Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-32 (2018) 343–348

# A New Approximate Method for Construction of the Normal Control. ⋆

**Evgeniy A. Krupennikov** *

* *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences (IMM UB RAS) 16 S.Kovalevskaya str., Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N.Yeltsin, 620002, 19 Mira str., Ekaterinburg, Russia (e-mail: krupennikov@imm.uran.ru)*

**Abstract:** The problem of construction of the normal control (namely, the control with the least norm in $L^2$ space) that generates a given trajectory of a control system is considered. A new method for constructing approximations of the normal control is suggested for a class of control systems with dynamics linear in controls and non-linear in state coordinates where the dimension of the control parameter is greater than or equal to the dimension of the state variables. This method relies on necessary optimality conditions in auxiliary variational problems.
An illustrating example is exposed. The results of numerical simulation are compared with the results obtained with another approach.



*Keywords:* normal control, Nonlinear control systems, calculus of variations, Hamiltonian systems.

## 1. INTRODUCTION

The problem of constructing the controls generating a given trajectory of a control system is considered in this paper. This problem occurs in many areas of mathematics such as optimal control theory, differential games and others and have applications in such areas as economics, medicine, robotics and others.

In some control construction problems the set of controls generating the given trajectory may contain many elements. The control that generates the given trajectory and has the least possible norm in $L^2$ space is called the normal control.

The problem of constructing the normal control was considered, for example, by Kryazhimskii and Osipov (1984); Osipov and Kryazhimskii (1995). The method suggested by A. V. Kryazhimskii and Yu. S. Osipov reconstructs the normal control by using a regularized (a variation of Tikhonov regularization, see Tikhonov (1943)) procedure of control with a guide. It is originated from the works of Krasovskii's school on the theory of optimal feedback control, see Krasovskii (1968); Krasovskii and Subbotin (1974).

In this paper we consider control systems linear in controls and non-linear in state coordinates with the dimension of the control parameter greater than or equal to the dimension of the state coordinates. The normal control is unique in the problem considered below.

A new approach to construction of approximations the normal control is presented in this paper. It is based on the method for solving dynamic reconstruction problems suggested by Subbotina and Krupennikov (2017); Krupennikov (2018). This approach relies on auxiliary variational problems on extremum of a regularized (Tikhonov (1943)) integral functional.

The suggested approximate method and an exact method providing the explicit expression for the normal control are discussed. Results of numerical simulations for both method are exposed and compared.

## 2. DYNAMICS

We consider control systems with dynamics of the form

$$\begin{gathered}\dot{x}(t) = G(x(t),t)u(t),\\ x(\cdot):[0,T]\to\mathbb{R}^n,\quad u(\cdot):[0,T]\to\mathbb{R}^m,\\ m\geq n,\quad t\in[0,T].\end{gathered}\tag{1}$$

Here $G(x,t)$ is an $n\times m$ matrix with elements $g_{ij}(x,t):\mathbb{R}^n\times[0,T]\to\mathbb{R}$, $i=1,...,n$, $j=1,...,m$ that have continuous derivatives

$$\begin{gathered}\frac{\partial g_{ij}(x,t)}{\partial t},\quad \frac{\partial g_{ij}(x,t)}{\partial x_k},\\ i=1,...,n,\quad j=1,...,m,\quad k=1,...,n,\\ x\in\mathbb{R}^n,\quad t\in[0,T].\end{gathered}\tag{2}$$

In (1) $x(t)$ is the vector of state coordinates and $u(t)$ is the vector of the control parameter.

⋆ This work was supported by the Russian Foundation for Basic Research (project no. 17-01-00074) and by the Program of the Presidium of the Russian Academy of Sciences 01 'Fundamental Mathematics and its Applications' under grant PRAS-18-01.


10.1016/j.ifacol.2018.11.407

The admissible controls are continuously differentiable functions satisfying the restriction

$$u(t) \in \mathbf{U}, \quad t \in [0,T], \tag{3}$$

where $\mathbf{U} \subset \mathbb{R}^m$ is a convex compact set.

## 3. THE NORMAL CONTROL PROBLEM

We assume that a trajectory $x^*(\cdot) : [0,T] \to \mathbb{R}^n$ of system (1) generated by some admissible control is given.

*Assumption 1.* There exists a constant $r > 0$ such that rows of the matrix $G(x,t)$ are linearly independent for any $\{x,t\} : t \in [0,T],\ x \in B_r[x^*(t)]$, where $B_r[x]$ is a closed ball of radius $r$ and with center in $x$.

Let

$$\begin{gathered}\mathbb{U}^* = \{u(\cdot) : [0,T] \to \mathbb{R}^m;\ u(\cdot) \in C^1([0,T]);\ u(t) \in \mathbf{U};\\ \dot{x}^*(t) = G(x^*(t),t)u(t),\ t \in [0,T]\}\end{gathered} \tag{4}$$

be the set of admissible controls that generate the trajectory $x^*(\cdot)$. This set is non-empty.

