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Numerical discretization for fractional differential equations with feedback control

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Abstract: In this paper, we introduce a numerical scheme for fractional differential equations with feedback control. Due to the possibility of dealing with a feedback control as a functional delay, we construct a numerical method based on Euler method accompanied with piecewise constant interpolation. The method is based on the idea of separating the current state and the prehistory function. The convergence of the method is stated and proved. Numerical experiments are given to clarify the good agreement between numerical and theoretical results.

Keywords: Fractional differential equation, Feedback control, Functional delay, Numerical scheme, Convergence analysis.

1. INTRODUCTION

In recent years, a high attention has been devoted to fractional calculus as a powerful tool for more precise modeling of real world phenomena. It has been shown that using this area of science in designing control systems results in controllers that are more efficient in comparison with traditional integer order controllers I.S. Jesus, J.T. Machado (2008); P. Lanusse, H. Benlaoukli, D. Nelson-Gruel, A. Oustaloup (2008). Due to that and because of the difficulty of finding analytical solutions for fractional differential equations(FDEs), we seek to introduce a simple numerical scheme for FDEs I. Podlubny (1999); K. Diethelm (2010) with feedback control

$$D^{(\beta)}x(t) = f(t, x(t), u), \quad t \in [t_0, \vartheta], \quad 0 < \beta \leq 1, \tag{1}$$

with initial conditions:

$$x(t_0) = x_0, \tag{2}$$

the time fractional derivative is defined in Caputo sense

$$D^{(\beta)}x(t) = \frac{1}{\Gamma(1-\beta)} \int\limits_{t_0}^{t} \frac{x\prime(\xi)}{(t-\xi)^{\beta}} d\xi.$$

By noticing FDEs (1) at each time point, we see that the system (1) can be conditioned by the prehistory of a function $x_t(\cdot) = \{x(t+s), s \in [t_0 - t, 0]\}$. So, if we suppose that the control gets out by the principle of feedback $u = u[x_t(\cdot)]$, we will receive the fractional equation with functional delay

$$D^{(\beta)}x(t) = f(t, x(t), u(x_t(\cdot))), \tag{3}$$

then equation (3) with an initial condition (2) can be written down in the form

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int\limits_{t_0}^{t} (t-\xi)^{1-\beta} f(\xi, x(\xi), u(x_\xi(\cdot)))d\xi. \tag{4}$$

We assume that the function $f(t, x, u)$ and the functional $u(x_t(\cdot))$ are chosen such that the problem (3) with initial conditions (2) has a unique solution $x(t)$, $t \in [t_0, \vartheta]$. The existence of a solution in similar problems were studied in Z. Yang, J. Cao (2013); Y. Zhou, J. Wang, L. Zhang (2014). We additionally assume that the function $f(t, x, u)$ is Lipschitz in the last two arguments, i.e., there exists a constants L_1 and L_2 such that, for all t, x^1, x^2, u^1, u^2, the following inequality holds:

$$| f(t, x^1, u^1) - f(t, x^2, u^2) | \leq L_1 | x^1 - x^2 | \tag{5}$$
$$+ L_2 | u^1 - u^2 | .$$

The functional $u(x_t(\cdot))$ is given on a set $Q = Q[t_0, t]$ of functions that are piecewise continuous on $[t_0, t]$ with a finite number of points of discontinuity of first kind and right continuous at the points of discontinuity, and the functional $u(x_t(\cdot))$ is Lipschitz, i.e., there exist a constant L_3, such that, for all t, $x_t^1(\cdot) \in Q$, $x_t^2(\cdot) \in Q$, the following inequality holds:

$$| u(x_t^1(\cdot)) - u(x_t^2(\cdot)) | \leq L_3 \|x_t^1(\cdot)) - x_t^2(\cdot))\|_Q, \tag{6}$$

where $\|x_t(\cdot))\|_Q = \max\limits_{t_0 \leq s \leq t} |x(s)|$.

2. NUMERICAL APPROACH

Now, we construct a numerical method for the solution of a problem (3) with initial condition (2). Numerical

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methods had been designed for the fractional differential equations without delay, see for example K. Diethelm (2010); C.P. Li, F.H. Zeng (2015), fractional differential equations with constant or variable concentrated delay Z. Wang, X. Huang, J. Zhou (2013); Y. Jia, Y. Xu, M. Lin (2017), functional differential equations of integer order C.W. Cryer, L. Tavernini (1972); A.V. Kim, V.G. Pimenov (2004), fractional variational problems M.M. Khader, A.S. Hendy (2013).

