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Regularization in the Problem of Minimization of Stochastic Sensitivity

Irina Bashkirtseva * **Lev Ryashko** **

* *Institute of Natural Sciences and Mathematics, Ural Federal University, Lenina, 51, 620000, Ekaterinburg, Russia (e-mail: Irina.Bashkirtseva@urfu.ru).*
** *Institute of Natural Sciences and Mathematics, Ural Federal University, Lenina, 51, 620000, Ekaterinburg, Russia (e-mail: Lev.Ryashko@urfu.ru)*

Abstract: We consider a problem of the construction of feedback regulator which synthesizes the assigned stochastic sensitivity of the equilibrium in stochastically forced nonlinear dynamic system. In the case of complete information, it is shown that this problem can be reduced to the solution of the matrix algebraic equation. A presence of noise in measurements deforms the stochastic sensitivity. We find conditions when such deformation is extremely large, and the considered problem is ill-posed. For this ill-posed problem, a regularization method is suggested. We propose an analytical approach which allows us to take into account a presence of noise in measurements when we construct an optimal feedback regulator. General theoretical results are illustrated by examples.

Keywords: Feedback regulators, random disturbances, stochastic sensitivity, ill-posedness, regularization.

1. INTRODUCTION

Problems of the stabilization of operating modes in stochastically forced nonlinear systems attract attention of many researchers (see, e.g. Krasovskii et al. (1961); Kushner (1967); Astrom (1970); Fleming et al. (1975); Guo et al. (2010) and bibliography therein). The theory of control and stabilization of linear stochastic systems is well elaborated (Athans (1971); Wonham (1979); Zhou et al. (2016)).

It is well known that many real systems are nonlinear, so a development of the nonlinear stochastic control theory is very important (see, e.g. Sun (2006); Zhao et al. (2016); Khrustalev et al. (2017); Rajpurohit et al. (2017); Homer et al. (2018)). A rigorous mathematical description of the stochastic dynamics in general nonlinear systems is given by the Kolmogorov-Fokker-Planck equation, however, a direct using of this equation is very difficult, especially in control problems. So, constructive approximations and asymptotics for probabilistic distributions are elaborated (Freidlin et al. (1984)). A semi-analytical approach for the approximation of the dispersion of random states near attractors forced by weak noise was suggested in Bashkirtseva et al. (2015). This approach based on the stochastic sensitivity function technique was successfully applied to analysis of noise-induced phenomena (Bashkirtseva et al. (2018)) and solution of related control problems (Ryashko et al. (2008); Bashkirtseva et al. (2017)). An undesired excitability of operating modes can be explained by the high sensitivity of the system to random disturbances, so, the main idea of the stabilization is to reduce this stochastic sensitivity by the appropriate feedback regulator. But in some circumstances, the desire to minimize the stochastic sensitivity can lead to unexpected ill-posed problem.

In the present paper, we discuss the problem of the stochastic sensitivity synthesis for systems with noise in measurements. It is shown that the presence of such noises in control systems with "idealized" optimal regulators minimizing the stochastic sensitivity can destroy the operating mode. To regularize this ill-posed problem, we suggest a method of the construction of the regulator taking into account the presence of the noise in measurements. Mathematically, for this regularization, it is required to solve quadratic matrix equations. A solvability analysis of this equation allows us to describe a set of attainable stochastic sensitivity matrices and find a minimal element in this set. The general theoretical results are illustrated by examples.

2. SYNTHESIS OF STOCHASTIC SENSITIVITY FOR EQUILIBRIUM

Consider a nonlinear stochastic system

$$\dot{x} = f(x) + g(x)u + \varepsilon\sigma(x)\xi(t), \tag{1}$$

where $x \in R^n$ is a state vector, $u \in R^l$ is a control vector, $f(x) \in R^n$ is a continuously differentiable vector-function, $g(x) \in R^{n\times l}$ is a matrix-function, $\xi(t) \in R^m$ is a δ-correlated white Gaussian noise vector satisfying $\mathrm{E}\xi(t) = 0$, $\mathrm{E}\xi(t)\xi^\top(\tau) = \delta(t-\tau)Q$, and Q is a non-negative definite $m \times m$-matrix. Here, $\sigma(x)$ is an $(n \times m)$-matrix-function that characterizes the dependence of disturbances on states, and ε is a scalar parameter of the noise intensity.

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We assume that the corresponding deterministic system (1) (with $\varepsilon = 0$ and $u = 0$ therein) has an equilibrium $\bar{x}$. The stability of $\bar{x}$ is not supposed.