*Remark 2.* To illustrate that the set $\mathbb{U}^*$ may contain more than one element let us consider the dynamics and the trajectory

$$\begin{gathered}\dot{x}(t) = u_1(t) + 2u_2(t),\\ x^*(t) \equiv 3t, \quad t \in [0,T].\end{gathered} \tag{5}$$

Such trajectory can be generated by a whole class of controls $\{\big(3 - 2f(\cdot), f(\cdot)\big) : f(\cdot) \in C^1([0,T])\}$.

*Definition 3.* An admissible control $u^*(\cdot) \in \mathbb{U}^*$ is called normal control if it has the least possible norm in $L^2[0,T]$ among all elements of the set $\mathbb{U}^*$,

$$\|u^*(\cdot)\|_{L^2[0,T]} = \sqrt{\int\limits_0^T \|u^*(t)\|^2 dt} = \min_{u(\cdot)\in\mathbb{U}} \|u(\cdot)\|_{L^2[0,T]}. \tag{6}$$

Hereinafter $\|f\| = \sqrt{\sum\limits_{i=1}^k f_i^2},\ f \in \mathbb{R}^k,\ k \in \mathbb{N}$ is Euclidean norm in $\mathbb{R}^k$.

Let us now prove that the normal control $u^*(\cdot)$ exists and is unique for a given trajectory $x^*(\cdot)$ of system (1). To do it, let us first prove an auxiliary lemma.

*Lemma 4.* For the trajectory $x^*(\cdot)$ of system (1) the following assertions are true:

**A1** The set $\mathbb{U}^*$ is closed in $C^1([0,T])$ space.

**A2** The set $\mathbb{U}^*$ is convex.

Proof of assertion **A1**. Let us consider an arbitrary sequence $\{u_1(\cdot), u_2(\cdot), \ldots\} \subset \mathbb{U}^*$ such that $\lim\limits_{i\to\infty} \|u_i(\cdot) - v(\cdot)\|_{C^1[0,T]} = 0,\ v(\cdot) \in C^1([0,T])$, where $\|f(\cdot)\|_{C^1[0,T]} = \max\limits_{t\in[0,T]} \|f(t)\| + \max\limits_{t\in[0,T]} \|\dot{f}(t)\|,\ f(\cdot) \in C^1[0,T]$ is the norm in $C^1[0,T]$ space. Let us prove that $v(\cdot) \in \mathbb{U}^*$.

Indeed, since $\dot{x}^*(\cdot) = G(x^*(\cdot),\cdot)u_i(\cdot),\ i = 1, 2, \ldots,$

$$\begin{gathered}\|G(x^*(\cdot),\cdot)v(\cdot) - \dot{x}^*(\cdot)\|_{C^1[0,T]}\\ = \lim_{i\to\infty} \|G(x^*(\cdot),\cdot)v(\cdot) - G(x^*(\cdot),\cdot)u_i(\cdot)\|_{C^1[0,T]}\\ \le nm\|G(x^*(\cdot),\cdot)\|_{max} \lim_{i\to\infty} \|v(\cdot) - u_i(\cdot)\|_{C^1[0,T]} = 0,\end{gathered} \tag{7}$$

where $\|G\|_{max} = \max\limits_{i=1,\ldots,n,j=1,\ \ldots,m} |g_{ij}|$ is the maximum norm of an $n\times m$ matrix. Therefore, $G(x^*(\cdot),\cdot)v(\cdot) = \dot{x}^*(\cdot)$.

The sequence $\{u_1(\cdot), u_2(\cdot), \ldots\}$, in particular, converges pointwise. As the set $\mathbf{U}$ is closed, $\lim\limits_{i\to\infty} u_i(t) = v(t) \in \mathbf{U},\ t \in [0,T]$.

We have obtained that the function $v(\cdot) \in C^1([0,T])$ generates $x^*(\cdot)$ (as a control) and that $v(t) \in \mathbf{U},\ t \in [0,T]$. So, $v(\cdot) \in \mathbb{U}^*$.

Thus, the set $\mathbb{U}^*$ is closed, since it contains all its limit points.

Proof of assertion **A2**. Let us consider two arbitrary elements $\{u_1(\cdot), u_2(\cdot)\} \subset \mathbb{U}^*$. Let us check that $u_1(\cdot) + (u_2(\cdot) - u_1(\cdot))\theta = u_3(\cdot) \in \mathbb{U}^*,\ \theta \in [0,1]$. Indeed,

$$\begin{gathered}G(x^*(t),t)u_3(t) = G(x^*(t),t)(u_1(t) + (u_2(t) - u_1(t))\theta)\\ = \dot{x}^*(t) + (\dot{x}^*(t) - \dot{x}^*(t))\theta = \dot{x}^*(t), \quad t \in [0,T].\end{gathered} \tag{8}$$

The set $\mathbf{U}$ is convex. Then, $u_3(t) = u_1(t) + (u_2(t) - u_1(t))\theta \in \mathbf{U}, \quad t \in [0,T]$.

So, $u_3(\cdot) \in \mathbb{U}^*$. Therefore, the set $\mathbb{U}^*$ is convex.□

The norm in $L^2[0,T]$ space is a strongly convex lower semi-continuous functional $\|f(\cdot))\|_{L^2[0,T]} = \left(\int\limits_0^T \|f(t)\|^2 dt\right)^{\frac{1}{2}}$. Then, it reaches a unique minimum on the convex compact set $\mathbb{U}^*$.

