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# Stochastic Sensitivity Synthesis in Discrete-Time Systems with Parametric Noise

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**Abstract:** Discrete nonlinear stochastic systems with general parametric noises are considered. To approximate the dispersion of random states, we propose an asymptotic approach based on the stochastic sensitivity analysis. This approach is used for the solution of the stabilization problem for the discrete controlled systems forced by parametric noise. A theory of the synthesis of the stochastic sensitivity by the feedback regulators is elaborated. Regulators minimizing the stochastic sensitivity are used in the problem of the structural stabilization of equilibrium regimes in population dynamics. The efficiency of this technique is demonstrated on the example of the suppression of undesired noisy large-amplitude regular and chaotic oscillations in the Hassell population model.



*Keywords:* Discrete systems, parametric noise, feedback regulators, stochastic sensitivity, structural stabilization.

## 1. INTRODUCTION

Studies devoted to the development of the control theory for nonlinear stochastic systems are actively developed (see e.g. Krasovskii et al. (1961); Astrom (1970); Fleming et al. (1975); Sun (2006); Kibzun et al. (2017)). An impact of random disturbances on the nonlinear dynamical systems can result in various unexpected phenomena (Horsthemke (1984); Anishchenko et al. (2007); Crutchfield et al. (1982); Schuster (1984); Fedotov et al. (2002, 2006)).

A full mathematical description of the dynamics of probabilistic distributions of stochastic discrete systems is given by the PerronFrobenius equation (Lasota et al. (1994)). Such type equations can be solved directly only in very special cases, and using these functional equations in control problems is very difficult technically. In these circumstances, approximations and asymptotics are very useful.

The asymptotic probabilistic analysis based on the stochastic sensitivity has been proposed in (Bashkirtseva et al. (2010a)), and developed in (Bashkirtseva et al. (2014); Ryashko et al. (2017); Bashkirtseva et al. (2018a,b)). In control problems, this approach was used in (Bashkirtseva et al. (2010b, 2011)).

In the present paper, we extend this technique on the case of discrete nonlinear stochastic systems with general parametric noises. A problem of the synthesis of the stochastic sensitivity is reduced to the solution of the quadratic matrix equation. Feedback regulators minimizing the stochastic sensitivity are constructed.

An application of the elaborated theory to the important problem of the structural stabilization of population systems in presence of parametric noise is given.

## 2. STOCHASTIC SENSITIVITY OF EQUILIBRIUM FORCED BY PARAMETRIC NOISE

Consider a general discrete-time nonlinear system

$$x_{t+1} = f(x_t, \eta_t), \tag{1}$$

where $x$ is an $n$-vector, and $f(x, \eta)$ is a smooth vector-function with $l$-dimensional vector of parameters $\eta$. It is supposed that parameters of system (1) are subject to random disturbances: $\eta_t = \varepsilon\xi_t$. Here, $\xi_t$ is an $l$-dimensional uncorrelated random process with parameters:

$$\mathrm{E}\xi_t = 0, \quad \mathrm{E}\xi_t\xi_t^\top = \mathrm{V}, \quad \mathrm{E}\xi_t\xi_k^\top = 0 \ (t \neq k).$$

Here, V is $l \times l$-matrix, and $\varepsilon$ is a scalar parameter of the noise intensity.

Let $\bar{x}$ be an exponentially stable equilibrium of the deterministic system (1) with $\eta = 0$: $\bar{x} = f(\bar{x}, 0)$.

Consider a solution $x_t^\varepsilon$ of the stochastic system (1) with initial data $x_0^\varepsilon = \bar{x} + \varepsilon z_0$, where the $n$-vector $z_0$ defines an initial deviation.

