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The HJB Approach and State Estimation for Control Systems with Uncertainty ⋆

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Abstract: This paper deals with the comparison principle for the first-order ODEs of the Hamilton - Jacobi - Bellman (HJB) type which describe solutions of the dynamical control system with a special structure, with combined nonlinearity of quadratic and bilinear kinds and with uncertainty in initial states and in system parameters. The uncertainty studied here is of set-membership type when only the bounding set for unknown items is given. The ellipsoidal estimates of reachable sets are derived using the special structure of studied control system. The techniques of generalized solutions of Hamilton - Jacobi - Bellman inequalities together with previously established results of ellipsoidal calculus are applied to find the external set-valued estimates of reachable sets as the level sets of a related cost functional.

Keywords: Control problems, Uncertainty, Nonlinearity, Estimation, Reachable sets.

1. INTRODUCTION

In this paper the nonlinear dynamical control systems with unknown but bounded uncertainties related to the case of a set-membership description of uncertainty (Kurzhanski (1974, 1977, 1991, 2010); Milanese et al. (1996); Milanese and Vicino (1991); Schweppe (1973)) are studied.

The main goal here is to construct for control systems with uncertainty related reachable sets or, if it is not possible to find the reachable sets precisely, to construct their outer (external) or inner (internal) estimates. Several approaches developed earlier may be used for these purposes but most of them are suitable only for the case of systems with linear dynamics (Chernousko (1994); Kurzhanski and Valyi (1997); Kurzhanski and Varaiya (2014); Polyak et al. (2004); Milanese et al. (1996); Kostousova and Kurzhanski (1996); Walter and Pronzato (1997)).

However, the nonlinear control systems constitute indisputably the important object of study both for the optimal control theory and for various applications (e.g., Apreutesei (2009); August and Koeppl (2012); Blanchini and Miani (2015); Boscain et al. (2013); Ceccarelli et al. (2004); Gusev (2016); Kuntsevich and Volosov (2015); Kishida and Braatz (2015)). The key issue in nonlinear set-membership estimation is to find suitable techniques, which are easy to interpret and which produce related estimates (of external and internal types) for the set of unknown system states without being too computationally demanding. Some results of this kind may be derived from discrete approximation techniques for differential inclusions (Baier et al. (2013); Häckl (1996); Kurzhanski and Filippova (1993); Filippova (1993)) but because of their generality they produce very complicated algorithms that require a huge amount of computer memory and very long and complex calculations.

In this paper the modified state estimation approaches which use the special bilinear–quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus (Kurzhanski and Valyi (1997); Kurzhanski and Varaiya (2014); Chernousko (1994); Mazurenko (2012); Sinyakov (2015)) are presented. Here techniques related to constructing external set-valued estimates of reachable sets for nonlinear control systems and based on results and on related techniques of the theory of generalized solutions of Hamilton - Jacobi - Bellman equations and inequalities are developed. Solutions to equations of the HJB type are rather difficult to calculate, and the design of related computational algorithms is important. However, for many applied problems one may often be satisfied with approximate solutions that may be achieved by substituting the original HJB equations with variational inequalities due to certain comparison principles (Bertsekas (1995); Kurzhanski (2006); Kurzhanski and Valyi (1997); Kurzhanski and Varaiya (2014); Gurman (1997); Gusev (2016)). Based on this key idea, it turns out to be possible to obtain estimates for the solutions of a nonlinear controlled system of the class under consideration.

The paper is organized as follows. After introducing some notations and definitions, the main problem is formulated in Section 2. The approaches related to estimates of reachable sets in nonlinear case and based on results of the theory of generalized solutions of Hamilton - Jacobi -

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Bellman inequalities are given in Section 3. Finally, some concluding remarks are given.

This study continues the researches presented in Filippova (2013, 2014, 2016); Filippova and Matviychuk (2014, 2015); Vzdornova and Filippova (2007) but here a more complicated case is considered, when the dynamical equations contain a nonlinearity of quadratic type and also contain bilinear terms defined by an uncertain matrix, this case is both of theoretical and of applied importance. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear dynamical systems with unknown but bounded disturbances and in modeling for applied problems of nonlinear control theory.

In this regard, as new challenges to the study of systems under consideration let us mention here new important issues, related to the recent theoretical and technological achievements on advanced modeling and control of various complex mechatronic systems with nonlinearity and uncertainty. Potential areas of possible theoretical interest or related applications include, in particular, modeling and identification of mechatronic (e.g., robotics, MEMS, motor actuation, hydraulic actuation) systems, nonlinear dynamic analysis of complex mechatronic systems, model-based advanced control of complex mechatronic systems, such as adaptive control, robust control, sliding-mode control, backstepping control, H-infinite control, etc.