This means that the normal control $u^*(\cdot)$ in problem (1),(3),(6) exists and is unique.

We call the problem of construction of the normal control $u^*(\cdot)$ for a given trajectory $x^*(\cdot)$ of system (1) *the normal control problem.*

## 4. THE EXACT METHOD

Let us first consider a known method providing an explicit formula for the exact solution of the normal control problem.

The problem of finding the normal control $u^*(\cdot)$ for a given trajectory $x^*(\cdot)$ can be formulated as a variational problem: to find such admissible control $u(\cdot) : [0,T] \to \mathbb{R}^m$ that provide a minimum to the convex functional

$$I(u(\cdot)) = \int\limits_0^T \frac{\|u(t)\|^2}{2} dt \tag{9}$$

and satisfy the equation

$$\dot{x}^*(t) - G(x^*(t),t)u(t) \equiv 0. \tag{10}$$

We can write the necessary optimality conditions for problem (9),(10) in Lagrange form (see, for example, Ioffe and Tikhomirov (1974)).

The Euler equation has the form

$$u(t) \equiv G^T(x^*(t),t)\lambda(t), \tag{11}$$

where $\lambda(t)$ is the Lagrange multipliers vector.

Substituting (11) into (10), we can express $\lambda(t)$ through the variable $t$. By substituting this expression back into (11), we finally get the solution of problem (9),(10)

$$u^*(t) \equiv G^T(x^*(t),t)\Big(G(x^*(t),t)G^T(x^*(t),t)\Big)^{-1}\dot{x}^*(t). \tag{12}$$

*Remark 5.* The matrix

$$G^g(x^*(t),t) = G^T(x^*(t),t)\Big(G(x^*(t),t)G^T(x^*(t),t)\Big)^{-1} \tag{13}$$

is the generalized inverse of the matrix $G(x^*(t),t)$. Since the rows of the matrix $G(x^*(t),t)$ are linearly independent on $[0,T]$ (Assumption 1), $G^g(x^*(t),t)$ exists on $[0,T]$ (see, for example, Byilov et al. (1966)).

## 5. THE APPROXIMATE NORMAL CONTROL PROBLEM.

In this paper another approach to the normal control problem is suggested. It relies on necessary optimality conditions in auxiliary variational problems and uses them as a base to construct approximations of the solution of the normal control problem.

We call the approximate normal control problem the problem of constructing such functions $u(\cdot,\alpha) = u^\alpha(\cdot) : [0,T] \to \mathbb{R}^m,\ \alpha > 0$ that

$$\lim_{\alpha\to 0} \|u^\alpha(\cdot) - u^*(\cdot)\|_{C[0,T]} = 0. \tag{14}$$

### 5.1 Auxiliary variational problem

To construct approximations (14) of the solution of the normal control problem, we introduce the auxiliary variational problem (AVP).

We consider the set of pairs of continuously differentiable functions $F_{xu} = \{\{x(\cdot),u(\cdot)\} : x(\cdot) : [0,T] \to \mathbb{R}^n,\ u(\cdot) : [0,T] \to \mathbb{R}^m\}$ such that these functions satisfy differential equations (1) and the following boundary conditions

$$\begin{aligned} &x(T) = x^*(T), \quad u(T) = G^T(x^*(T),T) \\ &\cdot\Big(G(x^*(T),T)G^T(x^*(T),T)\Big)^{-1}\dot{x}^*(T). \end{aligned} \tag{15}$$

AVP is to find a pair of such functions $x(\cdot,\alpha) = x^\alpha(\cdot) : [0,T] \to \mathbb{R}^n$ and $u(\cdot,\alpha) = u^\alpha(\cdot) : [0,T] \to \mathbb{R}^m$ that $\{x^\alpha(\cdot),u^\alpha(\cdot)\} \in F_{xu}$ and that they provide an extremum for the integral discrepancy functional

$$I(x(\cdot),u(\cdot)) = \int_0^T \left[-\frac{\|x(t)-x^*(t)\|^2}{2} + \alpha^2\frac{\|u(t)\|^2}{2}\right]dt. \tag{16}$$

Here $\alpha$ is a small regularising (see Tikhonov (1943)) parameter.

### 5.2 Necessary optimality conditions for the AVP

For AVP (1),(16),(15) the $n+m$ corresponding Euler equations have the form

$$\begin{gathered} \dot{\lambda}_i(t) + (x_i(t) - x_i^*(t)) \\ +\sum_{j=1}^n \left[\lambda_j(t)\sum_{k=1}^m \frac{\partial g_{jk}}{\partial x_i}(x(t),t)u_k(t)\right] = 0, \\ i = 1,\ldots,n, \\ u(t) = \frac{1}{\alpha^2}G^T(x^*(t),t)\lambda(t). \end{gathered} \tag{17}$$

We can substitute equations (17) into (1) to get the necessary optimality conditions for the AVP in the form of Hamiltonian system

$$\begin{aligned} &\dot{x}(t) = -\alpha^{-2}G(x(t),t)G^T(x(t),t)s(t), \\ &\dot{s}_i(t) = x_i(t) - x_i^*(t) \\ &+\frac{1}{\alpha^2}\langle s(t), \frac{\partial G}{\partial x_i}(x(t),t)G^T(x(t),t)s(t)\rangle, \quad i = 1,\ldots,n \end{aligned} \tag{18}$$

with boundary conditions

$$\begin{aligned} &x(T) = x^*(T), \\ &s(T) = -\alpha^2\big(G(x^*(T),T)G^T(x^*(T),T)\big)^{-1}\dot{x}^*(T), \end{aligned} \tag{19}$$

where the vector $s(t) = -\lambda(t)$ plays the role of the adjoint variables vector.

