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Generalized a bottleneck routing problem: dynamic programming and the start point optimization ⋆

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Abstract: One routing problem with constraints is considered. These constraints are reduced to precedence conditions which be to visiting sequence of megalopolises. This sequence is selected together with concrete trajectory and initial state for minimization of nonadditive criterion. These criterion is some generalization of known criterion for the bottleneck routing problem. The basis singularity of the used solving method consists of using of unique dynamic programming procedure for all initial states. The used criterion includes a controlled parameter influences on significance of different fragments of solution.

Keywords: Routing problem, nonadditive criterion, dynamic programming.

1. INTRODUCTION

We consider the routing problem consisting of sequential visiting for finite system of megalopolises. The used solution includes a index permutation name a route and a trajectory. This trajectory is realized by finite sequence of points of megalopolises. So, our solution is an ordered pair route-procession. The choice of route is subordinated to precedence conditions. These conditions can be generated by technological requirements for corresponding engineering problems. The concrete choice of a trajectory is subordinated to the route choice. The resulting solution in the form of the pars route-trajectory is estimated by criterion similar to the criterion for the bottleneck routing problem. This criterion includes some control parameter. So, we obtain some generalization of the bottleneck routing problem.

The known NP- hard-to-solve problem of traveling salesman (TSP) is a prototype of the considered problem; in this connection see Gutin (2002); Cook (2012); Gimadi (2016). In TSP the branch-and-bound method is used very widely (see Little (1963)). Two variants of dynamic programming (DP) for TSP are reduced in Bellman (1958); Held, Karp (1962). We use the DP-procedure similar Chentsov (2016). The new appearance of our investigation is connected with optimization of the initial state. In addition we use the economical DP-procedure introduced in (Chentsov , 2008, §4.9). For this procedure, total number of values of Bellman function is not required. We use only layers of this function (in the case when precedence conditions exists). The singularity connected with optimization of the initial state is realized only on last step of the Bellman function construction.

2. GENERAL DESIGNATIONS AND DEFINITIONS

In this section, some general notions and designations are introduced. We use quantors and propositional connectives; $\emptyset$ is empty set, $:=$ is equality by definition. For any objects p and q, by $\{p, q\}$ we denote the set containing p, q and not containing none other objects. Then Kuratovskii (1970), for any objects α and β, in the form of $(\alpha, \beta) =: \{\{\alpha\}; \{\alpha; \beta\}\}$, we have the ordered pair (OP) with first element α and second element β. For every OP z, by $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ we denote first and second elements of z respectively. If α, β, and γ are objects, then $(\alpha, \beta, \gamma) := ((\alpha, \beta), \gamma)$. For any sets A, B and C, we suppose (see Dieudonne (1960)) $A \times B \times C =: (A \times B) \times C$.

If H is a set, then $\mathcal{P}(H)$ is the family of all subsets of H and $\mathcal{P}'(H) := \mathcal{P}(H) \setminus \{\emptyset\}$; $\mathrm{Fin}(H)$ is the family of all finite sets of $\mathcal{P}'(H)$. For every nonempty sets A and B, by B^A we denote the set of all mappings from A into B; if $h \in B^A$ and $C \in \mathcal{P}(A)$, then $h^1(C) := \{h(x) : x \in C\}$ is image of C under operation of h. If A, B, C and D are nonempty sets, $h \in D^{A \times B \times C}$, $\mu \in A \times B$, and $\nu \in C$, then $h(\mu, \nu)$ is the value of h at the point $(\mu, \nu) \in A \times B \times C$; of course, $h(\mu, \nu) = h(\mathrm{pr}_1(\mu), \mathrm{pr}_2(\mu), \nu)$. As usual, permutation of a nonempty set H is a bijection of H onto itself; if α is

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a permutation of H, then the permutation α^{-1} (of H) inverse to α is defined by conditions

$$\alpha(\alpha^{-1}(h)) := \alpha^{-1}(\alpha(h)) = h \quad \forall h \in H.$$

Let $\mathbf{R}_+ =: \{\xi \in \mathbf{R} | 0 \leq \xi\}$, where $\mathbf{R}$ is real line. We suppose that $\mathbf{N} =: \{1; 2; ...\}$ and $\mathbf{N}_0 := \{0; 1; 2; ...\}$. For $p \in \mathbf{N}_0$ and $q \in \mathbf{N}_0$,

$$\overline{p,q} := \{k \in \mathbf{N}_0 \mid (p \leq k) \& (k \leq q)\}.$$

If K is a nonempty finite set, then by definition $|K| \in \mathbf{N}$ is cardinality of K and (bi)$[K]$ is the nonempty set of all bijections from $\overline{1, |K|}$ onto K. As usual, $|\emptyset| := 0$. For every nonempty set S, by $\mathcal{R}_+[S]$ we denote the set of all functions from S into $\mathbf{R}_+ : \mathcal{R}_+[S] := (\mathbf{R}_+)^S$.