2. PROBLEM FORMULATION

Introduce first some basic notations. Let $\mathbb{R}^n$ be the n–dimensional Euclidean space, $x'y = \sum_{i=1}^{n} x_i y_i$ be the inner product of $x, y \in \mathbb{R}^n$ with a prime indicating a transpose, $\|x\| = (x'x)^{1/2}$. Let $\operatorname{comp}\mathbb{R}^n$ be the set of all compact subsets of $\mathbb{R}^n$, $h(A, B)$ is the Hausdorff distance between $A, B \in \operatorname{comp}\mathbb{R}^n$. Denote also $B(a, r) = \{x \in R^n : \|x - a\| \le r\}$, a symbol I will stand for the identity $n\times n$-matrix.

Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$–matrices and denote by $E(a, Q)$ the ellipsoid in $\mathbb{R}^n$, $E(a, Q) = \{x \in \mathbb{R}^n : (Q^{-1}(x-a),(x - a)) \le 1\}$ with center $a \in \mathbb{R}^n$ and symmetric positive definite $n \times n$–matrix Q, for any $n \times n$–matrix $M = \{m_{ij}\}$ denote $Tr(M) = \sum_{i=1}^{i=n} m_{ii}$.

The following nonlinear control system is studied here

$$\dot{x} = A(t)x + f(x)d + u(t), \quad x_0 \in X_0, \quad t \in [t_0, T], \tag{1}$$

where $x, d \in \mathbb{R}^n$, $\|x\| \le K$ $(K > 0)$, $f(x)$ is the nonlinear function, which is quadratic in x, $f(x) = x'Bx$, with a symmetric and positive definite $n \times n$-matrix B.

The $n \times n$-matrix function $A(t)$ in (1) is assume to be of the form

$$A(t) = A^0 + A^1(t), \quad t \in [t_0, T], \tag{2}$$

where the $n \times n$-matrix A^0 is given and the measurable $n \times n$-matrix $A^1(t)$ is unknown but bounded, $A^1(t) \in \mathcal{A}^1$ $(t \in [t_0, T])$, that is

$$\begin{aligned} &A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1, \quad t \in [t_0, T], \\ &\mathcal{A}^1 = \big\{A=\{a_{ij}\}\in\mathbb{R}^{n\times n} : |a_{ij}|\le c_{ij}, \ i, j=1, \dots n\big\}, \end{aligned} \tag{3}$$

where the numbers $c_{ij} \ge 0$ $(i, j = 1, \dots n)$ are given.

Assume that X_0 in (1) is an ellipsoid,

$$X_0 = E(a_0, Q_0),$$

with a symmetric and positive definite matrix $Q_0 \in \mathbb{R}^{n\times n}$ and with a center a_0.

The set $\mathcal{U}$ of admissible controls $u(\cdot)$ in (1) consists of such functions $u(t)$ that are measurable in the sense of Lebesgue on $[t_0, T]$ and satisfy the following constraint $u(t) \in U$ for a.e. $t \in [t_0, T]$ (here U is a given set, $U \in \operatorname{comp}\mathbb{R}^n$).

Let the absolutely continuous function

$$x(t) = x\big(t; u(\cdot), A(\cdot), x_0\big)$$

be a solution to dynamical system (1)–(3) with initial state $x_0 \in X_0$, with admissible control $u(\cdot)$ and with a matrix $A(\cdot)$ satisfying (2)–(3). The *reachable set* $\mathcal{X}(t)$ at time t $(t_0 < t \le T)$ of system (1)–(3) is defined as follows,

$$\begin{aligned} \mathcal{X}(t) = \{x \in \mathbb{R}^n : \exists\, x_0\in\mathcal{X}_0, \ \exists\, u(\cdot)\in\mathcal{U}, \ \exists\, A(\cdot)\in\mathcal{A}, \\ x = x(t) = x\big(t; u(\cdot), A(\cdot), x_0\big)\big\}. \end{aligned}$$

It should be noted that the exact description of reachable sets $X(t)$ of a control system is a very difficult problem even in the case of linear dynamics of the control system. The estimation theory and related algorithms based on ideas of construction outer and inner ellipsoidal estimates of reachable sets for systems with a linear structure have been investigated and deeply developed in Kurzhanski and Valyi (1997); Chernousko (1994); Kurzhanski and Varaiya (2014). Some recent results devoted to problems of finding external (and in some cases internal) set-valued estimates of reachable sets $X(t)$ for special classes of nonlinear systems may be found in Filippova (2009, 2010, 2012, 2013, 2014, 2016); Mazurenko (2012); Filippova and Matviychuk (2014); Kostousova and Kurzhanski (1996); Kostousova (2013).