*Remark 6.* The suggested algorithm for solving the approximate normal control problem utilizes only necessary conditions (18),(19) which provide a stationary point for functional (16) irrespectively of whether the extremum is reached. Thus, it is not verified if an extremum is actually reached in the AVP.

### 5.3 A solution of the approximate normal control problem

We consider a linearized version of system (18):

$$\begin{aligned} &\dot{x}(t) = -\alpha^{-2}G(x^*(t),t)G^T(x^*(t),t)s(t), \\ &\dot{s}(t) = x(t) - x^*(t), \quad t \in [0,T] \end{aligned} \tag{20}$$

with boundary conditions (19).

It is a heterogeneous linear system of ODEs with variable coefficients which are continuous on $[0,T]$. So, the solution $(x^\alpha(\cdot),s^\alpha(\cdot)) : [0,T] \to \mathbb{R}^{2n}$ of (20), (19) exists and is unique and extendable on $[0,T]$.

Let us now consider the functions

$$u^\alpha(\cdot) = -\frac{1}{\alpha^2}G^T(x^*(\cdot),\cdot)s^\alpha(\cdot), \tag{21}$$

where $s^\alpha(\cdot)$ is the part of the solution of (22),(19).

The following theorem holds.

*Theorem 7.* For a trajectory $x^*(\cdot)$ of system (1) satisfying Assumption 1 the functions (21) satisfy condition (14).

The proof is based on the method of "freezing" the matrix $\mathbf{G}(\cdot) \triangleq G(x^*(\cdot),\cdot)G^T(x^*(\cdot),\cdot)$ (Byilov et al. (1966)). The main idea of this approach is considering values of the matrix $\mathbf{G}(t)$ for fixed points $t = t_0,t_1,\ldots,t_h,\ h = \lfloor T\alpha^{-1}\rfloor,\ t_0 = 0,\ t_i = T-(h-i)\alpha,\ i = 1,\ldots,h$ and obtaining consequent estimates for the solutions of the systems

$$\begin{aligned} &\dot{\bar{x}}_i(t) = -\alpha^{-2}\mathbf{G}(t_i)\bar{s}_i(t), \\ &\dot{\bar{s}}_i(t) = \bar{x}_i(t) - x^*(t), \\ &t \in [t_{i-1},t_i], \\ &\bar{x}_i(t_i) = \begin{cases} x^*(T), & i = h; \\ \bar{x}_{i+1}(t_i), & i < h, \end{cases} \\ &\bar{s}_i(t_i) = \begin{cases} -\alpha^2\big(\mathbf{G}(T)\big)^{-1}\dot{x}^*(T), & i = h; \\ \bar{s}_{i+1}(t_i), & i < h, \end{cases} \\ &i = 1,\ldots,h. \end{aligned} \tag{22}$$

Each system from (22) is a heterogeneous linear system of ODEs with constant coefficients. So, the solutions $\{\bar{x}_i(\cdot),\bar{s}_i(\cdot)\} : [t_{i-1},t_i] \to \mathbb{R}^{2n}$ of (22) exist and are unique and extendable on $[t_{i-1},t_i]$.

Let us now fix an $i \in \{0,\ldots,h\}$. We introduce the new variables

$$\begin{gathered}\bar z_i(\cdot) = (z_{i,1}(\cdot), z_{i,2}(\cdot),\ldots,z_{i,n}(\cdot), w_{i,1}(\cdot), w_{i,2}(\cdot),\ldots,w_{i,n}(\cdot)),\\ z_i(\cdot) = \bar x_i(\cdot) - x^*(\cdot),\\ w_i(\cdot) = \bar s_i(\cdot) + \alpha^2\Big(\mathbf{G}(t_i)\Big)^{-1}\dot x^*(\cdot).\end{gathered} \tag{23}$$

The $i$-th system from (22) can be rewritten in variables (23) as

$$\dot{\bar z}_i(t) = A\bar z_i(t) + \bar f(t), \tag{24}$$

where the $2n\times 2n$ matrix $A$ can be written in the block form

$$A = \begin{pmatrix} O_n & -\alpha^{-2}\mathbf{G}(t_i)\\ I_n & O_n\end{pmatrix}, \tag{25}$$

where $O_n$ is an $n\times n$ zero matrix, $I_n$ is an $n\times n$ identity matrix and

$$\begin{gathered}\bar f(\cdot) = \big((0,\ldots,0)_n, \alpha^2 f_1(\cdot), \alpha^2 f_2(\cdot),\ldots,\alpha^2 f_n(\cdot)\big),\\ f(\cdot) = \mathbf{G}^{-1}(t_i)\ddot x^*(\cdot).\end{gathered} \tag{26}$$

