t_2}\}$. Further, if $t \in T$, then following holds: $\Gamma \times Q \in \mathcal{D}_t \quad \forall \Gamma \in \mathcal{T}_t$. Following those constructions, we consider the set of measures, which are similar to Borel mappings from $[t, \vartheta_0]$ to Q, namely: $\mathcal{E}_t \triangleq \{\nu \in (\sigma - \mathrm{add})_+[\mathcal{D}_t] \mid \nu(\Gamma \times Q) = \lambda_t(\Gamma) \ \ \forall \Gamma \in \mathcal{T}_t\}$; we use $\mathcal{E}_t$ as set of generalized controls of player II. Moreover, we note that $D \stackrel{\times}{\_} P \triangleq \{(t, u, v) \in \Omega_t \mid (t, v) \in D\} \in \mathcal{C}_t \quad \forall D \in \mathcal{D}_t$. Therefore, we have the following (see (Chentsov, 2017a))

$$\Pi_t(\nu) \triangleq \{\eta \in \mathcal{H}_t \mid \eta(D \stackrel{\times}{\_} P) = \nu(D) \ \ \forall D \in \mathcal{D}_t\} \ \ \forall \nu \in \mathcal{E}_t. \tag{9}$$

Thus, the family of programs for second player on a interval $[t, \vartheta_0]$ was introduced.

We also need additional operations for generalized controls, namely gluing and constriction. Let us recall another notation and define corresponding designation: if X and Y - nonempty sets, where $h \in Y^X$ and $\tilde{X} \in \mathcal{P}'(X)$, then $(h|\tilde{X}) \triangleq (h(x))_{x \in \tilde{X}} \in Y^{\tilde{X}}$. As X a family can be used and as h — set function.

If $t_1 \in T$ and $t_2 \in [t_1, \vartheta_0]$, then (see (Chentsov, 2017a))

$$\mathcal{C}_{t_2} = \mathcal{C}_{t_1} |_{\Omega_{t_2}} = \{H \in \mathcal{C}_{t_1} \mid H \subset \Omega_{t_2}\},$$

$$\mathcal{C}_{t_1}^{t_2} = \mathcal{C}_{t_1} |_{[t_1, t_2[\times P \times Q} = \{H \in \mathcal{C}_{t_1} \mid H \subset [t_1, t_2[\times P \times Q\},$$

$$\mathcal{D}_{t_1}^{t_2} = \mathcal{D}_{t_1} |_{[t_1, t_2[\times Q} = \{D \in \mathcal{D}_{t_1} \mid D \subset [t_1, t_2[\times Q\}.$$

In this case we have σ-algebras of sets. It is useful to note that (see (Chentsov, 2017a))

$$\mathcal{H}_{t_2} = \{(\eta \mid \mathcal{C}_{t_2}) : \eta \in \mathcal{H}_{t_1}\}, \ \mathcal{E}_{t_2} = \{(\nu \mid \mathcal{D}_{t_2}) : \nu \in \mathcal{E}_{t_1}\}.$$

If $t_* \in T$, then we assume that $\tilde{A}_{t_*}$ is the set of all generalized multivalued non-anticipating strategies (see (Chentsov, 2017a)) for first player on a line $[t_*, \vartheta_0]$:

$$\begin{aligned}\tilde{A}_{t_*} \triangleq \{\alpha \in \prod_{\nu \in \mathcal{E}_{t_*}} \mathcal{P}'(\Pi_{t_*}(\nu)) \mid \forall \nu_1 \in \mathcal{E}_{t_*} \\ \forall \nu_2 \in \mathcal{E}_{t_*} \ \ \forall \theta \in [t_*, \vartheta_0] : ((\nu_1 \mid \mathcal{D}_{t_*}^{\theta}) \\ = (\nu_2 \mid \mathcal{D}_{t_*}^{\theta})) \Rightarrow (\{(\eta \mid \mathcal{C}_{t_*}^{\theta}) : \eta \in \alpha(\nu_1)\} \\ = \{(\eta \mid \mathcal{C}_{t_*}^{\theta}) : \eta \in \alpha(\nu_2)\})\}.\end{aligned} \tag{10}$$