Let the control input u in (1) be formed by the feedback regulator

$$u = K(x - \bar{x}) \tag{2}$$

with a constant $(l \times n)$-matrix K.

The closed-loop system (1), (2) can be written as

$$\dot{x} = f(x) + g(x)K(x - \bar{x}) + \varepsilon\sigma(x)\xi(t). \tag{3}$$

For the deviations of solutions $x^{\varepsilon}(t)$ of the stochastic system (3) from the equilibrium $\bar{x}$, the following asymptotics can be considered:

$$z(t) = \lim_{\varepsilon \to 0} \frac{x^{\varepsilon}(t) - \bar{x}}{\varepsilon}.$$

Dynamics of $z(t)$ is governed by the linear stochastic equation

$$\dot{z} = (A + BK)z + \sigma(\bar{x})\xi(t),$$

where

$$A = \frac{\partial f}{\partial x}(\bar{x}),\ B = g(\bar{x}).$$

The matrix $Z = \mathrm{E}zz^{\top}$ of the second moments of z is a solution of the linear deterministic matrix equation

$$\begin{aligned} \dot{Z} &= (A + BK)Z + Z(A + BK)^{\top} + S, \\ S &= \sigma(\bar{x})Q\sigma^{\top}(\bar{x}). \end{aligned} \tag{4}$$

A set of matrices K that provide an exponential stability of the equilibrium $\bar{x}$ of system (1) (with $\varepsilon = 0$ therein)can be written as

$$\mathbf{K} = \{K |\ Re\lambda_i(A + BK) < 0\},$$

where $\lambda_i(A + BK)$ are the eigenvalues of the matrix $A + BK$. Suppose that the pair (A, B) is stabilizable (see Wonham (1979)). This means that $\mathbf{K}$ is not empty.

For any $K \in \mathbf{K}$, the equation (4) has a unique exponentially stable stationary solution W. The matrix W satisfies the equation

$$(A + BK)W + W(A + BK)^{\top} + S = 0. \tag{5}$$

The stochastic sensitivity matrix W is a simple asymptotics of the dispersion of random states of system (3) around the equilibrium, $\bar{x}$: $\mathrm{cov}(\bar{x}^{\varepsilon}, \bar{x}^{\varepsilon}) \approx \varepsilon^2 W$.

Let the function $W(K)$ be a solution of the equation (5) for $K \in \mathbf{K}$. Consider an inverse problem: to find the feedback matrix K of the regulator (2) which provides the assigned stochastic sensitivity matrix $\bar{W}$ for system (3). Mathematically, this problem is reduced to the solution of the equation

$$W(K) = \bar{W}.$$

This equation can be written in a form

$$BK\bar{W} + \bar{W}K^{\top}B^{\top} + A\bar{W} + \bar{W}A^{\top} + S = 0. \tag{6}$$

A full analysis of the solvability of this equation can be found in Ryashko et al. (2008). In the case when $\mathrm{rank}(B) = n = l$, for any positive definite matrix $\bar{W}$, the equation (6) has the solution:

$$K = -B^{-1}\left[\frac{1}{2}S\bar{W}^{-1} + A\right]. \tag{7}$$

As one can see, this is a case when one can synthesize any, even arbitrarily small, stochastic sensitivity matrix choosing the appropriate K.

3. INFLUENCE OF NOISY DATA ON THE STOCHASTIC SENSITIVITY SYNTHESIS

It is well known, that for real systems, the data about the current state $x(t)$ contains random noise. Consider a case when one can measure only the vector

$$y(t) = x(t) + \varepsilon\varphi(x(t))\eta(t),$$

where $\varphi(x)$ is an $(n \times p)$-matrix-function, and $\eta(t)$ is a white uncorrelated Gaussian p-vector noise with parameters: $\mathrm{E}\eta(t) = 0$, $\mathrm{E}\eta(t)\eta^{\top}(\tau) = \delta(t - \tau)R$.