The approach presented here for estimating reachable sets of the system (1)–(3) uses the techniques of ellipsoidal calculus together with the techniques of Hamilton - Jacobi - Bellman (HJB) equations for nonlinear control systems with the nonlinearity described above and with the uncertainty in initial states, therefore this research establishes a connection between these two approaches to the estimation of unknown states of uncertain dynamical systems of the type considered.

3. MAIN RESULTS

Here the general scheme of the investigation is described with its implementation for the class of dynamical systems under study, also some illustrative examples are given.

3.1 Dynamic Programming Approach in General Case

Let us mention first some important results from the optimal control theory received in [Bertsekas (1995); Kurzhanski (2006); Kurzhanski and Varaiya (2014); Gusev (2016); Gurman (1997)] and related to the present study.

Consider the control system described by the ordinary differential equation

$$\dot{x} = f(t, x, u(t)), \quad t \in [t_0, T] \tag{4}$$

with a function $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ measurable in t and continuous in other variables. Here x stands for the state vector, t stands for time and control $u(\cdot)$ is a measurable function satisfying the constraints

$$u(\cdot) \in \mathcal{U} = \{u(\cdot) : u(t) \in U, \quad t \in [t_0, T]\} \tag{5}$$

where $U \in \mathrm{comp}\mathbb{R}^m$.

Let us assume that the initial condition $x(t_0)$ to the system (4) is unknown but bounded

$$x(t_0) = x_0, \quad x_0 \in X_0 \in \mathrm{comp}\mathbb{R}^n. \tag{6}$$

Let the absolutely continuous function

$$x(t) = x(t, u(\cdot), t_0, x_0)$$

be a solution to (4) with initial state x_0 satisfying (6) and with control function $u(t)$ satisfying (5).

Consider the control system (4)–(6) and assume in this Section that the function $f(t, x, u)$ in (4) is continuous in all variables and has continuous partial derivatives with respect to x. Suppose also that conditions providing the extendability of solutions to (4)–(6) on the interval $[t_0, T]$ are satisfied.

Denote by $X(t) = X(t;\ t_0, X_0)$ the reachable set of the system (4)–(6) at time t. It is known that the reachable set may be expressed as a level set of a value function for an auxiliary control problem (Kurzhanski (2006); Kurzhanski and Varaiya (2014); Gusev (2016)). The value function for the auxiliary problem is the solution to the following Hamilton - Jacobi - Bellman (HJB) equation

$$V_t(t, x) + \max_{u \in U} (V_x, f(t, x, u)) = 0. \tag{7}$$

In the common situation the value function may be not differentiable, in this case a solution to the HJB equation is treated as the viscosity (or minmax) solution (Lions (1982); Fleming and Soner (2006); Subbotin (1995)). The precise solutions to such equations of the HJB type are rather difficult to calculate. The use of corresponding variational HJB inequalities and related comparison theorems instead makes it possible to obtain approximate estimates of reachable sets (e.g., Kurzhanski (2006); Kurzhanski and Varaiya (2014); Gusev (2016); Gusev and Kurzhanski (2007)).

Consider the following auxiliary result.

Lemma 1. (Kurzhanski (2006)). Assume that there exists a function $\mu(t)$ integrable on $[t_0, T]$ and such that

$$V_t(t, x) + \max_{u \in U} (V_x, f(t, x, u)) \le \mu(t), \quad t_0 \le t \le T. \tag{8}$$

Then the following external estimate of the reachable set $X(t)$ of the system (4)–(6) is true

$$X(t) \subseteq \{\ x : V(t, x)\ \le\ \int_{t_0}^{t} \mu(s)ds\ +\ \max_{x \in X_0}\ V(t_0, x)\ \}, \quad t_0 \le t \le T. \tag{9}$$

Note that, without loss of generality, it is possible to take $\mu(s) = 0$ in (8) (Gusev and Kurzhanski (2007)).