If $\alpha \in \tilde{A}_{t_*}$ and $\nu \in \mathcal{E}_{t_*}$, then as $\alpha(\nu)$ we have nonempty subset of $\Pi_{t_*}(\nu)$; in particular, $\alpha(\nu) \subset \mathcal{H}_{t_*}$. Thus, we defined following union-set (see (Chentsov, 2017a))

$$\tilde{\Pi}_{t_*}(\alpha) \triangleq \bigcup_{\nu \in \mathcal{E}_{t_*}} \alpha(\nu) \in \mathcal{P}'(\mathcal{H}_{t_*}), \tag{11}$$

which is the set of all generalized controls-measures, generated by non-anticipating strategy α. From Proposition 1 we have following:

Proposition 3. If $(t_*, x_*) \in T \times \mathbb{R}^n$, then

$$\varepsilon_0(t_*, x_*) = \inf_{\alpha \in \tilde{A}_{t_*}} \sup_{\eta \in \tilde{\Pi}_{t_*}(\alpha)} \gamma_{t_*}(\varphi(\cdot, t_*, x_*, \eta)),$$

wherein $\exists \tilde{\alpha}_* \in \tilde{A}_{t_*} : \varepsilon_0(t_*, x_*) = \tilde{\Pi}_{t_*}(\tilde{\alpha}_*)$.

Therefore it is established that function $\varepsilon_0 : T \times \mathbb{R} \to [0, \infty[$, is equal to minimax of payoff function γ in terms of non-anticipating strategies for any fixed position.

Proposition 4. If $s \in \mathbb{N}_0$ and $(t, x) \in T \times \mathbb{R}^n$, then $\Sigma_0(t, x) \subset \Sigma_0^{(s)}(t, x)$.

From Proposition 4 we obtain

$$\varepsilon_0(t, x) \in \Sigma_0^{(s)}(t, x) \ \ \forall (t, x) \in T \times \mathbb{R}^n \ \ \forall s \in \mathbb{N}_0. \tag{12}$$

In particular, if $(t, x) \in T \times \mathbb{R}^n$ and $s \in \mathbb{N}_0$ as $\Sigma_0^{(s)}(t, x)$, we have non-empty subset of $[0, \infty[$, thus, $\inf(\Sigma_0^{(s)}(t, x)) \in [0, \infty[$ is defined.

Hereinafter, we assume

$$\varepsilon_0^{(s)}(t, x) \triangleq \inf(\Sigma_0^{(s)}(t, x)) \ \forall (t, x) \in T \times \mathbb{R}^n \ \ \forall s \in \mathbb{N}_0. \tag{13}$$

Using (13) with each $s \in \mathbb{N}_0$, we can define function

$$\varepsilon_0^{(s)} : T \times \mathbb{R}^n \to [0, \infty[. \tag{14}$$

Also, from (13) and Proposition 4, we get

$$\varepsilon_0^{(s)}(t, x) \leq \varepsilon_0(t, x) \ \ \forall (t, x) \in T \times \mathbb{R}^n \ \ \forall s \in \mathbb{N}_0. \tag{15}$$

Let us designate point-wise order in the set of all functions from $T \times \mathbb{R}^n$ into $[0, \infty[$ by $\leqq$. Then from (15) we have

$$\varepsilon_0^{(s)} \leqq \varepsilon_0 \ \ \forall s \in \mathbb{N}_0. \tag{16}$$

Proposition 5. If $s \in \mathbb{N}_0$ and $(t_*, x_*) \in T \times \mathbb{R}^n$, then $\varepsilon_0^{(s)}(t_*, x_*) \in \Sigma_0^{(s)}(t_*, x_*)$.

From (4) we have property $\varepsilon_0 \in \Sigma_0^{(s)}(t_*, x_*)$.

Also, from (15) we show, in particular, that exact upper bound is defined properly. That is,

$$\sup(\{\varepsilon_0^{(s)}(t, x) : s \in \mathbb{N}_0\}) \in [0, \varepsilon_0(t, x)] \ \forall (t, x) \in T \times \mathbb{R}^n.$$

Proposition 6. If $(t_*, x_*) \in T \times \mathbb{R}^n$, then $\varepsilon_0(t_*, x_*) = \sup(\{\varepsilon_0^{(s)}(t_*, x_*) : s \in \mathbb{N}_0\})$

We consider another proposition which will establish relations between $\varepsilon_0^{(s)}(t_*, x_*)$ and $\varepsilon_0^{(s+1)}(t_*, x_*)$.