3. STATEMENT OF PROBLEM

We fix a nonempty set X and its subset $X^0 \in \mathcal{P}'(X)$; so, $X^0 \neq \emptyset$ and $X^0 \subset X$. Elements of X^0 can be used as initial states of our problem. Moreover, we fix

$$M_1 \in \mathrm{Fin}(X), ..., M_N \in \mathrm{Fin}(X) \tag{1}$$

with the properties $(M_p \cap M_q = \emptyset \ \forall p \in \overline{1,N} \ \forall q \in \overline{1,N} \setminus \{p\}) \& (X^0 \cap M_j = \emptyset \ \forall j \in \overline{1,N})$. Moreover, fix

$$\mathbf{M}_1 \in \mathcal{P}'(M_1 \times M_1), ..., \mathbf{M}_N \in \mathcal{P}'(M_N \times M_N).$$

So, under $j \in \overline{1,N}$, we obtain $\mathbf{M}_j \neq \emptyset$ and $\mathbf{M}_j \subset M_j \times M_j$; in addition, by $z \in \mathbf{M}_j$ a variant of (interior) works under visiting of M_j is defined. Let $\mathbf{P} := (\mathrm{bi})[\overline{1,N}]$. If $x \in X^0$, then we can consider processes of the form

$$z_0 \longrightarrow z_1 \in \mathbf{M}_{\alpha(1)} \longrightarrow ... \longrightarrow z_N \in \mathbf{M}_{\alpha(N)}, \tag{2}$$

where $z_0 = (x, x)$ and $\alpha \in \mathbf{P}$. Of course, (2) is a consolidated process. Every visiting $z_j \in \mathbf{M}_{\alpha(j)}$ is representated as the movement from $\mathrm{pr}_1(z_j)$ into $\mathrm{pr}_2(z_j)$ with realization of interior works connected with M_j. In the following, elements of $\mathbf{P}$ are called routes. The choice of the concrete route must devoute to constraints in the form precedence conditions. In this connection, we introduce the set $\mathbf{K} \in \mathcal{P}(\overline{1,N} \times \overline{1,N})$ of all adress pairs of indexes (the case $\mathbf{K} = \emptyset$ is not excluded). We suppose that

$$\forall \mathbf{K}_0 \in \mathcal{P}'(\mathbf{K}) \, \exists z_0 \in \mathbf{K}_0 : \mathrm{pr}_1(z_0) \neq \mathrm{pr}_2(z) \quad \forall z \in \mathbf{K}_0; \tag{3}$$

in (Chentsov , 2008, part 2) concrete variants of setting with condition (3) are reduced. For every pair $(i, j) \in \mathbf{K}$, the megalopolis M_i must be visit before M_j (we introduce precedence conditions). Then

$$\mathbf{A} := \{\alpha \in \mathbf{P} \mid \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \quad \forall z \in \mathbf{K}\} \tag{4}$$

is the set of all admissible routes. According to (3), $\mathbf{A} \neq \emptyset$ (see (Chentsov , 2008, part 2)). So, $\mathbf{A} \in \mathcal{P}'(\mathbf{P})$. From (2), it is evident that a route not defines the process development. Therefore, we introduce tracks or trajectories.