Instead of (8), consider now the following inequality of a more general type

$$V_t(t, x) + \max_{u \in U} (V_x, f(t, x, u)) \le g(t, V(t, x)), \quad t_0 \le t \le T, \tag{10}$$

where $g(t, V)$ is integrable in $t \in [t_0, T]$ and is continuously differentiable in variable V.

Consider the following ordinary differential equation

$$\dot{U}(t) = g(t, U), \quad U(t_0) = U_0, \quad t_0 \le t \le T, \tag{11}$$

which is called a *comparison equation* for (4)–(6).

Theorem 1. (Gurman (1997); Gusev (2016)). Assume the relations (10) and (11) are fulfilled. Assume also that

$$\max_{x \in X_0}\ V(t_0, x)\ \le\ U_0. \tag{12}$$

Then the following upper estimate is valid

$$X(t) \subseteq \{x : V(t, x) \le U(t)\}, \quad t_0 \le t \le T. \tag{13}$$

The above results will be used further, in estimating the attainability sets of the system (1)–(3) with uncertainty and nonlinearity.

3.2 External Estimates of Reachable Sets of Control Systems under Uncertainty via HJB Results

Now rewrite the studied system (1)–(3) in the form of the corresponding differential inclusion (Filippov (1988); Kurzhanski and Filippova (1993); Filippova (2012)),

$$\dot{x} \in \mathcal{A}x + f(x)d + U, \quad x_0 \in X_0, \quad t \in [t_0, T], \tag{14}$$

and assume further that

$$U = E(\hat{a}, \hat{Q}), \tag{15}$$

with $\hat{a} \in \mathbb{R}^n$ and a symmetric and positive definite $n \times n$-matrix $\hat{Q}$.

The solution of the problem of estimating the unknown states of nonlinear differential inclusion (14) and the study of related problems of control synthesis may be reduced to the investigation of first order PDEs of the Hamilton-Jacobi-Bellman (HJB) type and of their modifications and generalizations. This approach was outlined earlier in the paper Filippova (2013), where it was studied however a system of a simpler type, when the uncertainty in the matrix coefficients was not assumed. The class of control systems of type (14)–(15) represents a more complicated case for the analysis and for the investigation, since this system can be referred as the type of nonlinear systems with bilinear-quadratic dynamics. Recent results related to different aspects of the study of such systems may be found also in Filippova (2016).

Consider the following HJB-type inequality

$$V_t(t, x) + \max_{u \in E(\hat{a}, \hat{Q})} \max_{A \in \mathcal{A}} (V_x, Ax + f(x)d + u) \le 0 \tag{16}$$

with the boundary condition

$$V(t_0, x) = \phi(x) \le 0 \tag{17}$$

where $\phi(x)$ is a given continuously differentiable function.

Denote by $k > 0$ the minimal positive number for which the inclusion

$$X_0 = E(a_0, Q_0) \subseteq E(a_0, k^2 B^{-1})$$

is true.

Define the functions $a^+(t)$ and $r^+(t)$ as the solutions of the following nonlinear ordinary differential equations,

$$\begin{gathered}
\dot{a}^+(t) = \tilde{A}^0 a^+(t) + a^{+\prime}(t) B a^+(t) d + r^+(t) d + \hat{a}, \\
\tilde{A}^0 = A^0 + 2d \cdot a_0' B, \\
\dot{r}^+(t) = \max_{\|l\|=1} \{ l' (2\tilde{B}_+(t) B^{1/2} \tilde{Q}(t) B^{1/2} + \\
B^{1/2} q_+^{-1}(t) G(t) B^{1/2}) l \} + q_+(t) r^+(t), \\
q_+(t) = \{ (n r^+(t))^{-1} \mathrm{Tr}(B G(t)) \}^{1/2}, \\
\tilde{B}_+(t) = B^{1/2} (A^0 + 2d \cdot (a^+(t))' B) B^{-1/2},
\end{gathered} \tag{18}$$

$$G(t) = \text{diag}\Big\{(n-v)\Big(\sum_{i=1}^{n} c_{ji}|a_i^+(t)| + \Big(\max_{\sigma=\{\sigma_{ij}\}} \sum_{p,q=1}^{n} Q_{pq}^+(t)c_{jp}c_{jq}\sigma_{jp}\sigma_{jq}\Big)^{1/2}\Big)^2\Big\}, \quad (19)$$

where the maximum in (19) is taken over all $\sigma_{ij} = \pm 1$, $i, j = 1, \ldots, n$, such that $c_{ij} \neq 0$ in (3) and v is a number of such indices i for which $c_{ij} = 0$ for all $j = 1, \ldots, n$, with initial conditions