Proposition 7. Let $(t_*, x_*) \in T \times \mathbb{R}^n$ and $s \in \mathbb{N}_0$. Then

$$\varepsilon_0^{(s)}(t_*, x_*) \leqq \varepsilon_0^{(s+1)}(t_*, x_*). \tag{17}$$

Finally, we have following property:

$$\varepsilon_0^{(s)} \leqq \varepsilon_0^{(s+1)} \ \ \forall s \in \mathbb{N}_0. \tag{18}$$

According to Proposition 3 in (Chentsov and Khachay, 2018), ε_0 is the minimax of special payoff in terms of non-anticipating strategies.

3. MAIN RESULTS

In this section we construct program operator, which will define for $s \in \mathbb{N}_0$ conversion from $\varepsilon_0^{(s)}$ to $\varepsilon_0^{(s+1)}$. To achieve this, we will use special type of construction, which is similar to one introduced in (Chentsov, 1978). The modification of program iterations method described in (Chentsov, 1978) corresponds to differential game with non-fixed moment of termination. Also, let us introduce new notations according to those in (Chentsov, 1978).

First of all, we introduce the function $\psi : T \times \mathbb{R}^n \to [0, \infty[$ by condition:

$$\psi(t, x) \triangleq \rho((t, x), \mathbf{M}) \ \ \forall (t, x) \in T \times \mathbb{R}^n. \tag{19}$$

According to the properties of the distance function from a point to a nonempty set $\mathbf{M}$ we have that $\psi \in C(T \times \mathbb{R}^n)$, where $C(T \times \mathbb{R}^n)$ - set of all continuous real-valued functions on $T \times \mathbb{R}^n$. We note that

$$\psi^{-1}([0, c]) \in \mathcal{F} \ \ \forall c \in [0, \infty[. \tag{20}$$

Thus, according to (20) and due to non-negativity of ψ, we have lower semi-continuity property.

Subsequently, for every non-empty set H as $\mathfrak{R}_+(H)$ we denote the set of all non-negative real-valued functions on H.

In terms of (20) we define, according to (Chentsov, 1978), the following set

$$\mathfrak{M} \triangleq \{g \in \mathfrak{R}_+(T \times \mathbb{R}^n) |\ g^{-1}([0, c]) \in \mathcal{F} \ \ \forall c \in [0, \infty[\}. \tag{21}$$

Besides, we require set with point-wise order

$$\mathfrak{M}_\psi \triangleq \{g \in \mathfrak{M} |\ g \leqq \psi\}; \tag{22}$$

where $\psi \in \mathfrak{M}_\psi$. Thus, (22) is non-empty subset of $\mathfrak{M}$.

3.1 Convergence operator

In this subsection, we construct special convergence operator for ε_0. For the sake of brevity, we skip the rigorous mathematical proofs of the presented results. Interested reader can find them in (Chentsov and Khachay, 2018). If $g \in \mathfrak{M}_\psi$ and $(t_*, x_*) \in T \times \mathbb{R}^n$, then (since $g \leqq \psi$) we obtain for some $c \in [0, \infty[$

$$\{g(t, x(t)) :\ t \in [t_*, \vartheta]\} \in \mathcal{P}'([0, c]) \ \ \forall v \in Q$$
$$\forall x(\cdot) \in \mathcal{X}_\pi(t_*, x_*, v)\ \forall \vartheta \in [t_*, \vartheta_0], \tag{23}$$

where $\mathcal{X}_\pi(t_*, x_*, v)$ is used in the sense of (Chentsov and Khachay, 2018). Following property (23), we have useful corollary:

$$\sup_{t \in [t_*, \vartheta]} g(t, x(t)) \leq c \ \ \forall v \in Q$$
$$\forall x(\cdot) \in \mathcal{X}_\pi(t_*, x_*, v)\ \forall \vartheta \in [t_*, \vartheta_0]. \tag{24}$$