2.1 Discretization of the problem

Let us fix some notations such that $\Delta = (\vartheta - t_0)/N$, where N is a positive integers, we introduce the time points $t_i = t_0 + i\Delta$, $i = 0, \dots, N$. Denote by x_i the approximations of functions $x(t_i)$ at the time points.

Let us introduce a discrete history for the time points t_i : $\{x_j\}_i = \{x_j, 0 \le j \le i\}$. The mapping $I : \{x_j\}_i \to x(t)$, $t \in [0, t_i]$ will be called the interpolation operator of discrete history. As we will construct a method of first order accuracy, we use the piecewise-constant interpolation

$$x(t) = x(t_{j-1}),\ t \in [t_{j-1}, t_j), \tag{7}$$

then, forward Euler method with piecewise-constant interpolation (7) is used to design the following algorithm for (4)

$$x_{i+1} = x_0 + \Delta^\beta \sum_{j=0}^{i} b_{j,i+1} f(t_j, x_j, u(x_{t_j}(\cdot))), \tag{8}$$

where

$$b_{j,i+1} = \frac{1}{\Gamma(\beta+1)}[(i-j+1)^\beta - (i-j)^\beta]. \tag{9}$$

Also the method can be written down in form

$$x_{i+1} = x_i + \frac{\Delta^\beta}{\Gamma(\beta+1)} f(t_i, x_i, u(x_{t_i}(\cdot))) \tag{10}$$

$$+\Delta^\beta \sum_{j=0}^{i-1} (b_{j,i+1} - b_{j,i}) f(t_j, x_j, u(x_{t_j}(\cdot))).$$

It is worth mentioning at $\beta = 1$ that the method (8) has an equivalence with Euler method for functional differential equations of first order which presented in C.W. Cryer, L. Tavernini (1972); A.V. Kim, V.G. Pimenov (2004).

3. CONVERGENCE ANALYSIS OF THE NUMERICAL SCHEME

Denote by $\varepsilon_i = | x(t_i) - x_i |$ is the absolute difference between the exact solution and the numerical solution of the scheme (8). The method converges with order p if there exists a constant C independent of Δ such that $\varepsilon_j^i \le C\Delta^p)$ for all $i = 0, 1, \cdots, N$.

Also, we define a prehistory of the error at the time t_i as follows $\{\varepsilon_j\}_i = \{\varepsilon_j, 0 \le j \le i\}$ with norm $\|\{\varepsilon_j\}_i\| = \max_{0 \le j \le i} \varepsilon_j$.

Lemma 1. Let the solution $x(t)$ of (3) with initial conditions (2) is continuously differentiable at $[t_0, \vartheta]$, then

$$\max_{t \in [t_0, t_i]} |x(t) - y(t)| \le \|\{\varepsilon_j\}_i\| + C\Delta, i = 0, 1, \cdots, N, \tag{11}$$

such that the constant C is independent of Δ.

Proof. Suppose that

$$\max_{t \in [t_0, t_i]} |x(t) - y(t)| = |x(t^*) - y(t^*)|,\ t^* \in [t_{j-1}, t_j), j \le i,$$

then

$$x(t^*) = x(t_{j-1}) + x'(c)(t^* - t_{j-1}),\ t_{j-1} \le ct^*,$$

then taking into account (7), we obtain

$$|x(t^*) - y(t^*)| \le \varepsilon_{j-1} + C_1\Delta,\ C_1 = \max_{t \in [t_0, \vartheta]} |x'(t)|,$$

and so the inequality (11) is achieved.

We introduce the discretized form of Gronwall inequality J. Dixon (1985); C.P. Li, F.H. Zeng (2013); Y. Zhou, J. Wang, L. Zhang (2014) which will be used in the proof of next theorem

Lemma 2. Let $a, b > 0$ and η_i satisfy

$$| \eta_n | \le b + ah \sum_{i=0}^{n-1} | \eta_i |, \quad n = k, k+1, ..., nh \le T,$$

then

$$| \eta_n | \le \exp(aT)(b + akhM_0), \quad n \le k, nh \le T,$$

where $M_0 = \max(| \eta_0 |, | \eta_1 |, ..., | \eta_{k-1} |)$.