We suppose that under $j \in \overline{1,N}$

$$(\widehat{\mathbf{M}}_j := \{\mathrm{pr}_1(z) : z \in \mathbf{M}_j\}) \& (\check{\mathbf{M}}_j := \{\mathrm{pr}_2(z) : z \in \mathbf{M}_j\}).$$

Moreover, suppose that

$$(\widetilde{\mathbf{X}} := X^0 \bigcup (\bigcup_{j=1}^{N} \widehat{\mathbf{M}}_j)) \& (\mathbf{X} := X^0 \bigcup (\bigcup_{j=1}^{N} \check{\mathbf{M}}_j)). \tag{5}$$

By $\mathbf{Z}$ we denote the set of all processions $(z_i)_{i \in \overline{0,N}} : \overline{0,N} \longrightarrow \widetilde{\mathbf{X}} \times \mathbf{X}$. Under $x \in X^0$ and $\alpha \in \mathbf{P}$, we obtain that

$$\mathcal{Z}_\alpha[x] := \{(z_i)_{i \in \overline{0,N}} \in \mathbf{Z} | (z_0 = (x, x)) \& (z_t \in \mathbf{M}_{\alpha(t)} \ \forall t \in \overline{1,N})\} \in \mathrm{Fin}(\mathbf{Z}) \tag{6}$$

is the set of all tracks coordinated with the route α. Of course, under $x \in X^0$, in the form of

$$\mathbf{D}[x] := \{(\alpha, \mathbf{z}) \in \mathbf{A} \times \mathbf{Z} | \mathbf{z} \in \mathcal{Z}_\alpha[x]\} \in \mathrm{Fin}(\mathbf{A} \times \mathbf{Z}), \tag{7}$$

we obtain the set of all admissible solutions defined as OP route-track.

Cost functions. We consider non additive variant of the cost aggregation. In this connection, we fix $\mathbf{c} \in \mathcal{R}_+[\mathbf{X} \times \widetilde{\mathbf{X}} \times \widehat{N}]$, $c_j \in \mathcal{R}_+[\widetilde{\mathbf{X}} \times \mathbf{X} \times \widehat{N}]$ $(j = 1, 2, ..., N)]$, where $\widehat{N} := \mathcal{P}'(\overline{1,N})$. By $\mathbf{c}, c_1, ..., c_N$ cost functions are denoted.

Moreover, we fix a parameter $\mathbf{a} \in \mathbf{R}_+ \setminus \{0\}$. Now, we introduce the nonadditive criterion. Namely, under $\alpha \in \mathbf{P}$ and $\mathbf{z} \in \mathbf{Z}$, let

$$\mathcal{B}_\alpha[\mathbf{z}] := \max_{t \in \overline{0,N-1}} \mathbf{a}^t [\mathbf{c}(\mathrm{pr}_2(\mathbf{z}(t)), \mathrm{pr}_1(\mathbf{z}(t+1)), \alpha^1(\overline{t+1,N})) + c_{\alpha(t+1)}(\mathbf{z}(t+1), \alpha^1(\overline{t+1,N}))]; \tag{8}$$

$\mathcal{B}_\alpha[\mathbf{z}] \in \mathbf{R}_+$. For the following in (8), the case $\alpha \in \mathbf{A}$ and $\mathbf{z} \in \mathcal{Z}_\alpha[x]$, where $x \in X^0$, is essential. So, for $x \in X^0$, we obtain the problem

$$\mathcal{B}_\alpha[\mathbf{z}] \longrightarrow \min, (\alpha, \mathbf{z}) \in \mathbf{D}[x]; \tag{9}$$

for problem (9), the value

$$V[x] := \min_{(\alpha, \mathbf{z}) \in \mathbf{D}[x]} \mathcal{B}_\alpha[\mathbf{z}] \in \mathbf{R}_+ \tag{10}$$

is defined and, moreover, $(\mathrm{SOL})[x] := \{(\alpha, \mathbf{z}) \in \mathbf{D}[x] | \mathcal{B}_\alpha[\mathbf{z}] = V[x]\} \in \mathcal{P}'(\mathbf{D}[x])$. Finally, it is important (see (10)) the following problem

$$V[x] \longrightarrow \inf, \quad x \in X^0. \tag{11}$$

So, in (11), we nave the problem of optimization of the initial state.