The solution of system (24) can be written with the help of Cauchy formula for solutions of heterogenous systems of linear ODEs with constant coefficients:

$$\bar z_i(t) = \Phi(t)\Big(z_i(t_i) - \int_t^{t_i}\Phi^{-1}(\tau)\bar f(\tau)d\tau\Big),\quad t\in[t_{i-1},t_i], \tag{27}$$

were $\Phi(\cdot)$ is a $2n\times 2n$ fundamental matrix of solutions of the homogenous part of system (24). It can be chosen as

$$\Phi(t) = \exp\left[-(T-t)A\right] \overset{def}{=} \sum_{k=0}^{\infty}\frac{1}{k!}\left(-(T-t)A\right)^k. \tag{28}$$

One can check after substituting (25) into (28) that

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left(-(T-t)A\right)^k \equiv \begin{pmatrix} B_{1,1}(t) & B_{1,2}(t)\\ B_{2,1}(t) & B_{2,2}(t)\end{pmatrix}, \tag{29}$$

were, in particular,

$$\begin{gathered}B_{1,2}(t) \triangleq\\ -\sum_{k=0}^{\infty}\frac{\alpha^{-(2k+2)}}{(2k+1)!}(-1)^k(T-t)^{(2k+1)}\left(\mathbf{G}(t_i)\right)^{(k+1)}.\end{gathered} \tag{30}$$

For any real matrix $B$ with linearly independent rows the product $BB^T$ is a positive definite matrix (see, for example, Byilov et al. (1966)). So, for a sufficiently small parameter $\alpha$ the matrix $\mathbf{G}(t_i) = G(t_i)G^T(t_i)$ is positive definite since Assumption 1 for any $i = 0,\ldots,h$. Therefore, it can be diagonalized by a unitary congruence $\mathbf{G}(t_i) = Q_i\Lambda_i Q_i^T$, where $Q_i$ is a real orthogonal matrix and $\Lambda_i$ is a real diagonal matrix with the eigenvalues $\Lambda_{i,j},\ j = 1,\ldots,n$ of the matrix $\mathbf{G}(t_i)$ on the diagonal, which are positive for any $i = 0,\ldots,h$ (Byilov et al. (1966)). Since $Q_i$ is orthogonal, $Q_i^T = Q_i^{-1}$. So,

$$\left(\mathbf{G}(t_i)\right)^k = Q_i\Lambda_i^k(t)Q_i^T,\quad k\in\mathbb{Z}. \tag{31}$$

Now, we introduce the notations

$$\begin{gathered}\Lambda_{cos}(t) \triangleq diag\big(\cos(\alpha^{-1}\sqrt{\Lambda_{i,j}}(T-t))\big),\\ \Lambda_{sin}(t) \triangleq diag\big(\sin(\alpha^{-1}\sqrt{\Lambda_{i,j}}(T-t))\big),\\ \Lambda_{sqrt} \triangleq diag\big(\sqrt{\Lambda_{i,j}}\big),\end{gathered} \tag{32}$$

where the notation $diag\big(a(j)\big)$ stands for diagonal $n\times n$ matrix with the elements $a(1), a(2),\ldots,a(n)$ on the diagonal.

We get by substituting (31) into (30) that in notations (32)

$$\begin{gathered}B_{1,2}(t) = -\frac{1}{\alpha}\Lambda_{sqrt}Q_i\sum_{k=0}^{\infty}\Big[\frac{\alpha^{-(2k+1)}}{(2k+1)!}\\ \cdot(-1)^k(T-t)^{(2k+1)}\Lambda_{sqrt}^{(2k+1)}\Big]Q_i^T.\end{gathered} \tag{33}$$

Therefore, applying the Maclaurin series formula for $\sin(\cdot)$, we get in notations (32) that

$$B_{1,2}(t) = \alpha^{-1}\Lambda_{sqrt}Q_i\Lambda_{sin}(t)Q_i^T. \tag{34}$$

We can obtain in the same way that

$$\begin{gathered}B_{1,1}(t) = B_{2,2}(t) = Q_i\Lambda_{cos}(t)Q_i^T,\\ B_{2,1}(t) = \alpha\Lambda_{sqrt}^{-1}Q_i\Lambda_{sin}(t)Q_i^T.\end{gathered} \tag{35}$$

It can be proved that

$$\lim_{\alpha\to 0}\left\|\int_t^{t_i}B_{2,2}(t)f(t)dt\right\| = 0,\quad t\in[t_{i-1},t_i]. \tag{36}$$

*Remark 8.* The prove is based on the fact that

$$\begin{gathered}\int_{t_1}^{t_2}\cos\big(\alpha^{-1}\sqrt{\Lambda_{i,j}}(T-t)\big)dt\\ = -\alpha\sqrt{\Lambda_{i,j}}^{-1}\sin\big(\alpha^{-1}\sqrt{\Lambda_{i,j}}(T-t)\big)\Big|_{t_1}^{t_2} = O(\alpha),\\ t_1\in[0,T),\quad t_2\in[0,T),\quad j = 1,\ldots,n.\end{gathered} \tag{37}$$

By using the scheme of proof of Theorem 1 in Krupennikov (2018), one can utilize (37) to obtain equality (36).