Theorem 3. Let the solution $x(t)$ of (3) with initial condition (2) is continuously differentiable at $t \in [t_0, \vartheta]$, then the method (8) converges with first order of accuracy.

Proof. Recalling definition of error ε_i, noticing eq.(4) and eq.(8), we get

$$\varepsilon_{i+1} = \tag{12}$$

$$| \tfrac{1}{\Gamma(\beta)} \int_{t_0}^{t_{i+1}} (t_{i+1} - \xi)^{1-\beta} f(\xi, x(\xi), u(x_\xi(\cdot)))d\xi -$$
$$\Delta^\beta \sum_{j=0}^{i} b_{j,i+1} f(t_j, y_j, u(y_{t_j}(\cdot))) | \le$$
$$| \tfrac{1}{\Gamma(\beta)} \int_{t_0}^{t_{i+1}} (t_{i+1} - \xi)^{1-\beta} f(\xi, x(\xi), u(x_\xi(\cdot)))d\xi -$$
$$\Delta^\beta \sum_{j=0}^{i} b_{j,i+1} f(t_j, x(t_j), u(x_{t_j}(\cdot))) | +$$
$$\Delta^\beta \sum_{j=0}^{i} b_{j,i+1} | f(t_j, x(t_j), u(x_{t_j}(\cdot))) - f(t_j, y_j, u(y_{t_j}(\cdot))) | .$$

For the first item in the right hand side of above inequality, the following inequality is proved in C.P. Li, F.H. Zeng (2015) p.33:

$$|\frac{1}{\Gamma(\beta)} \int_{t_0}^{t_{i+1}} (t_{i+1} - \xi)^{1-\beta} f(\xi, x(\xi), u(x_\xi(\cdot)))d\xi - \tag{13}$$

$$\Delta^\beta \sum_{j=0}^{i} b_{j,i+1} f(t_j, x(t_j), u(x_{t_j}(\cdot)))| \le C_2\Delta,$$

such that $C_2 = \frac{(\vartheta - t_0)^\beta}{\Gamma(\beta+1)} \max_{t \in [t_0, \vartheta]} |x'(t)|$.

We estimate the second term in the right hand side of (12) by using (5), (6) and (11):

$$\Delta^\beta \sum_{j=0}^{i} b_{j,i+1} |f(t_j, x(t_j), u(x_{t_j}(\cdot))) - f(t_j, y_j, u(y_{t_j}(\cdot)))| \le$$

$$\Delta^\beta \sum_{j=0}^{i} b_{j,i+1} (L_1|x(t_j) - y_j| + L_2|u(x_{t_j}(\cdot)) - u(y_{t_j}(\cdot))| \le$$

$$\Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}(L_1\varepsilon_j + L_2L_3\|x_{t_j}(\cdot) - y_{t_j}(\cdot)\|_Q) \le$$

$$\Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}(L_1\hat{\varepsilon}_j + L_2L_3(\hat{\varepsilon}_j + C_1\Delta))$$

$$= \Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}(L_1 + L_2L_3)\hat{\varepsilon}_j + C_1\Delta^{\beta+1}\sum_{j=0}^{i} b_{j,i+1}.$$

Also, note that

$$\sum_{j=0}^{i} b_{j,i+1} = \frac{(i+1)^{\beta}}{\Gamma(\beta+1)} \le \frac{(N)^{\beta}}{\Gamma(\beta+1)} = \frac{1}{\Gamma(\beta+1)}\frac{(\vartheta - t_0)^{\beta}}{\Delta^{\beta}},$$

we receive

$$\Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}|f(t_j, x(t_j), u(x_{t_j}(\cdot))) - f(t_j, y_j, u(y_{t_j}(\cdot)))| \quad (14)$$

$$\le C_3\Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}\hat{\varepsilon}_j + C_4\Delta,$$

where

$$C_3 = L_1 + L_2L_3,\ C_4 = \frac{C_1(\vartheta - t_0)^{\beta}}{\Gamma(\beta+1)}.$$

Collecting (12), (13) and (14), it follows

$$\varepsilon_{i+1} \le C_3\Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}\hat{\varepsilon}_j + C_5\Delta,\ C_5 = C_2 + C_4,$$

and so the induction inequality is proved.

$$\hat{\varepsilon}_{i+1} \le C_3\Delta^{\beta}\sum_{j=0}^{i} b_{j,i+1}\hat{\varepsilon}_j + C_5\Delta. \quad (15)$$