Let

$$\mathbf{V} := \inf_{x \in X^0} V[x].$$

4. ALGORITHMIC VARIANT OF DYNAMIC PROGRAMMING

In this section, an economical variant of the DP procedure is considered. This variant is similar to Chentsov (2016). At the beginning, we introduce essential lists of tasks and layers of the position space. So,

$$\mathcal{G} := \{K \in \widehat{N} | \forall z \in \mathbf{K} \ (\mathrm{pr}_1(z) \in K) \Rightarrow (\mathrm{pr}_2(z) \in K)\}; \tag{12}$$

we obtain the set of all essential lists (of tasks). Moreover, $\mathcal{G}_s := \{K \in \mathcal{G} | s = |K|\} \ \forall s \in \overline{1,N}$. In addition, $\mathcal{G}_N = \{\overline{1,N}\}$. Let $\mathbf{K}_1 := \{\mathrm{pr}_1(z) : z \in \mathbf{K}\}$; then $\mathcal{G}_1 = \{\{t\} : t \in \overline{1,N} \setminus \mathbf{K}_1\}$. Finally Chentsov (2013),

$$\mathcal{G}_{s-1} = \{K \setminus \{t\} : K \in \mathcal{G}_s, \ t \in \mathbf{I}(K)\} \quad \forall s \in \overline{2,N}.$$

We obtain recurrent procedure $\mathcal{G}_N \longrightarrow ... \longrightarrow \mathcal{G}_1$. Now, we construct the sets $D_0, D_1, ..., D_N$; we call these sets layers of the position space. Suppose that

$$\widetilde{\mathcal{M}} := \bigcup_{j\in\overline{1,N}\setminus\mathbf{K}_1} \check{\mathbf{M}}_j$$

and $D_0 := \widetilde{\mathcal{M}} \times \{\emptyset\} = \{(x,\emptyset) : x \in \widetilde{\mathcal{M}}\}$. Moreover, $D_N := X^0 \times \{\overline{1,N}\} = \{(x,\overline{1,N}) : x \in X^0\}$. Then, D_0 and D_N are extreme layers of the position space. Consider the construction of immediate layers. Namely, at the beginning, for $s \in \overline{1,N-1}$ and $K \in \mathcal{G}_s$, we suppose sequentially that

$$J_s(K) := \{j \in \overline{1,N}\setminus K | \{j\} \cup K \in \mathcal{G}_{s+1}\},$$

$$\mathcal{M}_s[K] := \bigcup_{j\in J_s(K)} \check{\mathbf{M}}_j,$$

$$\widetilde{\mathbf{D}}_s[K] := \mathcal{M}_s[K] \times \{K\} = \{(x,K) : x \in \mathcal{M}_s[K]\}.$$

Then, under $s \in \overline{1,N-1}$,

$$D_s := \bigcup_{K\in\mathcal{G}_s} \widetilde{\mathbf{D}}_s[K].$$

As a result, we have the sets $D_0 \neq \emptyset, D_1 \neq \emptyset, ..., D_N \neq \emptyset$. In addition, by (Chentsov, 2013, (6.11)) we obtain the following property:

$$(\mathrm{pr}_2(z), K\setminus\{j\}) \in D_{s-1} \quad \forall s \in \overline{1,N}$$
$$\forall (x,K) \in D_s \ \forall j \in \mathbf{I}(K) \ \forall z \in \mathbf{M_j}. \tag{13}$$

Using (13), we construct sequentially the functions $v_0 \in \mathcal{R}_+[D_0]$, $v_1 \in \mathcal{R}_+[D_1], ..., v_N \in \mathcal{R}_+[D_N]$;

$$v_0 \longrightarrow v_1 \longrightarrow ... \longrightarrow v_N. \tag{14}$$

So, we define $v_0 \in \mathcal{R}_+[D_0]$ by the rule $v_0(x,\emptyset) := 0 \, \forall x \in \widetilde{\mathcal{M}}$. If $s \in \overline{1,N}$ and $v_{s-1} \in \mathcal{R}_+[D_{s-1}]$ was already constructed, then (see (13))

$$v_s(x,K) = \min_{j\in\mathbf{I}(K)} \min_{z\in\mathbf{M}_j} \sup(\{\mathbf{c}(x,\mathrm{pr}_1(z),K)$$
$$+c_j(z,K); \mathbf{a}v_{s-1}(\mathrm{pr}_2(z), K\setminus\{j\})\}) \quad \forall (x,K) \in D_s. \tag{15}$$

By (15) we realize all chain (14).