An asymptotics

$$z_t = \lim_{\varepsilon \to 0} \frac{x_t^\varepsilon - \bar{x}}{\varepsilon}$$

of the deviation of $x_t^\varepsilon$ from $\bar{x}$ is governed by the following linear stochastic equation:

$$z_{t+1} = Fz_t + G\xi_t, \tag{2}$$

where

$$F = \frac{\partial f}{\partial x}(\bar{x}, 0), \quad G = \frac{\partial f}{\partial \eta}(\bar{x}, 0).$$

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The deterministic characteristics

$$m_t = \mathrm{E}z_t, \quad M_t = \mathrm{E}z_t z_t^\top$$

of the random discrete-time process $z_t$ are solutions of the equations:

$$m_{t+1} = Fm_t, \tag{3}$$

$$M_{t+1} = FM_tF^\top + S, \quad S = GG^\top. \tag{4}$$

Due to the exponential stability of $\bar{x}$, the spectral radius of the matrix $F$ satisfies the inequality $\rho(F) < 1$. Hence, the system (3) has a unique stable stationary solution $m = \lim_{t\to\infty} m_t = 0$. The system (4) also has a unique stable stationary solution $M = \lim_{t\to\infty} M_t$ satisfying the following matrix equation

$$M = FMF^\top + S. \tag{5}$$

The matrix $M$ is the stochastic sensitivity matrix of the equilibrium $\bar{x}$. This matrix allows us to approximate a dispersion of stationary distributed random states $x_t^\varepsilon$ of nonlinear stochastic system (1) near $\bar{x}$:

$$\mathrm{cov}(x_t^\varepsilon, x_t^\varepsilon) \approx \varepsilon^2 M.$$

Further, consider how one can control this distribution by the synthesis of the appropriate stochastic sensitivity matrix $M$.

## 3. SYNTHESIS OF STOCHASTIC SENSITIVITY

Consider a discrete-time nonlinear system with parametric noise and control

$$x_{t+1} = f(x_t, u_t, \eta_t), \quad \eta_t = \varepsilon\xi_t, \tag{6}$$

where $x$ is an $n$-dimensional vector, $u$ is $r$-dimensional vector of the control input, $f(x, u, \eta)$ is a smooth vector-function, and $\xi_t$ is an $l$-dimensional vector of random disturbances with parameters:

$$\mathrm{E}\xi_t = 0, \quad \mathrm{E}\xi_t\xi_t^\top = \mathrm{V}, \quad \mathrm{E}\xi_t\xi_k^\top = 0 \ (t \neq k).$$

Here, V is $l \times l$-matrix, and $\varepsilon$ is a scalar parameter of the noise intensity.

Let $\bar{x}$ be an equilibrium of the deterministic system (6) with $u = 0$, $\eta = 0$. Note that the stability of $\bar{x}$ is not supposed.

We will consider the feedback control $u = u(x)$. It is assumed that $u(\bar{x}) = 0$, and $\bar{x}$ is exponentially stable in the closed-loop deterministic system

$$x_{t+1} = f(x_t, u(x_t), 0). \tag{7}$$

For the asymptotics $z_t = \lim_{\varepsilon\to 0} \frac{x_t^\varepsilon - \bar{x}}{\varepsilon}$ of deviations of solutions $x_t^\varepsilon$ of the closed-loop stochastic system

$$x_{t+1} = f(x_t, u(x_t), \varepsilon\xi_t) \tag{8}$$

from the deterministic equilibrium $\bar{x}$, one can write

$$z_{t+1} = (F + BK)z_t + G\xi_t. \tag{9}$$

Here,

$$F = \frac{\partial f}{\partial x}(\bar{x}, 0, 0), \quad G = \frac{\partial f}{\partial \eta}(\bar{x}, 0, 0),$$
$$B = \frac{\partial f}{\partial u}(\bar{x}, 0, 0), \quad K = \frac{\partial u}{\partial x}(\bar{x}).$$

First two moments $m_t = \mathrm{E}z_t$, $M_t = \mathrm{E}z_t z_t^\top$ satisfy the equations

$$m_{t+1} = (F + BK)m_t, \tag{10}$$

$$M_{t+1} = (F + BK)M_t(F + BK)^\top + S. \tag{11}$$

The set $\mathbf{K}$ of matrices $K$ that provide an exponential stability of the equilibrium $\bar{x}$ of system (7) can be written as

$$\mathbf{K} = \{K |\ \rho(F + BK) < 1\}.$$

We suppose that the set $\mathbf{K}$ is not empty.