It can be proved in the same way that

$$\lim_{\alpha\to 0}\left\|\alpha\int_t^{T}B_{1,2}(t)f(t)dt\right\| = 0,\quad t\in[0,T]. \tag{38}$$

One can check that

$$\Phi^{-1}(t) = \begin{pmatrix} B_{1,1}(t) & -B_{1,2}(t)\\ -B_{2,1}(t) & B_{2,2}(t)\end{pmatrix}. \tag{39}$$

Therefore, we get by substituting (26),(29) and (39) into Cauchy formula (27) that

$$\begin{gathered}w_i(t) = -\alpha^2 B_{2,1}(t)\Big(z_i(t_i) + \int_t^T B_{1,2}(\tau)f(\tau)d\tau\Big)\\ +\alpha^2 B_{2,2}(t)\Big(w_i(t_i) + \int_t^T B_{2,2}(\tau)f(\tau)d\tau\Big).\end{gathered} \tag{40}$$

For $i = h$ the boundary conditions are $w_h(T) = z_h(T) = \overrightarrow{0}$. Therefore, Applying (35),(36),(38) and with regard to continuity of the elements of $B_{2,1}(\cdot)$ and $B_{2,2}(\cdot)$, we obtain that

$$\lim_{\alpha\to 0}\|\alpha^{-2}w_h(\cdot)\|_{C[T-\alpha,T]} = 0. \tag{41}$$

We get by applying result (41) to expressions (23) that

$$\lim_{\alpha\to 0}\|\alpha^{-2}\bar s_h(\cdot) - \Big(\mathbf{G}(T)\Big)^{-1}\dot x^*\|_{C[T-\alpha,T]} = 0. \tag{42}$$

Since elements of $\mathbf{G}(\cdot)$ are continuously differentiable, there exists a constant $\bar G > 0$ such that

$$\|\mathbf{G}_{kj}(T) - \mathbf{G}_{kj}(\cdot)\|_{C[T-\alpha,T]} \le \alpha\bar G,\quad k,j = 1,\ldots,n. \tag{43}$$

So, we get from (42) that

$$\lim_{\alpha\to 0}\|\alpha^{-2}\bar s_h(\cdot) - \Big(\mathbf{G}(\cdot)\Big)^{-1}\dot x^*\|_{C[T-\alpha,T]} = 0. \tag{44}$$

Now let us return to the solutions of system (20). We introduce the discrepancies $\triangle x(\cdot) = x^\alpha(\cdot) - \bar{x}_h(\cdot)$, $\triangle s(\cdot) = s^\alpha(\cdot) - \bar{s}_h(\cdot)$ for the solutions of (22),(19) and (20),(19).

$$\begin{aligned} \dot{\triangle} x(\cdot) =& -\alpha^{-2}\mathbf{G}(t)\triangle s(t) \\ &+ \alpha^{-2}\big(\mathbf{G}(T) - \mathbf{G}(t)\big)\bar{s}(t), \\ \dot{\triangle} s(t) =& \triangle x(t), \\ \triangle s(T) =& \triangle x(T) = \overrightarrow{0}, \quad t \in [T-\alpha, T]. \end{aligned} \tag{45}$$

The solution of linear system (45) is continuous in parameter $\alpha$. Therefore, since (41) and (43), when $\alpha \to 0$ it converges pointwise to the solution of the homogenous part of system (45), which is identical zero due to the zero boundary conditions. So, the solutions $(x^\alpha(\cdot), s^\alpha(\cdot))$ of (20),(19) converge pointwise to the solutions $(\bar{x}_h(\cdot), \bar{s}_h(\cdot))$ of system (22),(19) on $[T-\alpha, T]$. Therefore, since (42),

$$\lim_{\alpha \to 0} \|\alpha^{-2} s^\alpha(t) - \Big(\mathbf{G}(t)\Big)^{-1} \dot{x}^*\|_{C[T-\alpha,T]} = 0. \tag{46}$$

This result can be now expanded for the cases of $i = h-1, h-2, \ldots, 0$.

*Remark 9.* In the proof the estimates for the boundary conditions for each step are consequently obtained through the estimates obtained on the previous step. The proof also considers the signs of these conditions to prove that the final estimate is not exponential. For briefness, the accurate proof is left out and will be published in future papers.

So, one can obtain that

$$\lim_{\alpha \to 0} \|\alpha^{-2} s^\alpha(t) - \Big(\mathbf{G}(t)\Big)^{-1} \dot{x}^*\|_{C[0,T]} = 0. \tag{47}$$

Finally, let us consider functions (21). It follows from (47) that

$$\begin{aligned} &\lim_{\alpha \to 0} \|u^\alpha(\cdot) - G^T(x^*(\cdot), \cdot) \\ &\cdot \big(G(x^*(\cdot), \cdot)G^T(x^*(\cdot), \cdot)\big)^{-1} \dot{x}^*(\cdot)\|_{C[0,T]} = 0, \end{aligned} \tag{48}$$

where

$$G^T(x^*(\cdot), \cdot)\left(G(x^*(\cdot), \cdot)G^T(x^*(\cdot), \cdot)\right)^{-1} \dot{x}^*(\cdot) = u^*(\cdot), \tag{49}$$

which was proved in section 4, (9)–(12).