As a result, we obtain procession $(v_0, v_1, ..., v_N)$. In addition, v_0 is defined by the simplest rule and

$$V[x^0] = v_N(x^0, \overline{1,N}) \quad \forall x^0 \in X^0. \tag{16}$$

The property (16) is established similarly to (Chentsov , 2016, (4.15)) (we note that, in our case, the proposition similar to (**?**, Theorem 1) is valid; but, in this proposition, parameter $\mathbf{a}$ is used). Of course, by (15) and (16)

$$V[x^0] = \min_{j\in\mathbf{I}(\overline{1,N})} \min_{z\in\mathbf{M}_j} \sup(\{\mathbf{c}(x^0,\mathrm{pr}_1(z),\overline{1,N})$$
$$+c_j(z,\overline{1,N}); \mathbf{a}v_{N-1}(\mathrm{pr}_2(z), \overline{1,N}\setminus\{j\})\}) \quad \forall x^0 \in X^0. \tag{17}$$

So, by (17) the problem (11) takes place

$$v_N(x^0, \overline{1,N}) \longrightarrow \inf, \ x^0 \in X^0. \tag{18}$$

We note that, under $x^0 \in X^0$, construction of $(\alpha^0, z^0) \in (\mathrm{SOL})[x^0]$ is realized similarly to (Chentsov , 2016, (4.20)—(4.27)) with application of parameter $\mathbf{a}$; compare (15) and (Chentsov , 2016, Proposition 1).

5. THE CASE, WHEN X^0 IS EQUIPPED WITH A METRIC

Let $\widetilde{\mathbf{M}}$ be the union of all sets $\widehat{\mathbf{M}}_j$, $j \in \overline{1,N}$. In the following, we suppose that X^0 is equipped with a metric ρ; so, (X^0, ρ) is a metric space. For $x \in X^0$ and $\varepsilon \in \mathbf{R}_+\setminus\{0\}$,

$$B^0_\rho(x,\varepsilon) := \{y \in X^0 | \rho(x,y) < \varepsilon\} \in \mathcal{P}'(X^0)$$

is the corresponding open ball with the center x and radius ε. We suppose that (X^0, ρ) is completely bounded space:

$$\forall \varepsilon \in \mathbf{R}_+\setminus\{0\} \ \exists K \in \mathrm{Fin}(X^0) : X^0 = \bigcup_{x\in K} B^0_\rho(x,\varepsilon). \tag{19}$$

Remark If X^0 is a bounded set in a finite-dimensional arithmetic space, then (19) is fulfilled.

Condition 4.1. $\forall \varepsilon \in \mathbf{R}_+\setminus\{0\} \exists \delta \in \mathbf{R}_+\setminus\{0\} \ \forall x_1 \in X^0 \ \forall x_2 \in X^0$

$$(\rho(x_1,x_2) < \delta) \Rightarrow (|\mathbf{c}(x_1,y,\overline{1,N}) - \mathbf{c}(x_2,y,\overline{1,N})| < \varepsilon$$
$$\forall y \in \widetilde{\mathbf{M}}).$$

In the following, we suppose that Condition 4.1 is fulfilled.

Proposition 4.1. Let $\varepsilon^0 \in \mathbf{R}_+\setminus\{0\}$. Moreover, let $\delta^0 \in \mathbf{R}_+\setminus\{0\}$ such that

$$(\rho(x_1,x_2) < \delta^0) \Rightarrow (|\mathbf{c}(x_1,y,\overline{1,N}) - \mathbf{c}(x_2,y,\overline{1,N})| < \varepsilon_0$$
$$\forall y \in \widetilde{\mathbf{M}}).$$

Finally, let $\mathcal{K} \in \mathrm{Fin}(X^0)$ satisfies to equality

$$X^0 = \bigcup_{x\in\mathcal{K}} B^0_\rho(x,\delta_0).$$

Then, the following chain of inequalities is fulfilled:

$$\mathbf{V} \le \min_{x\in\mathcal{K}} v_N(x,\overline{1,N}) \le \mathbf{V} + \varepsilon_0. \tag{20}$$

The correspondence proof is reduced to immediate combination of (17), (19), and Condition 4.1.

So, in our case, for determination of the global extremum with every precision, discretizations of X^0 can be used. In addition, the universal variant of DP procedure similar to Chentsov (2016) is used (see (14), (15)).

Now, by Proposition 4.1 we can be restricted to consideration of variants for problems similar to (11) and using instead of X^0 finite subsets of X^0. Do, we consider one such problem fixing

$$\mathcal{K} \in \mathrm{Fin}(X^0) \tag{21}$$

(of course, we are oriented on realization of relations similar to (20)). We suppose that functions (14) were constructed. We choose $x_0 \in \mathcal{K}$ such that

$$v_N(x_0,\overline{1,N}) = \min_{x\in\mathcal{K}} v_N(x,\overline{1,N}) \tag{22}$$

(of course, $x_0 \in X^0$ and $V[x_0] = v_N(x_0,\overline{1,N})$). In (22), we use the property (21). Now, we consider the natural Bellman procedure for construction of solution of the set $(\mathrm{SOL})[x_0]$ (see analogous constructing of (Chentsov , 2016, §4)).