By (24), with $g \in \mathfrak{M}_\psi$ and $(t_*, x_*) \in T \times \mathbb{R}^n$, we obtain $\exists \tilde{c} \in [0, \infty[$:

$$\sup(\{\sup_{t \in [t_*, \bar{\vartheta}]} g(t, \bar{x}(t)); \psi(\bar{\vartheta}, \bar{x}(\vartheta))\}) \in [0, \tilde{c}]$$
$$\forall v \in Q\ \forall \bar{x}(\cdot) \in \mathcal{X}_\pi(t_*, x_*, v)\ \forall \bar{\vartheta} \in [t_*, \vartheta_0].$$

Then for $g \in \mathfrak{M}_\psi$ and $(t_*, x_*) \in T \times \mathbb{R}^n$, we obtain following

$$\sup_{v \in Q} \inf_{(x(\cdot), \vartheta) \in \mathcal{X}_\pi(t_*, x_*, v) \times [t_*, \vartheta_0]} \sup(\{\sup_{t \in [t_*, \vartheta]} g(t, x(t));$$
$$\psi(\vartheta, x(\vartheta))\}) \in [0, \infty[. \tag{25}$$

Considering (25), we define the convergence operator $\Gamma : \mathfrak{M}_\psi \to \mathfrak{R}_+[T \times \mathbb{R}^n]$ as follows:

$$\Gamma(g)(t_*, x_*) \triangleq \sup_{v \in Q} \inf_{(x(\cdot), \vartheta) \in \mathcal{X}_\pi(t_*, x_*, v) \times [t_*, \vartheta_0]} \sup(\{\sup_{t \in [t_*, \vartheta]}$$
$$g(t, x(t));\ \psi(\vartheta, x(\vartheta))\})\ \forall g \in \mathfrak{M}_\psi \ \ \forall (t_*, x_*) \in T \times \mathbb{R}^n.$$

If $g \in \mathfrak{M}_\psi$, $(t_*, x_*) \in T \times \mathbb{R}^n$ and $v \in Q$, then by $\mathfrak{h}[g; t_*; x_*; v]$ we denote the functional

$$(x(\cdot), \vartheta) \longmapsto \sup(\{\sup_{t \in [t_*, \vartheta]} g(t, x(t)); \psi(\vartheta, x(\vartheta))\}) :$$
$$\mathcal{X}_\pi(t_*, x_*, v) \times [t_*, \vartheta_0] \to [0, \infty[; \tag{26}$$

also, we consider following Lebesgue sets: for $b \in [0, \infty[$

$$\mathcal{Y}_b[g; t_*; x_*; v] \triangleq \{(x(\cdot), \vartheta) \in \mathcal{X}_\pi(t_*, x_*, v) \times [t_*, \vartheta_0] |$$
$$\mathfrak{h}[g; t_*; x_*; v](x(\cdot), \vartheta) \leq b\}.$$

Theorem 8. If $k \in \mathbb{N}_0$, then $\varepsilon_0^{(k+1)} = \Gamma(\varepsilon_0^{(k)})$.

3.2 Fixed point property of operator Γ

In the sequel, we consider an important property of operator Γ, namely its fixed point.

Theorem 9. Function ε_0 is the fixed point of operator Γ: $\varepsilon_0 = \Gamma(\varepsilon_0)$.

Let us consider the set of all fixed points of operator Γ, namely $\mathfrak{M}_\psi^{(\Gamma)} \triangleq \{g \in \mathfrak{M}_\psi \mid g = \Gamma(g)\}$. In regards to previous theorem, we obtain $\varepsilon_0 \in \mathfrak{M}_\psi^{(\Gamma)}$. Let $\tilde{\mathfrak{M}}_\psi^{(\Gamma)} \triangleq \{g \in \mathfrak{M}_\psi^{(\Gamma)} \mid \varepsilon_0^{(0)} \leqq g\} = \{g \in \mathfrak{M}_\psi^{(\Gamma)} \mid \rho(\cdot, \mathbf{N}) \leqq g\}$.

Theorem 10. Function ε_0 is the smallest element of $\tilde{\mathfrak{M}}_\psi^{(\Gamma)}$, in other words, $\varepsilon_0 \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}$ and $\varepsilon_0 \leqq \mathbf{g} \ \ \forall \mathbf{g} \in \tilde{\mathfrak{M}}_\psi^{(\Gamma)}$.