Theorem proved. □

## 6. EXAMPLE

In this section we illustrate the work of the approximate method suggested in section 5 and compare the results of it's numerical simulation with the results obtained for the exact method described in section 4.

Let us consider the following dynamic control system

$$\begin{gathered} \dot{x}(t) = G(x(t), t)u(t), \\ x(\cdot) : [0,T] \to \mathbb{R}^4, \ u(\cdot) : [0,T] \to \mathbb{R}^5, \\ \|u\| \le 10, \ t \in [0,1], \end{gathered} \tag{50}$$

$$G(x,t) =$$

$$\begin{pmatrix} x_1^2 - t & 2\sqrt{t+1} & \sin(x_3) & x_2^2 + x_3^2 & \dfrac{x_4}{x_1^2} \\ 2 - 3tx_3 & -\dfrac{x_2 x_4}{t+1} & 4x_1 \cos & -7t & (x_1 + x_3)x_2 \\ \dfrac{x_1 + tx_2}{x_3} & t^2 - 1 & x_3 + \sqrt{t} & x_3 - 4x_2^2 & \dfrac{4}{t^2+1} \\ \sqrt{t + x_2} & 4 & tx_1 + x_3 x_4 & x_2^2 - x_3 & 5t \end{pmatrix}$$

and the trajectory

$$x^*(t) \equiv \left(\sqrt{t+1}, 2 - t^2, -\cos(t) - 2, t - 4\right). \tag{51}$$

One can check that dynamics (50) and trajectory (51) satisfy Assumption 1.

We consider the approximate normal control problem (14) for dynamics (50) and trajectory (51).

System (20) was integrated numerically by classical Runge-Kutta method to construct the numerical approximations $u_N^\alpha(t)$, $t \in [0,T]$ of the functions $u^\alpha(\cdot)$ (21) for various values of the parameter $\alpha$ and various numbers $N$ of the steps of numerical integration.

The time needed to construct the graph of $u_N^\alpha(\cdot)$ vor various approximation parameters is presented in Table 1. The last row in this table is the norm $\|u_N^\alpha(t) - u_{200}^*(\cdot)\|_{C[0,T]}$, where $u_{200}^*(\cdot)$ is an approximation of $u^*(\cdot)$ constructed as a linear interpolation of the set of sample points

$$\begin{aligned} \big\{(t_i, u^*(t_i)) = \big(t_i, G^T(x^*(t_i), t_i)\big[G(x^*(t_i), t_i) \\ \cdot G^T(x^*(t_i), t_i)\big]^{-1} \dot{x}^*(t_i)\big) : t_i = iT/M, \ i = 0, \ldots, M\big\}, \end{aligned} \tag{52}$$

taking $M = 200$.

*Remark 10.* The function $u_{200}^*(\cdot)$ was chosen to measure the discrepancy for all considered approximations because increasing the number of sample points $M$ in (52) beyond 200 results in very small changes. Namely, $\|u_{200}^*(\cdot) - u_M^*(\cdot)\|_{C[0,T]} \le 10^{-6}$, $M > 200$.

We consider the linear interpolations $u_M^*(\cdot)$ of the sets of sample points (52) as numerical approximations of the exact solution of the normal control $u^*(\cdot)$, which was obtained by the exact method described in section 4.

Table 1. Calculation time of the numerical approximations $u_N^\alpha(t)$, $t \in [0,T]$ of the approximate solutions $u^\alpha(\cdot)$.

| $\alpha$ | 0.5 | **0.05** | **0.02** | 0.01 | 0.001 |
|---|---|---|---|---|---|
| $N$ | 20 | **200** | **500** | 1000 | 10000 |
| calculation time(sec) | 0.20 | **0.31** | **0.45** | 0.73 | 9.03 |
| $\|u_N^\alpha(t) - u_{200}^*(\cdot)\|_{C[0,T]}$ | 0.57 | **0.22** | **0.07** | 0.015 | $10^{-6}$ |

The time needed to construct the graphs of the functions $u_M^*(\cdot)$ and the discrepancies $\|u_M^*(t) - u_{200}^*(\cdot)\|_{C[0,T]}$ are presented in Table 2.

Table 2. Calculation time of the numerical approximations $u_M^*(t)$, $t \in [0,T]$ of the exact solution $u^*(\cdot)$

| $M$ | 3 | **10** | **50** | **200** |
|---|---|---|---|---|
| calculation time(sec) | 0.83 | **4.18** | **24.06** | **161,2** |
| $\|u_M^*(t) - u_{200}^*(\cdot)\|_{C[0,T]}$ | 0.25 | **0.14** | **0.006** | **0** |

The graphs of the 5-th coordinates of the approximations $u_{N,5}^\alpha(t)$, $u_{M,5}^*(t)$, whose approximation parameters are marked with bold font in tables 1,2, are shown on pictures 1,2.