For any $K \in \mathbf{K}$, the equation (11) has a unique stable stationary solution $M = \lim_{t\to\infty} M_t$ which satisfies the matrix equation

$$M = (F + BK)M(F + BK)^\top + S. \tag{12}$$

Consider now how the stochastic sensitivity matrix $M$ depends on the feedback $u(x)$. As one can see, $M$ is fully defined only by the local parameters $K = \frac{\partial u}{\partial x}(\bar{x})$. So, we can consider the regulators of a simple linear structure

$$u(x) = K(x - \bar{x}). \tag{13}$$

We now consider the problem of the synthesis of the assigned stochastic sensitivity matrix $M$ by an appropriate regulator (13).

Let $\mathbf{M} = \{M \in \mathbf{R}^{n\times n} | M \succ 0\}$ be a set of the admissible stochastic sensitivity matrices. Here, $M \succ 0$ means that the matrix $M$ is symmetric and positive definite. We denote by $M_K$ the solution of the equation (12) for the fixed matrix $K \in \mathbf{K}$. The aim of control is the synthesis of the assigned stochastic sensitivity matrix.

Let $W$ be the required stochastic sensitivity matrix of the system (8), (13). In order to find the matrix $K \in \mathbf{K}$ which provides the equality $M_K = W$, we have to solve the following quadratic matrix equation

$$\begin{aligned} W = {} & FWF^\top + FWK^\top B^\top + BKWF^\top + \\ & + BKWK^\top B^\top + S. \end{aligned} \tag{14}$$

In some cases, this equation is unsolvable, therefore, a preliminary attainability analysis Bashkirtseva et al. (2010b) should be carried out.

In the one-dimensional case ($n = r = l = 1$), the equation (14) is written as

$$B^2WK^2 + 2BFWK + F^2W + S - W = 0.$$

For this quadratic equation, the discriminant is $D = 4B^2W(W - S)$, therefore the inequality $W \geq S$ is a condition of attainability. For $B \neq 0$, we obtain an explicit formula for the feedback coefficient

$$K = -\frac{1}{B}\left(-F \pm \sqrt{1 - \frac{S}{W}}\right). \tag{15}$$

The function $M_K$ has the form

$$M_K = \frac{S}{1 - (F + BK)^2}. \tag{16}$$

Note that the value $M_K$ is minimal ($M_K = S$) for $K = -F/B$.

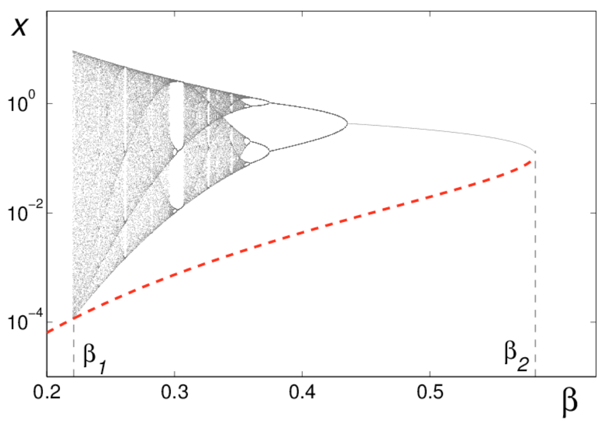



Fig. 1. Bifurcation diagram of the deterministic Hassell model without control

Consider now how this general theory can be applied to the problem of the structural stabilization of the dynamics of population systems.

## 4. STRUCTURAL STABILIZATION OF THE STOCHASTIC HASSELL MODEL

In population dynamics theory, an important role is played by the simple so-called conceptual models. Along with the well-known logistic model, the discrete system proposed by Hassell (see Hassell (1975)) is widely used. This model exhibits a variety of dynamical regimes, both regular and chaotic (Geritz et al. (2004); Bascompte et al. (1994)). Here, we consider Hassell-type discrete model with embedded Allee effect.

In this model, there exists a trivial stable equilibrium corresponding to the extinction. Mathematically, the population dynamics in persistence regime is described by the nontrivial attractor in the form of equilibrium, periodic, and chaotic oscillations.