$$a^+(t_0) = a_0, \quad r^+(t_0) = (k_0^+)^2. \quad (20)$$

Theorem 2. Let the function $V(t, x)$ be defined as follows

$$V(t,x) = (x - a^+(t))'(r^+(t))^{-1}B(x - a^+(t)) - 1 \quad (21)$$

with $a^+(t)$ and $r^+(t)$ defined in (18)-(19). Then $V(t, x)$ satisfies the HJB inequality (16) and the following boundary condition of type (17) is valid

$$V(t_0, x) = (x - a_0)'k^{-2}B(x - a_0) - 1 \leq 0. \quad (22)$$

Moreover, also the following inclusion is true

$$X(t) \subseteq \{x : V(t,x) \leq 0\}, \quad t_0 \leq t \leq T. \quad (23)$$

Proof. The proof of the theorem is based on results of Filippova (2016) and is carried out according to the scheme of Theorem 1 with necessary corrections due to the more complicated structure of constraints on the initial data and parameters. Note here also that the existence of solutions of the system (18)-(19) follows from the classical results of the differential equations theory and may be established also similar to the reasonings given in (18)-(19). □

Remark 1. Theorem 2 allows to find the solution of HJB inequality (16)-(17) explicitly. It follows from the special form of the chosen initial function $V(t_0, x)$ (22) and a special type of studied control system. In more general cases the use of appropriate approximations gives the way to establish a similar connection between the techniques of ellipsoidal calculus for dynamic control systems with uncertainties and results based on comparison theorems of theory of Hamilton-Jacobi-Bellman equations and inequalities.

3.3 Numerical Simulation

Consider two examples of the nonlinear systems of type (14)-(15).

Example 1. Consider the following control system

$$\begin{cases} \dot{x}_1 = c_1 x_2 + u_1, \\ \dot{x}_2 = c_2 x_1 - 0.1x_1^2 - x_2^2 + u_2, \end{cases} \quad (24)$$

Assume here $x_0 \in X_0 = B(0, 1)$, $0 \leq t \leq 0.5$ and $U = B(0, 0.1)$. Assume also that $|c_1| \leq 0.05$ and $|c_2| \leq 1$. The trajectory tube $\mathcal{X}(t)$ is shown in Figure 1. It can be seen here that the cross-sections of the tube are nonconvex.

Consider one more example of the nonlinear system of type (14)-(15) with another nonlinear term in system equations.

Example 2. Consider the following control system

$$\begin{cases} \dot{x}_1 = c_1 x_2 + u_1, \\ \dot{x}_2 = c_2 x_1 + x_1^2 + x_2^2 + u_2, \end{cases} \quad (25)$$

Here $x_0 \in \mathcal{X}_0 = B(0, 1)$, $0 \leq t \leq 0.9$ and $\mathcal{U} = B(0, 0.5)$. Assume that $|c_1| \leq 0.05$ and $|c_2| \leq 1$. The

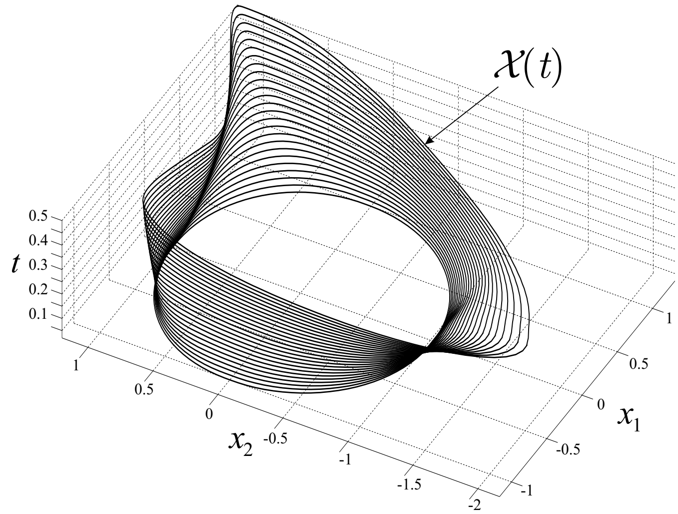

Fig. 1. Trajectory tube X(t).