According to the statement of a lemma (3.1) in C.P. Li, F.H. Zeng (2013), the coefficients $b_{j,i+1}$ defined in (9) have the following property

$$b_{j,i+1} \le \frac{1}{\Gamma(\beta+1)}C_6(i-j+1)^{\beta-1}, \quad (16)$$

such that $C_6 = \max\{1, \beta 2^{1-\beta}\}$. From (15) and (16), we obtain

$$\hat{\varepsilon}_{i+1} \le C_7\Delta^{\beta}\sum_{j=0}^{i}(i-j+1)^{\beta-1}\hat{\varepsilon}_j + C_5\Delta, \quad (17)$$

such that $C_7 = \frac{1}{\Gamma(\beta+1)}C_3C_6$. Using the generalized discretized Gronwall's inequality which proposed in lemma.2, the statement of the theorem is achieved.

4. NUMERICAL EXPERIMENT

Some numerical examples are introduced to ensure accuracy of the theoretical results

Example. Consider the following fractional differential equations with feedback control

$$D^{(\beta)}x(t) = ax + u,\ t \in [0,1], \quad 0 < \beta \le 1, \quad (18)$$

$$x(0) = 1, \quad (19)$$

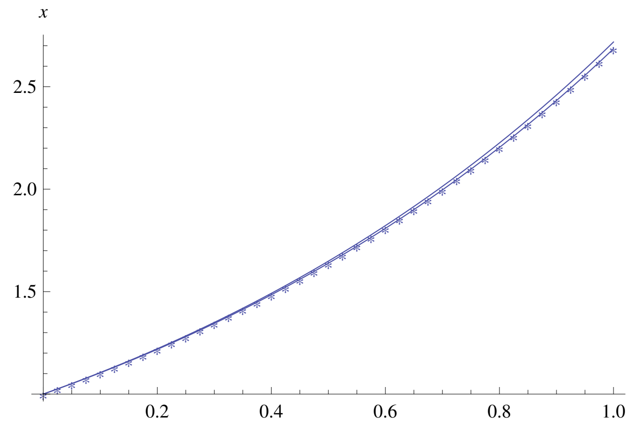

Fig. 1. State variable $x(t)$ for $\beta = 1$ and $h = 1/40$(solid line with plot marker) and the exact solution (solid line) and $\min(J) = 4.105$

with

$$\min J[u], \quad J = \frac{1}{2}\int_0^1 (x^2(t) + u^2)dt.$$

For the first case of of control term $u = 0$, the exact solution is

$$x(t) = ae(t), \quad (20)$$

such that $e(t) = a^q E_{\beta}(at^{\beta}) + \frac{1}{t}\sum_{l=1}^{q-1}\frac{(at^{\beta})^l}{\Gamma(l\beta)}, \quad q = \frac{1}{\beta}$, where

$$E_{\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}.$$

The experimental order of convergence (EOC) of the fractional forward Euler method (8) is used to verify the first order of accuracy for the first case of the example $u = 0$. The EOC is computed by the formula $\log_2 \frac{E(h,T)}{E(h/2,T)}$, where $E(h,T) =| y(T) - y_{T/h} |$ such that $y_{T/h}$ and $y(T)$ are the numerical and exact values of the solution at the end of interval T respectively.

Table 1. Absolute errors and EOC for the first case of control

h	$E(0.25,T)$	EOC(0.25)	$E(0.5,T)$	EOC(0.5)
1/10	1.7×10^{4}	0.98	5.3×10^{3}	0.97
1/20	0.8×10^{4}	0.99	2.7×10^{3}	0.978
1/40	0.4×10^{4}	1.01	6.9×10^{4}	0.985
1/80	0.2×10^{4}	1.025	3.4×10^{4}	0.997
1/160	0.1×10^{4}		0.2×10^{5}	

Table 2. Absolute errors and EOC for the second case of control

h	$E(0.1,T)$	EOC(0.1)	$E(0.9,T)$	EOC(0.9)
1/10	3.12×10^{3}	1.03	2.9×10^{3}	0.98
1/20	1.5×10^{3}	1.01	1.4×10^{3}	0.985
1/40	0.7×10^{4}	0.98	7.4×10^{4}	0.993
1/80	3.8×10^{4}	0.99	3.7×10^{4}	0.999
1/160	1.9×10^{4}		1.8×10^{4}	