Consider the following Hassell-type population model with Allee effect:

$$x_{t+1} = \frac{\alpha x_t^2}{(\beta + x_t)^6}. \tag{17}$$

Here, $x_t$ is a population size at the time $t$, the positive parameter $\alpha$ stands for the intrinsic growth rate, and $\beta$ defines the carrying capacity of the environment. The trivial equilibrium $\bar{x}_0 = 0$ of this system is stable for any parameters. Along with $\bar{x}_0 = 0$, the system (17) can possess two more equilibria $\tilde{x}, \ \bar{x} : \bar{x}_0 < \tilde{x} < \bar{x}$. The equilibrium $\tilde{x}$ is always unstable, and the stability of $\bar{x}$ depends on system parameters.

In what follows, we study system dynamics under the variation of the parameter $\beta$ for fixed $\alpha = 1$. In Fig. 1, the bifurcation diagram of the model (17) is presented. Here, the unstable equilibrium $\tilde{x}$ is shown by red dashed line, and nontrivial attractors (Feigenbaum's tree) are plotted by grey. As one can see, the persistence $\beta$-zone for system (17) is bounded by the interval $\beta_1 < \beta < \beta_2$, where $\beta_1 = 0.2202$ marks the crisis bifurcation, and $\beta_2 = 0.5824$ marks the saddle-node bifurcation. Outside this interval, the population is extinct for all initial values $x_0$.

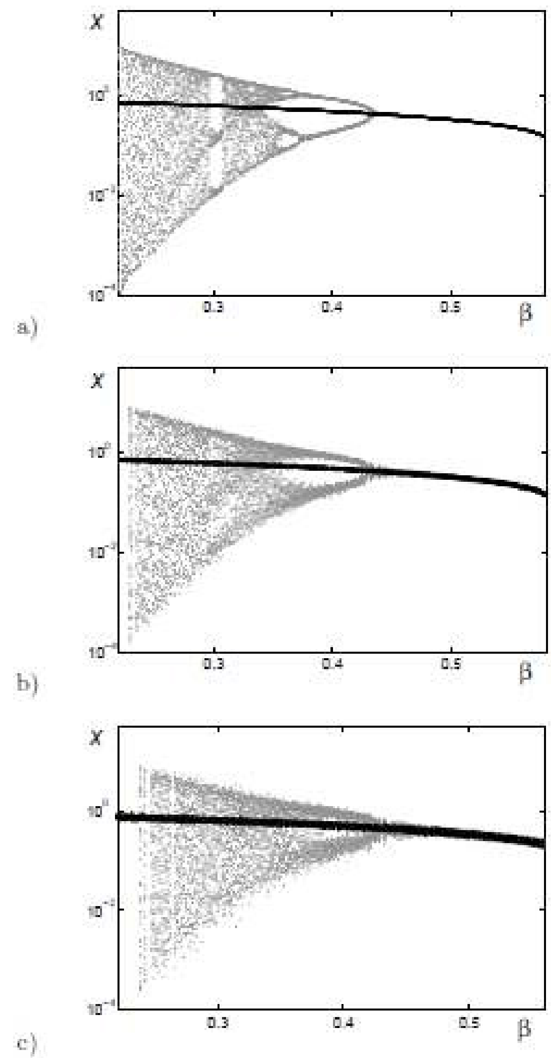



Fig. 2. Random states of the stochastic Hassell model without control (grey) and with control (black) for a) $\varepsilon = 0.001$, b) $\varepsilon = 0.005$, c) $\varepsilon = 0.01$.

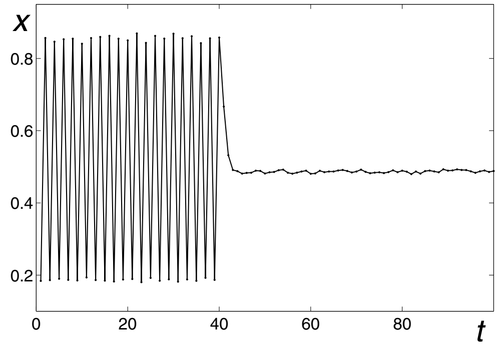
Note that in the interval $\beta_1 < \beta < \beta_2$, the system is bistable. One of the coexisting attractors (trivial equilibrium) is associated with the regime of extinction, and another one corresponds to the persistence regime. This persistence regime exists in the form of equilibria, regular, or chaotic oscillations. The unstable equilibrium playing a role of the separatrix for basins of attraction is a dangerous border between regimes of persistence and extinction.