So, we suppose $\mathbf{z}^{(0)} := (x_0, x_0) \in \widetilde{\mathbf{X}} \times \mathbf{X}$ and choose (see (17)) $\mathbf{j}_1 \in \mathbf{I}(\overline{1,N})$ and $\mathbf{z}^{(1)} \in \mathbf{M}_{\mathbf{j}_1}$ for which

$$V[x_0] = \sup(\{\mathbf{c}(x^0,\mathrm{pr}_1(\mathbf{z}^{(1)}),\overline{1,N})$$

$$+c_{\mathbf{j}_1}(\mathbf{z}^{(1)}, \overline{1,N}); \mathbf{a}v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\mathbf{j}_1\})\}). \qquad (23)$$

Then, by (13) the following inclusion

$$(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\mathbf{j}_1\}) \in D_{N-1} \qquad (24)$$

is realized, where $N-1 \geq 1$. By (15) and (24)

$$v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\mathbf{j}_1\})$$
$$= \min_{j\in\mathbf{I}(\overline{1,N}\setminus\{\mathbf{j}_1\})} \min_{z\in\mathbf{M}_j} \sup(\{\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(z), \overline{1,N}\setminus\{\mathbf{j}_1\})$$
$$+c_j(z, \overline{1,N}\setminus\{\mathbf{j}_1\}); \mathbf{a}v_{N-2}(\mathrm{pr}_2(z), \overline{1,N}\setminus\{\mathbf{j}_1; j\})\}).$$

Using this equality, we choose $\mathbf{j}_2 \in \mathbf{I}(\overline{1,N}\setminus\{\mathbf{j}_1\})$ and $\mathbf{z}^{(2)} \in \mathbf{M}_{\mathbf{j}_2}$ for which

$$v_{N-1}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\mathbf{j}_1\})$$
$$= \sup(\{\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\mathbf{j}_1\})$$
$$+c_{\mathbf{j}_2}(\mathbf{z}^{(2)}, \overline{1,N}\setminus\{\mathbf{j}_1\}); \mathbf{a}v_{N-2}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\mathbf{j}_1; \mathbf{j}_2\})\}), \qquad (25)$$

where by (13) $(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\mathbf{j}_1; \mathbf{j}_2\}) \in D_{N-2}$. From (23) and (25) we obtain the following equality:

$$V[x_0] = v_N(x_0, \overline{1,N})$$
$$= \sup(\{\max_{t\in\overline{0,1}} \mathbf{a}^t[\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(t)}), \mathrm{pr}_1(\mathbf{z}^{(t+1)}), \overline{1,N}\setminus\{\mathbf{j}_k : k \in \overline{1,t}\})$$
$$+c_{\mathbf{j}_{t+1}}(\mathbf{z}^{(t+1)}, \overline{1,N}\setminus\{\mathbf{j}_k : k \in \overline{1,t}\})];$$
$$\mathbf{a}^2 v_{N-2}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\mathbf{j}_k : k \in \overline{1,2}\})\}), \qquad (26)$$

where the obvious equality $\overline{1,0} = \emptyset$ is used. Under $N = 2$, by (26), we obtain the optimal solution $((\mathbf{j})_{k\in\overline{1,2}}, (\mathbf{z}^{(k)})_{k\in\overline{0,2}})$. For $N > 2$, it is required to continue the choice procedure similar to (23) and (25) intil exhaustion of the index set $\overline{1,N}$. Then, we obtain $\mathbf{j}_1 \in \overline{1,N}, ..., \mathbf{j}_N \in \overline{1,N}$, $\mathbf{z}^{(0)} \in \widetilde{\mathbf{X}} \times \mathbf{X}, \mathbf{z}^{(1)} \in \widetilde{\mathbf{X}} \times \mathbf{X}, ..., \mathbf{z}^{(N)} \in \widetilde{\mathbf{X}} \times \mathbf{X}$ for which

$$\eta_0 := (\mathbf{j}_k)_{k\in\overline{1,N}} \in \mathbf{A},\ \mathbf{z}_0 = (\mathbf{z}^{(k)})_{k\in\overline{0,N}} \in \mathcal{Z}_{\eta_0}[x_0],$$

and $\mathcal{B}_{\eta_0}[\mathbf{z}_0] = v_N(x_0, \overline{1,N}) = V[x_0]$. So, $(\eta_0, \mathbf{z}_0) \in (\mathrm{SOL})[x_0]$.