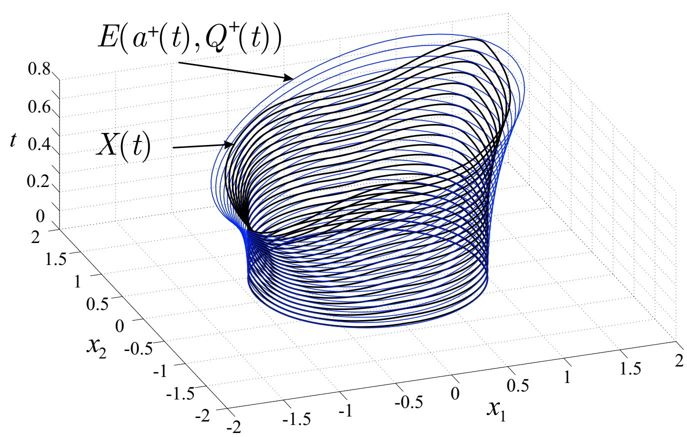

Fig. 2. Trajectory tube X(t) and its upper estimate $E(a^*(t), Q^*(t))$.

reachable sets $\mathcal{X}(t)$ and their external ellipsoidal estimates $E(a^*(t), Q^*(t))$ are shown in Figure 2.

These two examples are included here to illustrate the main approach, more details on numerical algorithms basing on Theorem 1 and Theorem 2 and producing the external (and in some cases internal) ellipsoidal tubes of type $E^+(t) = E(a^*(t), Q^*(t))$ under different assumptions on system parameters can be found in Filippova (2012); Filippova and Matviychuk (2014).

Because the study presented here is motivated not only by theoretical interest but also by applied problems of controlling the movement of objects under conditions of uncertainty, the presented algorithm may be used in concrete control problems including for example the optimization of maneuvers of an artificial Earth satellite with thrusters in a strong gravitational field (Kuntsevich and Volosov (2015); Malyshev and Tychinskii (2005)).

Remark 2. Earlier some approaches had been proposed to obtain differential equations describing dynamics of external ellipsoidal estimates for reachable sets of control system under uncertainty, e.g., in Chernousko and Rokityanskii (2000) the authors studied estimation problems for systems with uncertain matrices in dynamical equations, but additional nonlinear terms in dynamics were not considered there, it is done in this paper.

Remark 3. In Filippova (2010) differential equations of ellipsoidal estimates for reachable sets of a nonlinear dynamical control system were derived for the case when system state velocities contain quadratic forms but in that case the uncertainty in matrix coefficients was not assumed. In Filippova (2016) differential equations for external ellipsoidal estimates of reachable sets of a control system with nonlinearity and with uncertain matrix were derived.

Remark 4. The special case when the quadratic form in the equations of dynamics of the controlled system may be not positive definite was studied in Filippova (2017).

4. CONCLUSIONS

The paper deals with the problems of state estimation for uncertain dynamical control systems under the assumption that the initial system state is unknown but bounded with given constraints.

The modified state estimation approach based on HJB equations technique which uses the special quadratic structure of nonlinearity of the control system and allow to construct the external ellipsoidal estimates of reachable sets was presented.

Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also given. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

REFERENCES

Apreutesei, N. C. (2009). An optimal control problem for a prey-predator system with a general functional response. *Appl. Math. Lett.*, 22(7), 1062-1065.

Asselborn, L., Groß, D. and Stursberg, O. (2013). Control of uncertain nonlinear systems using ellipsoidal reachability calculus. In: *Proceedings of the 9th IFAC Symposium on Nonlinear Control Systems, Toulouse, France, September 4-6, 2013*. IFAC, 50–55.

August, E., Lu, J. and Koeppl, H.(2012). Trajectory enclosures for nonlinear systems with uncertain initial conditions and parameters. In: *Proceedings of the 2012 American Control Conference, 27-29 June 2012, Montreal, QC.* IEEE Computer Soc., 1488-1493.

Baier, R., Gerdts, M. and Xausa, I. (2013). Approximation of reachable sets using optimal control algorithms. *Numerical Algebra, Control and Optimization*, 3(3), 519–548.

Bellman, R.(1997). *Introduction to Matrix Analysis: Second Edition. Classics in Applied Mathematics (Book 19).* SIAM, Philadelphia.

Bertsekas, D. P.(1995). *Dynamic Programming and Optimal Control.* Athena Scientific, V.I,II. Belmont, MA.

Blanchini, F. and Miani, S. (2015). *Set-Theoretic Methods in Control. Systems & Control: Foundations & Applications.* Birkhäuser, Basel.