Then the closed-loop system (1) with the noisy control $u = K(y - \bar{x})$ can be written as

$$\begin{aligned} \dot{x} &= f(x) + g(x)K(x - \bar{x}) + \\ &+ \varepsilon g(x)K\varphi(x)\eta(t) + \varepsilon\sigma(x)\xi(t). \end{aligned} \tag{8}$$

For the asymptotics $z(t)$, one can write the following system

$$\dot{z} = (A + BK)z + BK\varphi(\bar{x})\eta + \sigma(\bar{x})\xi.$$

The matrix $Z = \mathrm{E}zz^{\top}$ of the second moments of z is a solution of the following matrix equation

$$\dot{Z} = (A + BK)Z + Z(A + BK)^{\top} + BK\Phi K^{\top}B^{\top} + S,$$

$$\Phi = \varphi(\bar{x})R\varphi^{\top}(\bar{x}),\ \ S = \sigma(\bar{x})Q\sigma^{\top}(\bar{x}).$$

For any $K \in \mathbf{K}$, this equation has a unique exponentially stable stationary solution W. The stochastic sensitivity matrix W satisfies the following equation

$$(A + BK)W + W(A + BK)^{\top} + BK\Phi K^{\top}B^{\top} + S = 0. \tag{9}$$

So, a presence of the noises in the measurements results in the additional term in the equation for the stochastic sensitivity matrix (compare (9) and (5)).

Consider how the appearance of this additional term changes the stochastic sensitivity matrix W. Let $\bar{W}$ be an assigned stochastic sensitivity matrix of the equilibrium $\bar{x}$ for the system (3), and $\bar{K}$ be a feedback matrix of the regulator (2) that synthesizes this $\bar{W}$. So, the equation

$$(A + B\bar{K})\bar{W} + \bar{W}(A + B\bar{K})^{\top} + S = 0 \tag{10}$$

holds.

Let us find out now what will be the stochastic sensitivity matrix of the equilibrium $\bar{x}$ for system (8) with the same $\bar{K}$. For the corresponding stochastic sensitivity matrix $\hat{W}$ (see (9)), we have

$$(A + B\bar{K})\hat{W} + \hat{W}(A + B\bar{K})^{\top} + B\bar{K}\Phi\bar{K}^{\top}B^{\top} + S = 0. \tag{11}$$

An addition of the noise into the observation changes the stochastic sensitivity matrix from $\bar{W}$ to $\hat{W} = \bar{W} + V$. As it follows from (10), (11), the matrix $V = \hat{W} - \bar{W}$ satisfies the equation

$$(A + B\bar{K})V + V(A + B\bar{K})^{\top} + B\bar{K}\Phi\bar{K}^{\top}B^{\top} = 0. \tag{12}$$

So, the presence of noise in the observations increases the stochastic sensitivity matrix by value V. It is natural to expect that small noise in the observations results in the

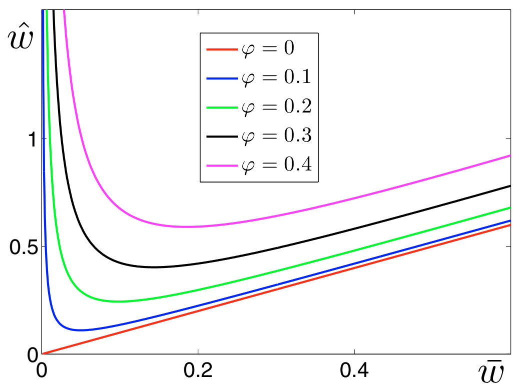

Fig. 1. Dependence of the stochastic sensitivity $\hat{w}$ of system (13), (15) on $\bar{w}$ for $k = \bar{k}$, $a = \sigma = 1$.

small V. However, there are circumstances in which V becomes significantly large. Such ill-posedness arises when we try to reduce the stochastic sensitivity by decreasing the matrix $\bar{W}$.

We will illustrate this ill-posed problem on the simple example of one-dimensional system.

Consider the system

$$\dot{x} = f(x) + u + \varepsilon\sigma\xi(t) \tag{13}$$

with the control

$$u = k(x - \bar{x}). \tag{14}$$

Here, $\xi(t)$ is a scalar white Gaussian noise with parameters $\mathrm{E}\xi(t) = 0$, $\mathrm{E}\xi(t)\xi(\tau) = \delta(t-\tau)$. For this system, the equation (5) looks like

$$2(a + k)w + \sigma^2 = 0,$$

where $a = f^{'}(\bar{x})$, and w is a scalar stochastic sensitivity of the equilibrium $\bar{x}$ for system (13),(14). For any assigned stochastic sensitivity $\bar{w}$ one can find

$$\bar{k} = -a - \frac{\sigma^2}{2\bar{w}}$$

which provides this $\bar{w}$. As one can see, in this case, the regulator (14) can synthesize an arbitrarily small stochastic sensitivity $\bar{w}$.

Consider now a case when the regulator uses noisy measurements:

$$u = k(y - \bar{x}), \quad y = x + \varepsilon\varphi\eta, \tag{15}$$

where $\eta(t)$ is a white uncorrelated Gaussian scalar noise with parameters: $\mathrm{E}\eta(t) = 0$, $\mathrm{E}\eta(t)\eta(\tau) = \delta(t-\tau)$.