Proposition 4.1 determines the theoretical possibility of discretizing the set X^0 from the considerations of a given accuracy for the realization of $\mathbf{V}$. In the next section, the results of a computational experiment are presented in the case when the set X^0 itself is finite and its discretization is not required. Thus, the constructions of the next section can be considered independently of Proposition 4.1.

6. COMPUTATIONAL EXPERIMENT

The optimal algorithm which provides for the determination of the initial state, the route (satisfying the conditions of precedence) and the trajectory conforming with this route was constructed on the basis of the above-mentioned theoretical constructions. As X we use the plane $\mathbf{R} \times \mathbf{R}$. The cost of external displacements was determined using the Euclidean distance: the values of the function $\mathbf{c}$ are Euclidean distances between points on the plane. Functions $c_1, ..., c_N$ that evaluated the costs when performing work on target sets (megalopolises) were determined each time by summing two Euclidean distances: from the point of arrival to a fixed point and connected to a megalopolis, and from this point to the point of departure. We assume that when a M_j, where $j \in \overline{1,N}$ megalopolis is visited, the performer selects the arrival point (among the points M_j) from which it is directed to the point m_j that belongs to the convex hull M_j, performs the required operation there, and then moves to the departure point. As $\mathbf{M}_j$, we use the Cartesian product $M_j \times M_j$ each time. The parameter $\mathbf{a}$ was changed during the experiment. As X^0, we used a set containing 4 elements. The term "total costs" below is the value of $\mathbf{V}$. The algorithm ensures its finding and constructing the optimal permissible solution in the form of a set consisting of an initial state, a route and a trajectory.

So, we are considering a generalized version of the bottleneck routing problem, for which a global extremum and an optimal solution are determined. The calculations were carried out for three values of the parameter $\mathbf{a} : \mathbf{a} = 1$ (the usual bottleneck routing problem), $\mathbf{a} = 1,2$ (a problem in which the final fragments of the solution, defined in the form of a route-trajectory pair, are more significant), $\mathbf{a} = 0,8$ (the case of the main problem, when the initial fragments of the solution are more significant).

The algorithm is implemented as a program for PC in the programming language C ++, running in the 64-bit operating system of the Windows family, starting with Windows 7. The computational part of the program is implemented in a separate stream from the user interface. For the case of solving a problem on a plane, it is possible to graphically represent the route and the route and increase the individual sections of the graph; the image can be saved to a bmp image file. The source data and the results of the program account are stored in a text file of a special structure.

The computer experiment was carried out on a computer with a central processor Intel Core i7, 64 GB of RAM with the operating system Windows 7 Maximum SP1.

Suppose that 33 sets are given in the form of uniform grids with 12 points on the circles. The number of address pairs is 33. The set of initial positions is represented by the following points (4 points): (-70, -95); (0.0); (40.10); (90.35). The distance is estimated by the Euclidean norm.

Case $\mathbf{a} = 1$. "Total costs": 56,038. Starting point $x^0 = (-70, -95)$. Final point — $(11, -65, 66)$. Time of calculation — 7 hours 24 min. 26 seconds.

Case $\mathbf{a} = 1,2$. "Total costs": 6680,837. Starting point $x^0 = (-70, -95)$. Final point — $(16, -67)$. Time of calculation — 6 hours 45 min. 13 seconds.

Case $\mathbf{a} = 0.8$. "Total costs": 22,17. Starting point $x^0 = (0, 0)$. Final point — $(59, 61, -84)$. Time of calculation — 6 hours 46 min. 53 seconds.