Along with the deterministic system (17), consider the stochastic system

$$x_{t+1} = \frac{\alpha x_t^2}{(\beta + \varepsilon\xi_t + x_t)^6}, \tag{18}$$

with the stochastically forced parameter $\beta$ of the carrying capacity. Here, $\xi_t$ is a standard uncorrelated discrete-time Gaussian process, $\varepsilon$ is the noise intensity.

Under random perturbations, a thin structure of the Feigenbaum's tree is washed out (see Fig. 2, grey color). With increasing noise, the trajectory of system (18) can intersect the separatrix (dashed line) and fall into the




a)

b)

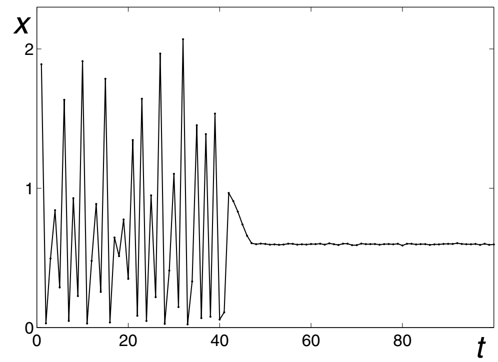



Fig. 3. Time series of the stochastic system with $\varepsilon = 0.005$ and a) $\beta = 0.4$ (deterministic 2-cycle), b) $\beta = 0.32$ (chaotic attractor). The control is switched on at $t = 40$.

basin of attraction of the trivial equilibrium $\bar{x}_0 = 0$. This behavior corresponds to the noise-induced extinction.

To prevent these undesirable ecological shifts caused by noise, consider an appropriate control procedure. The model (18) with control looks like

$$x_{t+1} = \frac{\alpha x_t^2}{(\beta + \varepsilon\xi_t + x_t)^6} + u_t, \;\; u_t = K(x_t - \bar{x}(\beta)). \quad (19)$$

Here, we use the feedback regulator that forms the control input proportional to the deviation of the state $x_t$ from the equilibrium $\bar{x}(\beta)$. To stabilize the stochastic dynamics of system (19) near $\bar{x}(\beta)$, we will use the theory of the stochastic sensitivity synthesis. Here, we will use the regulator which provides the minimal value of the stochastic sensitivity. An effectiveness of the proposed control approach is illustrated in Figs. 2, 3.

In Fig. 3, we compare uncontrolled and controlled stochastic dynamics for two fixed values $\beta = 0.4$ and $\beta = 0.32$. For $\beta = 0.4$, the deterministic uncontrolled system (17) has regular oscillations (2-cycle). Under the random disturbances with $\varepsilon = 0.005$, we observe noisy oscillations (Fig. 3a for $0 \le t < 40$). At $t = 40$, we switch on the control synthesizing the minimal stochastic sensitivity of the equilibrium $\bar{x}$. Stochastic dynamics of system with such control is shown in Fig. 3a for $t \ge 40$. Here, one can observe small-amplitude stochastic oscillations. Note that here this regulator not only stabilizes the unstable equilibrium $\bar{x}$ but also provides a small dispersion of random states around $\bar{x}$.

For $\beta = 0.32$, the deterministic uncontrolled system (17) demonstrates chaotic oscillations. Under the random disturbances with $\varepsilon = 0.005$, we observe noisy chaos (see Fig. 3b for $0 \le t < 40$). At $t = 40$, our optimal regulator minimizing stochastic sensitivity of the equilibrium $\bar{x}$ is switched on. Stochastic dynamics of the controlled system is shown in Fig. 3b for $t \ge 40$. Again, we observe the suppression of large-amplitude stochastic oscillations.

Results of the optimal control for the whole parametric $\beta$-zone are shown in Fig. 2 by black color for three values of the noise intensity. As one can see, this control provides the structural stabilization in the wide range of system parameters.

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