On a condition of existence of non-anticipating selections

Dmitrii A. Serkov*,**

* Krasovskii Institute of Mathematics and Mechanics Ural Branch of the Russian Academy of Sciences
** Ural Federal University named after the first President of Russia

Yekaterinburg, Russia (e-mail: serkov@imm.uran.ru)

Abstract: The hereditary selections of multi-functions play an important role in the theory of differential games in connection with the construction of resolving quasi-strategies. The existence of a non-anticipating selection of a non-anticipating multi-function is considered. In most cases important for applications, it is known that any non-anticipating multi-function with non-empty compact values has a non-anticipating selection. Namely, the result is valid when the non-anticipation property is defined by a totally ordered family in the domain of "time" variable. In this note, we show that the condition is essential: when the family is not totally ordered, there exists a hereditary multi-function with non-empty compact values that has no non-anticipating selections.

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1. INTRODUCTION

2. NOTATION AND DEFINITIONS

The non-anticipating selections of multi-functions play an important role in the theory of differential games in connection with the construction of idealized resolving strategies — quasi-strategies. In the early works (see Nardzewski (1964); Varaiya and Lin (1969); Roxin (1969); Elliott and Kalton (1972), etc.), quasi-strategies were defined as operators on the functional spaces of control realizations or trajectories with the property of physical feasibility or non-anticipation. On the other hand, in some game-theoretical constructions, non-anticipating multi-functions and, consequently, multi-valued quasi-strategies arise in a natural way (see Chentsov (1976)). In the same years (see Chentsov (1978)), the question of the existence of a single-valued selector of the multi-function preserving the property of non-anticipation was considered for mappings on the spaces of generalized controls, where specific properties of measures were essentially used. Recently (see Serkov and Chentsov (2018)) a rather general statement of the problem was studied: it was shown for the most important cases that every non-anticipating multi-function with non-empty and compact values has a non-anticipating selection. Namely, the result is valid when the non-anticipation property is defined by a totally ordered set family in the domain of "time" variable. In this paper studying the existence of a non-anticipative selection as an independent problem, we establish that the above condition of total ordering is essential; otherwise there are examples of non-anticipating multi-functions with non-empty compact values that have no non-anticipating selections.

Hereinafter, we use standard set-theoretic notation (quantifiers, relations, ∅ as the empty set); ≡ means "is equal by definition." Any set whose elements are sets is called a family. Let \( \mathcal{P}(T) \) denote the family of all (all non-empty) subsets of an arbitrary set \( T \). If \( A \) and \( B \) are non-empty sets, then \( B^A \) denotes the set of all mappings from \( A \) to \( B \). If \( f \in B^A \) and \( C \in \mathcal{P}(A) \), then \((f \upharpoonright C) \in B^C \), by definition, the restriction of \( f \) to the set \( C \): \( f \upharpoonright C(x) \equiv f(x) \forall x \in C \). For \( F \in \mathcal{P}(B^A) \), we set \((F \upharpoonright C) \equiv \{ (f \upharpoonright C) : f \in F \} \). For any set \( X \not= \emptyset \) and a partial order relation \( \leq \in \mathcal{P}(X \times X) \), we denote by \( (X, \leq) \) the corresponding partially ordered set (or poset). A set \( C \subseteq X \) is called a chain in \( (X, \leq) \) if it is totally ordered by \( \leq: (x \leq y) \lor (y \leq x) \forall x, y \in C \).

Choose non-empty sets \( T, X, \) and \( Y \), and fix non-empty sets \( T, \Omega, \) and \( Z \), such that \( T \in \mathcal{P}(\mathcal{P}(T)), \Omega \in \mathcal{P}(\mathcal{P}(Y)), \) and \( Z \in \mathcal{P}(\mathcal{P}(X)) \).

We call a multi-function \( \alpha \in \mathcal{P}(Z)^\Omega \) non-anticipating if the relation

\[ ((\omega_1 \mid H) = (\omega_2 \mid H)) \Rightarrow ((\alpha(\omega_1) \mid H) = (\alpha(\omega_2) \mid H)) \tag{1} \]

is fulfilled for all \( \omega_1, \omega_2 \in \Omega, H \in T \) and denote by \( \mathbf{n} \) the set of all non-anticipating multi-functions from \( \Omega \) to \( Z \).

We call a function \( \beta \in (Z^\Omega) \) non-anticipating if the relation

\[ ((\omega_1 \mid H) = (\omega_2 \mid H)) \Rightarrow ((\beta(\omega_1) \mid H) = (\beta(\omega_2) \mid H)) \tag{2} \]

is fulfilled for all \( \omega_1, \omega_2 \in \Omega, H \in T \) and denote by \( \mathbf{n} \) the set of all non-anticipating functions from \( \Omega \) to \( Z \).