7. CONCLUSION

The method of solving the generalized problem of routing of displacements with performance of works connected with points of visits is constructed in the article. The quality criterion is similar to the criterion in the well-known "bottleneck problem", but includes an additional parameter that allows one to change the significance of the fragments of the trajectory (amplifying the influence of the

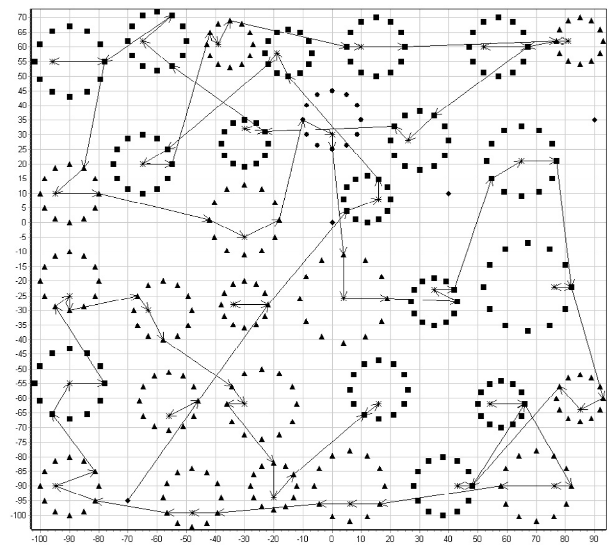

Fig. 1. Case $\mathbf{a} = 1$

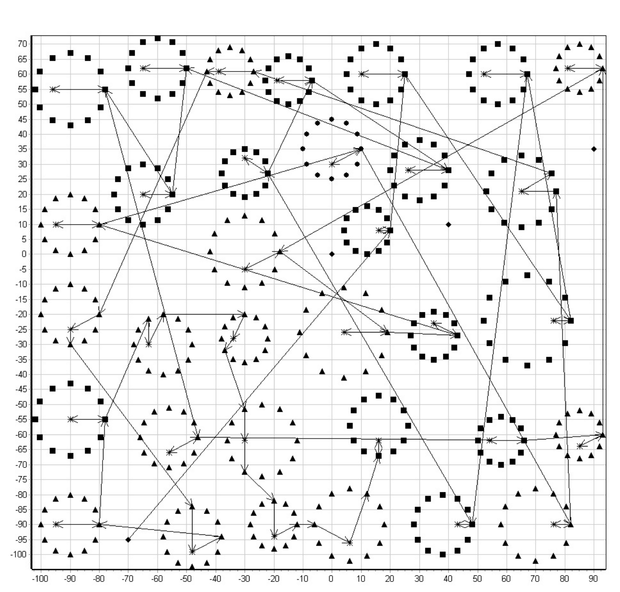

Fig. 2. Case $\mathbf{a} = 1.2$

initial, or, on the contrary, final fragments). The statement of the problem allows multivariance of displacements, as a result of which the problem of visiting megalopolises (some finite sets) is obtained. As a result, the route and the track (or trajectory) of the movement are allotted as part of each joint solution. The choice of the route (ie, permutation of the indices) can be constrained by the precedence conditions (one-after-another condition). The cost functions aggregated are not additive, they allow dependence on the list of tasks (performed at the time of moving or, conversely, not executed). The article constructs a variant of dynamic programming that delivers a global extremum and a concrete optimal solution that realizing it. This solution involves choosing the initial state in the interest of optimization. The corresponding optimal algorithm is implemented on a PC; some results of the computational experiment are given.

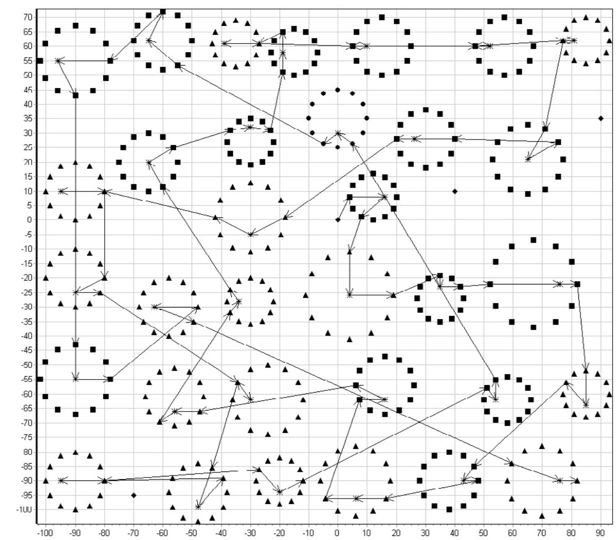

Fig. 3. Case $\mathbf{a} = 0.8$

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