Noticing table. 1 and table. 2, one can see that EOC approaches to the first order of accuracy which agrees with the theoretical results. Also, for the first case of control, see figures (1) and (2). For the 2.nd and 3.rd cases of control

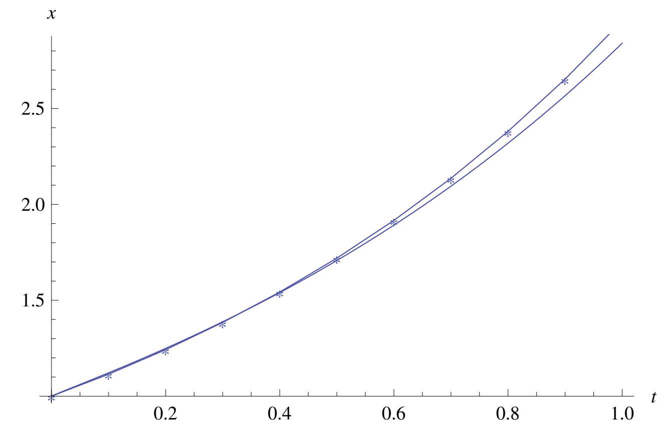

Fig. 2. State variable $x(t)$ for $\beta = 0.95$ and $h = 0.1$(solid line with plot marker) and the exact solution (solid line) and $\min(J) = 4.871$

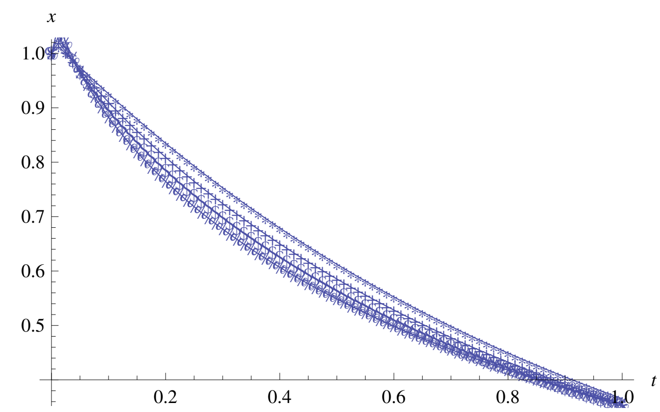

Fig. 3. State variable $x(t)$ for different values of β and $h = 1/80$ starting from right to left with $\beta = 1, \beta = 0.9, \beta = 0.8, \beta = 0.7$ and $\min(J)$ corresponding to $x(t)$ starting from right to left is $2.834, 2.816, 2.809, 2.815$ respectively.

terms which have the forms $u = r(t)x(t), \quad r(t) = -2$ and $u = r(t)x(t) + \int_0^t p(s)x(s)ds, \quad r(t) = -2, \quad p(s) = -1$ respectively, we have no exact solutions for theses cases. We introduce some figures to illustrate the efficiency of the proposed scheme. For the 2.nd case of control, figures (3), (4) are shown. For the 3.rd case of control, figures (5), (6) and (7) are prepared. Numerical results shown in these figures ensure that the approximate solutions converge as the grid size is decreased. This supports that the numerical scheme is stable. Further as β approaches close to 1, the numerical solutions of the state variable approach the analytical solutions for $\beta = 1$. Thus, in the limit, the solution for the integer order optimal control problem is recovered.

5. CONCLUSION

A first order accuracy forward Euler method is designed to obtain a numerical solution of fractional differential equation with feedback control in a simple way. A detailed converge analysis of the proposed scheme is discussed step by step. We plan to study the possibility of introducing high order methods for this sort of equations as a future work.

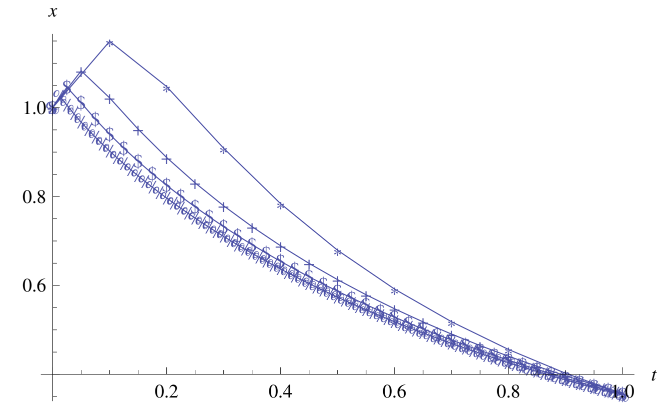

Fig. 4. State variable $x(t)$ for different values of h and $\beta = 0.85$ starting from right to left with $h = 1/10, h = 1/20, h = 1/40, h = 1/80$ and $\min(J)$ corresponding to $x(t)$ starting from right to left is $2.817, 2.820, 2.816, 2.812$ respectively.