For any \( \alpha \in \mathcal{P}(Z)^\Omega \), we define the subset \( \mathbf{n}[\alpha] \in \mathcal{P}(\mathbf{n}) \) of functions that are also selections of \( \alpha \):

\[ \mathbf{n}[\alpha] \equiv \mathbf{n} \cap \prod_{\omega \in \Omega} \alpha(\omega). \tag{3} \]
Thus, $n[\alpha] \in \mathcal{P}(Z^\Omega)$ is the set of all non-anticipating selections of a multi-function $\alpha$.

Let the set $X$ be equipped with the Hausdorff topology $\tau_X$. Then we assume that the set $Z \in \mathcal{P}(X^T)$ is equipped with the topology $\tau_Z$ induced by the Tikhonov topology $\otimes^T(\tau_X)$ on the product $\prod_{t \in T} X_t$, $X_t \cong X$; we also assume that the set $Z^\Omega = \prod_{\omega \in \Omega} Z_\omega$, $Z_\omega \cong Z$ is equipped by the Tikhonov product topology $\tau_{Z^\Omega} = \otimes^\Omega(\tau_Z)$.

3. THEOREM AND EXAMPLES

In Serkov and Chentsov (2018) the following theorem is proved.

Theorem 1. Let $T$ be a chain in the poset $(\mathcal{P}(T), \subseteq)$. Let $\alpha \in \mathcal{N}$ and $\alpha(\omega)$ be non-empty compact in $(Z, \tau_Z)$ for every $\omega \in \Omega$. Then $n[\alpha]$ is a non-empty compact in $(Z^\Omega, \tau_{Z^\Omega})$.

In particular, every non-anticipating multi-function with non-empty compact values has a non-anticipating selection.

In the following examples, we choose the domain of "time" variable as a segment of the real line that is typical for control problems. The first example shows that the total ordering condition on the family $T$ in Theorem 1 is essential. The second one shows that replacing this condition with the condition «$T$ forms a base of a filter» does not improve the situation.

Example 1. Let $T \cong [-\pi, \pi]$, $X = Y \cong \mathbb{R}$ and the set $X$ be equipped with the topology $\tau_X$ generated by $| \cdot |$-metric. So, $(X, \tau_X)$ is $T_2$-topological space. Let $A \cong [-\pi, -\frac{3\pi}{4}]$, $B \cong [\frac{3\pi}{4}, \pi]$ and $T \cong \{A, B\}$. Then, $T \in \mathcal{P}(\mathcal{P}(T))$. Let elements $\omega_r, \omega_b, \omega_g, \omega_y \in Y^T$ be defined by the relations (see Fig. 1):

$$\omega_r(t) = \max\{\omega_b(t), \omega_g(t), \frac{\pi}{2} - \cos(t)\},$$
$$\omega_b(t) \leqslant t, \quad \omega_g(t) \leqslant -\omega_r(t), \quad \omega_y(t) \leqslant -\omega_b(t)$$

for all $t \in T$ and let $\Omega \cong \{\omega_r, \omega_b, \omega_g, \omega_y\}$. It is easy to verify that the restrictions of elements (4) to the sets $A$ and $B$ satisfy only the following relations:

$$(\omega_r | B) = (\omega_b | B),$$
$$(\omega_b | A) = (\omega_y | A),$$
$$(\omega_g | B) = (\omega_y | B),$$
$$(\omega_y | A) = (\omega_r | A).$$

Let elements $h_{r_1}, h_{b_1}, h_{g_1}, h_{b_2}, h_{g_2} \in X^T$ be defined by the relations (see Fig. 2; in the definitions of $h_{r_1}, h_{r_2}$ we use extensions of $\omega_r$ provided by (4)):

$$h_{r_1}(t) = \omega_r(t + \pi/4) + \pi/4, \quad h_{r_2}(t) = \omega_r(t - \pi/4) + \pi/4,$$
$$h_{b_1}(t) = \omega_b(t), \quad h_{b_2}(t) = \omega_b(t + \pi/2),$$
$$h_{g_1}(t) = \omega_g(t), \quad h_{g_2}(t) = \omega_g(t + \pi/2),$$
$$h_{y_1}(t) = \omega_y(t), \quad h_{y_2}(t) = \omega_y(t + \pi/2),$$