Let us find the stochastic sensitivity which is synthesized by this regulator with the same $k = \bar{k}$. Here, the general equation (11) looks like

$$2(a + \bar{k})\hat{w} + \bar{k}^2\varphi^2 + \sigma^2 = 0.$$

So, $\hat{w} = \bar{w} + v$, where

$$v = \left(a^2\bar{w} + a\sigma^2 + \frac{\sigma^4}{4\bar{w}}\right) \cdot \frac{\varphi^2}{\sigma^2}$$

In Figure 1, we show plots of the function $\hat{w} = \bar{w} + v$ for $a = \sigma = 1$ and various values of φ.

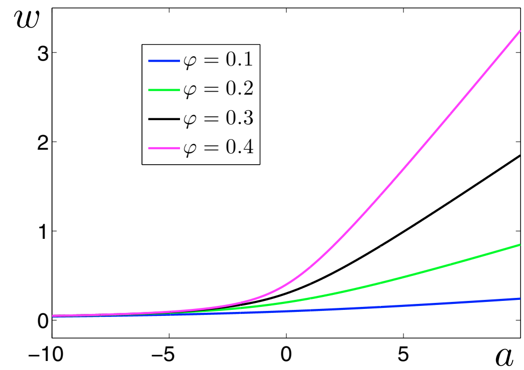

Fig. 2. Dependence of the minimal stochastic sensitivity w_* of system (13), (15) on a for $k = k_*$, $\sigma = 1$.

As one can see, if $\bar{w}$ tends to zero, the value v unlimitedly increases. So, in presence of noise in observations, the desire to use very small values of $\bar{w}$ leads to ill-posedness in the form of strong instability.

For the regularization of this problem, we suggest the following method. Choosing coefficients of the regulator that synthesizes the assigned stochastic sensitivity, one has to take into account a presence of the noise in measurements.

This means that we have to find a feedback matrix K as a solution of the equation (9) for the assigned stochastic sensitivity matrix W. Note that (9) is the quadratic matrix equation which can be rewritten in the following form:

$$\begin{aligned}&(BK\Phi^{\frac{1}{2}} + W\Phi^{-\frac{1}{2}})(BK\Phi^{\frac{1}{2}} + W\Phi^{-\frac{1}{2}})^\top - \\ &-W\Phi^{-1}W + AW + WA^\top + S = 0.\end{aligned} \tag{16}$$

The inequality

$$W\Phi^{-1}W - AW - WA^\top - S \succeq 0 \tag{17}$$

is the necessary condition for the solvability of this quadratic equation. Here, $Q \succeq 0$ means that the matrix Q is positive semi-definite. If $\mathrm{rank}(B) = n$, then the inequality (17) is also sufficient. This means that for any positive definite matrix W satisfying (17), one can find the solution of the quadratic equation (16) in the form Bashkirtseva et al. (2017)

$$K = -B^{-1}\left(Q^{\frac{1}{2}}Z\Phi^{-\frac{1}{2}} - W\Phi^{-1}\right).$$

Here, $Q = W\Phi^{-1}W - AW - WA^\top - S$, and Z is an arbitrary orthogonal $(n \times n)$-matrix.

Note that the inequality (17) plays a role of the additional condition that provides a regularization of our problem. This condition does not allow us to assign extremely small values of the stochastic sensitivity matrix. Minimal values of the stochastic sensitivity matrix W have to satisfy this restriction.

Let us consider how to use it in one-dimensional case. For system (13), (15), the inequality (17) can be written as

$$\frac{w^2}{\varphi^2} - 2aw - \sigma^2 \geq 0.$$

A minimal value $w = w_*$ satisfying this condition is as follows:

$$w_* = \varphi^2 \left[a + \sqrt{a^2 + \frac{\sigma^2}{\varphi^2}} \right].$$

The corresponding optimal regulator which synthesizes such minimal stochastic sensitivity has a feedback coefficient

$$k_* = -a - \sqrt{a^2 + \frac{\sigma^2}{\varphi^2}}.$$

In Figure 2, we show plots of the function $w_*(a)$ for $\sigma = 1$ and various values of φ. As one can see, the minimal stochastic sensitivity behaves quite regularly.

CONCLUSION

In some circumstances, control problems can be ill-posed and require an additional regularization. We have found that the problem of the synthesis of the assigned stochastic sensitivity of the equilibrium is ill-possed in presence of noise in measurements. We have suggested a regularization method and constructed an optimal feedback regulator. An efficiency of the proposed approach was demonstrated.

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