Let us consider one special case of our setting. Namely, let

$$(\mathbf{M} \triangleq T \times \mathcal{M})\ \&\ (\mathbf{N} \triangleq T \times \mathcal{N}), \tag{27}$$

where $\mathcal{M}$ and $\mathcal{N}$ are closed non-empty sets in $\mathbb{R}^n$ with usual topology generated by euclidean norm $\|\cdot\|$. Moreover, let $\mathcal{M} \subset \mathcal{N}$. If $H \in \mathcal{P}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, then we introduce

$$(\|\cdot\| - \inf)[x; H] \triangleq \inf(\{\|x - h\| :\ h \in H\}) \in [0, \infty[. \tag{28}$$

We obtain the following property: for $H \in \mathcal{P}'(\mathbb{R}^n)$ and $x_* \in \mathbb{R}^n$,

$$\rho((t, x_*), T \times H) = (\|\cdot\| - \inf)[x_*; H] \ \ \forall t \in T. \tag{29}$$

Indeed, we fixate $t_* \in T$. Then,

$$\rho((t_*, x_*), T \times H) = \inf(\{\rho((t_*, x_*), (t, h)) : (t, h) \in T \times H\}).$$

In addition, for $h \in H$, we obtain that $(t, h) \in T \times H$, where $t \in T$, and

$$\|x_* - h\| \leq \rho((t_*, x_*), (t, h)). \tag{30}$$

Also, $(t_*, h) \in T \times H$ and $\rho((t_*, x_*), (t_*, h)) = \|x_* - h\|$. Wherein, $\rho((t_*, x_*), T \times H) \leq \rho((t_*, x_*), (t_*, h))$. As a corrollary,

$$\rho((t_*, x_*), T \times H) \leq \|x_* - h\|. \tag{31}$$

Since the selection of h was arbitrary, by (28)

$$\rho((t_*, x_*), T \times H) \leq (\|\cdot\| - \inf)[x_*; H]. \tag{32}$$

On the other hand, from (30), we have that

$$(\|\cdot\| - \inf)[x_*; H] \leq \rho((t_*, x_*), (t, \tilde{h}))\ \forall t \in T\ \forall \tilde{h} \in H.$$

Then $(\|\cdot\| - \inf)[x_*; H] \leq \rho((t_*, x_*), T \times H)$, and by (32), we obtain $\rho((t_*, x_*), T \times H) = (\|\cdot\| - \inf)[x_*; H]$. Since the selection of t_* was arbitrary, the property (29) is established. So, by (27) and (29), $\forall (t, x) \in T \times \mathbb{R}^n$

$$(\rho((t, x), \mathbf{M}) = (\|\cdot\| - \inf)[x; \mathcal{M}])\ \&$$
$$(\rho((t, x), \mathbf{N}) = (\|\cdot\| - \inf)[x; \mathcal{N}]).$$

Therefore, we obtain that

$$\omega(t, x(\cdot), \theta) \triangleq \sup(\{(\|\cdot\| - \inf)[x(\theta); \mathcal{M}];\ \max_{t \leq \xi \leq \theta} (\|\cdot\| - \inf)$$
$$[x(\xi); \mathcal{N}]\}) \quad \forall t \in T\ \forall x(\cdot) \in C_n([t, \vartheta_0])\ \forall \theta \in [t, \vartheta_0].$$

As a corollary, in considered case (27), we obtain

$$\gamma_t(x(\cdot)) \triangleq \min_{\theta \in [t, \vartheta_0]} \sup(\{(\|\cdot\| - \inf)[x(\theta); \mathcal{M}];$$
$$\max_{t \leq \xi \leq \theta} (\|\cdot\| - \inf)[x(\xi); \mathcal{N}]\}) \quad \forall t \in T\ \forall x(\cdot) \in C_n([t, \vartheta_0]).$$

Thus we have natural functional, which defines quality of pursuit.

CONCLUSION

In this paper, we have considered a relaxation of the non-linear zero-sum pursuit-evasion differential game. For player I, we have constructed the optimal strategy, namely, we have defined minimax function, which values are represented as guaranteed result for special type of payoff. Then, we have constructed special iterative procedure in space of positions by finding convergence operator Γ and proving that minimax function is its fixed point.

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