Boscain, U., Chambrion, T. and Sigalotti, M.(2013). On some open questions in bilinear quantum control. In: *European Control Conference (ECC), Jul 2013, Zurich, Switzerland*, IEEE, 2080-2085.

Ceccarelli, N., Di Marco, M., Garulli, A. and Giannitrapani, A. (2004). A set theoretic approach to path planning for mobile robots. In: *Proceedings of the 43rd IEEE Conference on Decision and Control, Atlantis, Bahamas, December 14-17*, 147-152.

Chernousko, F. L. (1994). *State Estimation for Dynamic Systems.* CRC Press, Boca Raton.

Chernousko, F. L. and Rokityanskii, D. Ya. (2000). Ellipsoidal bounds on reachable sets of dynamical systems with matrices subjected to uncertain perturbations. *Journal of Optimiz. Theory and Appl.*, 104(1), 1-19.

Crandall, M.G., Evans, L.C. and Lions, P-L. (1984). Some Properties of Solutions of Hamilton-Jacobi Equations. *Transactions of the American Mathematical Society*, 282, 487-502.

Filippov, A. F. (1988). *Differential Equations with Discontinuous Right-Hand Sides.* Kluwer Academic Publishers, Dordrecht.

Filippova, T. F. (1993). A note on the evolution property of the assembly of viable solutions to a differential inclusion. *Computers & Mathematics with Applications, Elsevier Science Publishing Company, Inc.*, 25(2), 115–121.

Filippova, T. F. (2009). Estimates of trajectory tubes in control problems under uncertainty. In: *Proc. of the 4th International Conf. "Physics and Control - 2009", PhysCon2009, Italy, Catania, 1-4 September*, 1–6.

Filippova, T. F. (2010). Differential equations for ellipsoidal estimates for reachable sets of a nonlinear dynamical control system. *Proceedings of the Steklov Institute of Mathematics*, 271(1, Supplement), 75–84.

Filippova, T. F. (2012). Set-valued dynamics in problems of mathematical theory of control processes. *International Journal of Modern Physics B (IJMPB)*, 26(25), 1–8.

Filippova, T. F. (2013). State estimation for a class of nonlinear dynamic systems with uncertainty through dynamic programming technique. In: *Proc. of the 6th International Conf. "Physics and Control - 2013", PhysCon2013, August 26-29, 2013, San Lois Potosi, Mexico*, 1–6.

Filippova, T. F. (2014). Estimates of reachable sets of control systems with nonlinearity and parametric perturbations. *Trudy Inst. Mat. i Mekh. UrO RAN*, 20 (4), 287–296.

Filippova, T. F. (2016). Differential equations of ellipsoidal state estimates for bilinear-quadratic control systems under uncertainty. *9th CHAOS Conference Proceedings, 23-26 May 2016, Senate House, University of London, UK*, 97–106.

Filippova, T. F. (2017). Ellipsoidal estimates of reachable sets for control systems with nonlinear terms. IFAC-PapersOnLine: 20th World Congress of the International Federation of Automatic Control (IFAC-2017), Toulouse, France, July 9-14, 2017: proceedings, 50(1), 15925–15930.

Filippova, T. F. and Matviychuk, O. G. (2014). Algorithms of estimating reachable sets of nonlinear control systems with uncertainty. In: *Proceedings of the 7th Chaotic Modeling and Simulation International Conference (Lisbon, Portugal: 7-10 June, 2014). Published by ISAST:*

International Society for the Advancement of Science and Technology, Christos H Skiadas (Ed.), 115–124.

Filippova, T. F. and Matviychuk, O. G. (2015). Algorithms of estimating reachable sets of nonlinear control systems with uncertainty. *Journal of Chaotic Modeling and Simulation (CMSIM)*, 3, 205–214.

Fleming, W. H. and Soner, H.M. (2006). *Controlled Markov Processes and Viscosity Solutions.* 2nd edn. Springer, New York.

Gurman, V. I. (1997). *The Extension Principle in Problems of Control.* Fizmatlit. Moscow. [in Russian]

Gusev, M. I. (2016). Application of penalty function method to computation of reachable sets for control systems with state constraints. *AIP Conference Proceedings*, 1773(050003), 1–8.

Gusev, M. I. and Kurzhanski, A. B. (2007). On the Hamiltonian techniques for designing nonlinear observers under set-membership uncertainty. In: *Preprints of the 7-th IFAC Symposium on Nonlinear Control Systems, Pretoria, South Africa, August 21-24, 2007*, 343–348.