for $t \in T$ and let $Z \cong \{h_{r_1}, h_{b_1}, h_{g_1}, h_{b_2}, h_{g_2}, h_{y_2}\}$. It is easy to verify that the restrictions of the elements from $Z$ to the sets $A$ and $B$ satisfy the relations below:

$$(h_{r_1} | B) = (h_{b_2} | B), \quad (h_{r_2} | B) = (h_{b_1} | B),$$
$$(h_{b_1} | A) = (h_{g_1} | A), \quad (h_{b_2} | A) = (h_{g_2} | A),$$
$$(h_{g_1} | B) = (h_{b_1} | B), \quad (h_{g_2} | B) = (h_{b_2} | B),$$
$$(h_{g_1} | A) = (h_{r_1} | A), \quad (h_{g_2} | A) = (h_{r_2} | A).$$

Fig. 1. The set $\Omega$: the case $A \cap B = \varnothing$.

Fig. 2. The set $Z$: the case $A \cap B = \varnothing$.

We consider the multi-function $\alpha \in \mathcal{P}(Z^\Omega)$ of the form

$$\alpha(\omega_r) \cong \{h_{r_1}, h_{r_2}\},$$
$$\alpha(\omega_b) \cong \{h_{b_1}, h_{b_2}\},$$
$$\alpha(\omega_g) \cong \{h_{g_1}, h_{g_2}\},$$
$$\alpha(\omega_y) \cong \{h_{y_1}, h_{y_2}\}.$$

It is clear that the values of the multi-function are non-empty and compact in $(Z, \tau_Z)$ (as, in fact, in any other topology on $Z$). From relations (9)–(12) and (13)–(16), we derive the following equalities:

$$\alpha(\omega_r) | B = (\alpha(\omega_b) | B),$$
$$\alpha(\omega_b) | A = (\alpha(\omega_y) | A),$$
$$\alpha(\omega_g) | B = (\alpha(\omega_y) | B),$$
$$\alpha(\omega_y) | A = (\alpha(\omega_r) | A).$$

It means (see (5)–(8), (1)) that $\alpha$ is non-anticipative, or $\alpha \in \mathcal{N}$. Thus, $\alpha$ satisfies all the conditions of Theorem 1.

Let us show that $n[\alpha] = \varnothing$, i.e. there is no non-anticipating selections of the multi-function $\alpha$. Suppose the contrary, that there exists a function $\beta \in n[\alpha]$. Hence (see (3), (13)–(16)), $\beta$ satisfies (2) and the inclusions
\[
\beta(\omega_r) \in \{h_{r1}, h_{r2}\}, \quad \beta(\omega_b) \in \{h_{b1}, h_{b2}\}, \quad \beta(\omega_y) \in \{h_{y1}, h_{y2}\}, \quad \beta(\omega_y) \in \{h_{y1}, h_{y2}\}.
\] (21) (22) (23) (24)

Let us suppose (see (21)) that
\[
\beta(\omega_r) = h_{r1}.
\] (25)

From (2), (5), (25), we have \((h_{r1} | B) = (\beta(\omega_r) | B) = (\beta(\omega_b) | B). Then (see (22), (9)),
\[
\beta(\omega_b) = h_{b2}.
\] (26)

From (2), (6), (26), we have \((h_{b2} | A) = (\beta(\omega_b) | A) = (\beta(\omega_g) | A). Then (see (23), (10)),
\[
\beta(\omega_g) = h_{y2}.
\] (27)

From (2), (7), (27), we have \((h_{y2} | B) = (\beta(\omega_y) | B) = (\beta(\omega_y) | B). Then (see (24), (11)),
\[
\beta(\omega_y) = h_{y2}.
\] (28)

From (2), (8), (28), we have \((h_{y2} | A) = (\beta(\omega_y) | A) = (\beta(\omega_r) | A). Then (see (21), (12)),
\[
\beta(\omega_r) = h_{r2},
\] (29)

which contradicts equality (25). So, assuming \(\beta(\omega_r) = h_{r1}\), we get a contradiction. In view of (21), there is only one possibility: to assume that \(\beta(\omega_r) = h_{r2}\). But in this case, by analogous arguments, we obtain the equality \(\beta(\omega_r) = h_{r1}\), which contradicts this assumption. Thus, the assumption of the existence of a non-anticipative selection of the multi-function \(\alpha\) leads to contradiction, that is, the statement of Theorem 1 does not hold.

**Example 2.** A similar example holds in the case when the family \(T\) forms a base of a filter. Recall (see, for example, (Engelking, 1985, §1.6.7)) that a family \(\mathcal{V} \in \mathfrak{F}(\mathfrak{P}(\mathcal{V}))\) is a filter base if, for any \(v, v' \in \mathcal{V}\), there exists a set \(v'' \in \mathcal{V}\) such that \(v'' \subset v \cap v'\).