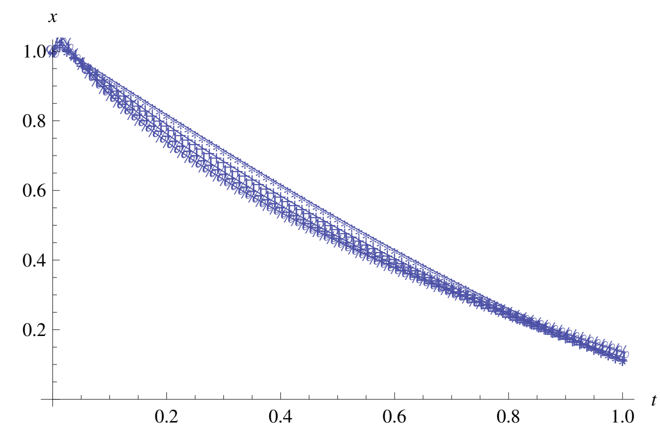

Fig. 5. State variable $x(t)$ for different values of β and $h = 1/80$ starting from right to left with $\beta = 1, \beta = 0.9, \beta = 0.8, \beta = 0.7$ and $\min(J)$ corresponding to $x(t)$ starting from right to left is $2.0397, 2.041, 2.053, 2.0798$ respectively.

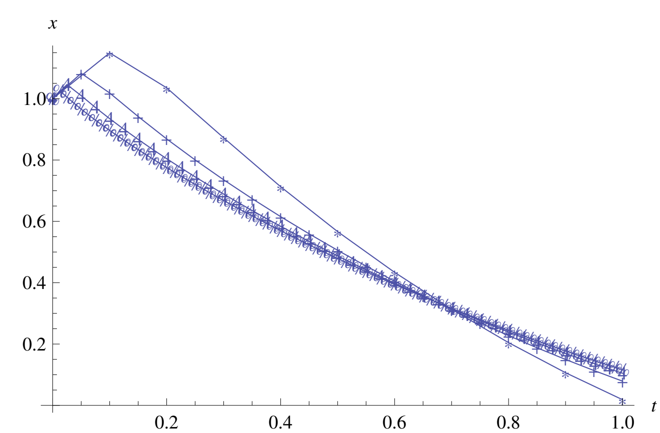

Fig. 6. State variable $x(t)$ for different values of h and $\beta = 0.85$ starting from right to left with $h = 1/10, h = 1/20, h = 1/40, h = 1/80$ and $\min(J)$ corresponding to $x(t)$ starting from right to left is $1.899, 1.985, 2.026, 2.046$ respectively.

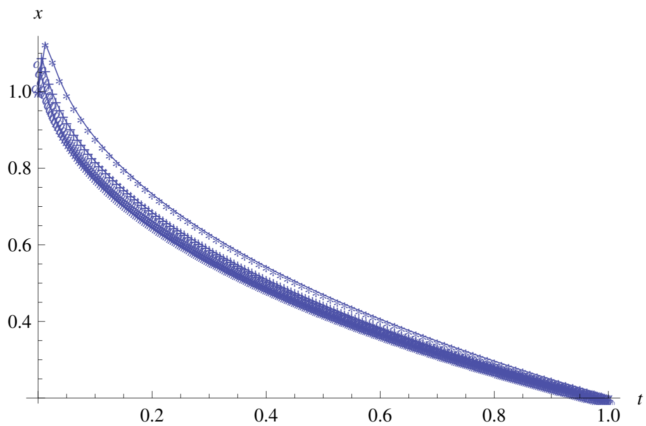

Fig. 7. State variable $x(t)$ for different values of h and $\beta = 0.5$ starting from right to left with $h = 1/80, h = 1/160, h = 1/320$ and $\min(J)$ corresponding to $x(t)$ starting from right to left is $2.2097, 2.193, 2.1797$ respectively.

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