## 7. CONCLUSION

Comparing the exact method, described in section 4, and the approximate method, suggested in section 5, we can see that the second one reduces the task of finding a

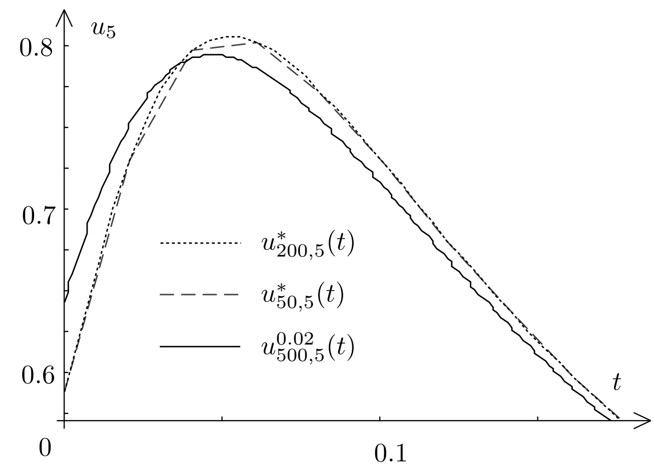


Fig. 1. Graphs of $u^{0.02}_{500,5}(t)$ and $u^*_{50,5}(t)$ in comparison with $u^*_{200,5}(t)$ (the graph is scaled).

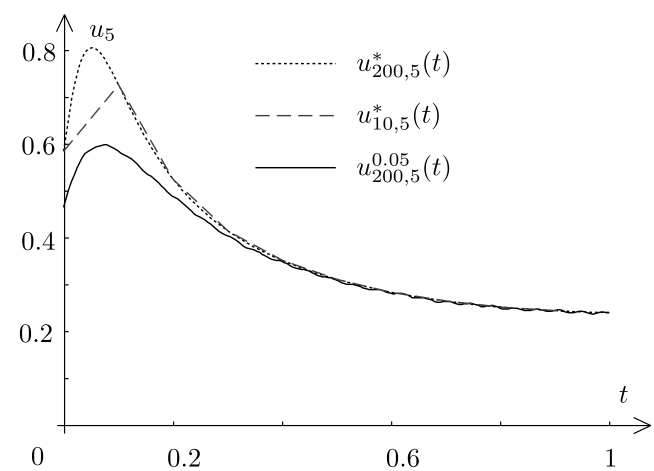


Fig. 2. Graphs of $u^{0.05}_{200,5}(t)$ and $u^*_{10,5}(t)$ in comparison with $u^*_{200,5}(t)$.

generalized inverse of a variable $n \times m$ matrix $G(x^*(t), t)$ to the task of solving a system of $2n$ linear ODEs with variable coefficients. In some applications numerical integration of ODE systems may be more preferable than matrix inversing.

In this paper numerical simulation of both methods is presented. The quality of approximation of the normal control in $C[0, T]$ space for the methods was exposed for various approximation parameters. It is shown that the suggested method may allow to obtain the same quality of approximation as the approach, based on counting matrix inverses, but in less computation time.

The estimates of the speed of the suggested approximate method's convergence and it's comparison with another methods for constructing normal controls is the matter of the future research.

## ACKNOWLEDGEMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 17-01-00074) and by the Program of the Presidium of the Russian Academy of Sciences 01 'Fundamental Mathematics and its Applications' under grant PRAS-18-01.

## REFERENCES

Byilov, B.F., Vinograd, R.E., Grobman, D.M., and Nemickii, V.V. (1966). *Teoriya pokazateley Lyaponova i eyo prilozheniya k voprosam ustoichivosti [Theory of Lyapunov exponents and its application to the problems of stablity]*. Nauka, Moscow. (in Russian).

Ioffe, A.D. and Tikhomirov, V.M. (1974). *Teoriya extremalnih zadach [Theory of Extremum Problems]*. Nauka, Moscow. (in Russian).

Krasovskii, N.N. (1968). *Teoriya upravleniya dvizheniem [Theory of motion control]*. Nauka, Moscow. (in Russian).

Krasovskii, N.N. and Subbotin, A.I. (1974). *Positcionnie differentcialnie igri [Positional differential games]*. Nauka, Moscow. (in Russian).

Krupennikov, E.A. (2018). On control reconstruction problems for dynamic systems linear in controls. In *Static & Dynamic Game Theory: Foundations & Applications*. Birkhauser. Unpublished.

Kryazhimskii, A.V. and Osipov, Y.S. (1984). Modelling of a control in a dynamic system. In *Engrg. Cybernetics*, 21(2), 38–47. Scripta Pub. Co., Washington.

Osipov, Y.S. and Kryazhimskii, A.V. (1995). *Inverse problems for ordinary differential equations: dynamical solutions*. Gordon and Breach, London.

Subbotina, N.N. and Krupennikov, E.A. (2017). The method of characteristics in an identification problem. In *Proceedings of the Steklov Institute of Mathematics*, 299, Suppl. 1, 205–216. Pleiades Publishing.

Tikhonov, A.N. (1943). Ob ustoichivosti obratnih zadach [on the stability of inverse problems]. In *Doklady Academii Nauk SSSR*, 39, 195–198. USSR Academy of Sciences, Moscow. (in Russian).