Let \(T, X, Y\) and \(\tau_X\) be defined as in Example 1. Let \(A = [-\pi, \pi/3], B = [-\pi/3, \pi], C = [-\pi/3, \pi/3]\) and \(\mathcal{T} \triangleq \{A, B, C\}. Then \(\mathcal{T} \in \mathfrak{F}(\mathfrak{F}(\mathcal{T}))\), and the family \(\mathcal{T}\) is a filter base. Let the elements \(\omega_r, \omega_b, \omega_g, \omega_y \in Y^T\) be of the form (see Fig. 3; for clarity, these functions are shown slightly apart from each other):
\[
\omega_r(t) \triangleq \max\{\min\{t + \pi/3, 0\}, t - \pi/3\}, \quad \omega_b(t) \triangleq -\omega_g(t), \quad \omega_g(t) \triangleq -\omega_r(t), \quad \omega_y(t) \triangleq -\omega_r(t),
\]

\(t \in T\), and let \(\Omega \triangleq \{\omega_r, \omega_b, \omega_g, \omega_y\}\). It is easy to verify that the restrictions of the elements of \(\Omega\) to the set \(C\) coincide:
\[
(\omega_r | C) = (\omega_b | C) = (\omega_g | C) = (\omega_y | C),
\] (30)

and the restrictions to the sets \(A\) and \(B\) (as in Example 1) satisfy only relations (5)–(8).

Let the elements \(h_{r1}, h_{b1}, h_{b2}, h_{y1}, h_{y2} \in \mathfrak{X}^T\) be of the form (see Fig. 4):
\[
h_{r1}(t) \triangleq \max\{-0.7 \times (t + \pi/3), 0, t - \pi/3\},
\]
\[
h_{r2}(t) \triangleq \min\{-0.7 \times (t - \pi/3), 0, t - \pi/3\},
\]
\[
h_{b1}(t) \triangleq \max\{\min\{t + \pi/3, 0\}, 0.7 \times (t - \pi/3)\},
\]
\[
h_{b2}(t) \triangleq -h_{b1}(t),
\]
\[
h_{g1}(t) \triangleq \omega_y(t),
\]
\[
h_{g2}(t) \triangleq 0.7 \times \omega_y(t),
\]
\[
h_{g1}(t) \triangleq \max\{-0.7 \times (t + \pi/3), \min\{0, \pi/3 - t\}\},
\]
\[
h_{g2}(t) \triangleq -h_{b1}(t).
\] (31)

Fig. 3. The set \(\Omega\): \(T\) is a base of a filter.

\[
t \in T, \quad \text{and let } Z \triangleq \{h_{r1}, h_{b1}, h_{b2}, h_{y1}, h_{y2}, h_{b2}, h_{y2}\}. Then the restrictions of elements (31) to the set \(C\) coincide:
\[
(\omega_r | C) = (h_{b2} | C) = (h_{y1} | C) = (h_{b2} | C) = (\omega_g | C) = (h_{y2} | C),
\] (32)

and the restrictions to the sets \(A\) and \(B\) (as in Example 1) satisfy relations (9)–(12). Let a multi-function \(\alpha \in \mathfrak{F}(\mathfrak{Z})^T\)

be defined by relations (13)–(16). It is clear, that the values of \(\alpha\) are non-empty and compact in \((\mathfrak{Z}, \tau_Z)\). From (9)–(12), (32) and (13)–(16), we have equalities (17)–(20) and
\[
(\alpha(\omega_r) | C) = (\alpha(\omega_b) | C) = (\alpha(\omega_g) | C) = (\alpha(\omega_y) | C).
\] (33)

From relations (5)–(8), (30), (17)–(20), (33) and definition (1), it follows that the multi-function \(\alpha\) is non-anticipating, or \(\alpha \in \mathfrak{N}\). We have verified that the multi-function \(\alpha\) satisfies all the conditions of Theorem 1.

To show that \(\mathfrak{n}[\alpha] = \mathfrak{E}\), let us suppose the contrary: there is a function \(\beta \in \mathfrak{n}[\alpha]\). Then, as in Example 1, assumption (25) implies equality (29), contradicting this assumption, and vice versa. So, the assumption \(\beta \in \mathfrak{n}[\alpha]\) was wrong.

Thus, the weakening of the total ordering condition to the filter base condition in Theorem 1 is also impossible.
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