Häckl, G. (1996). *Reachable sets, control sets and their computation.* Augsburger Mathematisch-Naturwissenschaftliche Schriften, 7. PhD Thesis, University of Augsburg, Augsburg.

Kishida, M. and Braatz, R. D. (2015). Ellipsoidal bounds on state trajectories for discrete-time systems with linear fractional uncertainties. *Optim Eng*, 16, 695-711.

Kostousova, E. K. (2013). On polyhedral estimates for trajectory tubes of differential systems with a bilinear uncertainty *Springer Proc. in Mathematics & Statistics*, 47, 465–473.

Kostousova, E. K. and Kurzhanski, A. B. (1996). Theoretical framework and approximation techniques for parallel computation in set-membership state estimation. In: *Proc. Symposium on Modelling Analysis and Simulation.* Lille, France, July 9-12. 2, 849–854.

Kuntsevich, V. M. and Volosov, V. V. (2015). Ellipsoidal and interval estimation of state vectors for families of linear and nonlinear discrete-time dynamic systems. *Cybernetics and Systems Analysis*, 51(1), 64–73.

Kurzhanski, A. B. (1974). Programmed control with incomplete information. *Differ. Uravn.* 10(12), 2162-2172.

Kurzhanski, A. B. (1977). *Control and Observation under Conditions of Uncertainty.* Nauka. Moscow. [in Russian]

Kurzhanski, A. B. (1991). The identification problem Theory of guaranteed estimates. *Autom. Remote Control*, 52(4), 447–465.

Kurzhanski, A. B. (2006). Comparison principle for equations of the Hamilton-Jacobi Type. *Proceedings of the Steklov Institute of Mathematics, Suppl. 1*, 185–195.

Kurzhanski, A. B. (2010). Hamiltonian techniques for the problem of set-membership state estimation. *International Journal of Adaptive Control and Signal Processing*, 25(3), 249-263.

Kurzhanski, A. B. and Filippova, T. F. (1993). On the theory of trajectory tubes a mathematical formalism for uncertain dynamics, viability and control. In: *Advances in Nonlinear Dynamics and Control: a Report from Russia, Progress in Systems and Control Theory, (ed. A.B. Kurzhanski)*, Birkhäuser, Boston. 17, 22–188.

Kurzhanski, A. B. and Valyi, I. (1997). *Ellipsoidal Calculus for Estimation and Control.* Birkhäuser, Boston.

Kurzhanski, A. B. and Varaiya, P. (2014). *Dynamics and Control of Trajectory Tubes. Theory and Computation.* Springer-Verlag, New York.

Lions, P. L. (1982). *Generalized Solutions of Hamilton-Jacobi Equations.* Research Notes in Mathematics, Vol. 69, Pitman Advanced Publishing Program. Boston, MA.

Malyshev, V. V. and Tychinskii, Yu. D. (2005). Construction of attainability sets and optimization of maneuvers of an artificial Earth satellite with thrusters in a strong gravitational field, *Proc. of RAS, Theory and Control Systems*, 4, 124-132.

Mazurenko, S. S. (2012). A differential equation for the gauge function of the star-shaped attainability set of a differential inclusion. *Doklady Math., Moscow*, 86(1), 476-479.

Milanese, M., Norton, J. P., Piet-Lahanier, H. and Walter, E. (Eds.) (1996). *Bounding Approaches to System Identication.* Plenum Press. New York.

Milanese, M. and Vicino, A. (1991). Estimation theory for nonlinear models and set membership uncertainty. *Automatica*, 27(2), 403–408.

Polyak, B. T., Nazin, S. A., Durieu, C., and Walter, E. (2004). Ellipsoidal parameter or state estimation under model uncertainty. *Automatica J., IFAC*, 40, 1171–1179.

Schweppe, F. (1973). *Uncertain Dynamic Systems.* Prentice-Hall, Englewood Cliffs, New Jersey.

Sinyakov, V. V. (2015). Method for computing exterior and interior approximations to the reachability sets of bilinear differential systems. *Differential Equations*, 51(8), 1097-1111.

Subbotin, A. I. (1995). *Generalized Solutions of First-Order PDEs. The Dynamic optimization Perspective.* Birkhauser, Boston.

Vzdornova, O.G. and Filippova, T.F. (2007). Pulse control problems under ellipsoidal constraints: constraint parameters sensitivity. *Automation and Remote Control*, 68(11), 2015–2028.

Walter, E. and Pronzato, L. (1997). *Identification of parametric models from experimental data.* Springer-Verlag, Heidelberg.