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Differential Equations for Ellipsoidal Estimates of Reachable Sets for a Class of Control Systems with Nonlinearity and Uncertainty ⋆

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Abstract: The problem of estimating reachable sets of nonlinear dynamical control systems with combined nonlinearity of quadratic and bilinear types and with uncertainty in initial states and in system parameters is studied. We assume that the uncertainty is of set-membership kind when only the bounding sets for unknown parameters and functions are given. We find the ellipsoidal estimates of reachable sets using the special bilinear-quadratic structure of studied control system. The main result consists in deriving the related differential equations which describe the dynamics of the ellipsoidal estimates of reachable sets of the control system under study, related numerical simulation results are also given.

Keywords: Control problems, Uncertainty, Nonlinearity, Estimation, Reachable sets.

1. INTRODUCTION

In the paper we study control systems with unknown but bounded uncertainties with a set-membership description of uncertain parameters and functions (Kurzhanski and Valyi (1997); Kurzhanski and Varaiya (2014); Kurzhanski and Filippova (1993)). The motivation for this study may be found in modeling problems for systems under uncertainty appearing in different applications such as physical engineering problems, economical modeling, ecological problems (Apreutesei (2009); Blanchini and Miani (2015); Boscain et al. (2013); Ceccarelli et al. (2004); Asselborn et al. (2013); August and Koeppl (2012); Malyshev and Tychinskii (2005)). The approaches to the solution of linear control problems under uncertainty and of some special classes of nonlinear control problems with uncertain parameters based on the set-membership approach have been developed very intensively during last decades (Kurzhanski and Valyi (1997); Kurzhanski and Varaiya (2014); Chernousko (1994); Chernousko and Rokityanskii (2000); Polyak et al. (2004); Sinyakov (2015); Walter and Pronzato (1997); Milanese et al. (1996); Mazurenko (2012); Häckl (1996); Kishida and Braatz (2015); Baier et al. (2013)).

The solutions of many control and estimation problems under uncertainty are based on the construction, investigation and estimation of reachable (or attainable) sets of control systems or on studying set-valued estimates of reachable sets if the exact description of these sets is difficult, especially in the case of nonlinearity of dynamical systems. In this paper the modified state estimation approach is further developed, it uses the special structure of nonlinearity and uncertainty of studied control system and uses also the advantages of ellipsoidal calculus (Kurzhanski and Valyi (1997); Kurzhanski and Varaiya (2014)). The studies in the direction presented here are motivated by applications related, e.g., to satellite control problems (Kuntsevich and Volosov (2015)) with nonlinearity and disturbances in the model description.

This study continues the researches Filippova (2010, 2013, 2014, 2016, 2017a,b, 2018) and considers a more complicated case, when the dynamical equations contain both nonlinearity of quadratic type and also bilinear terms defined by uncertain matrix. This special case of control systems with uncertainty and nonlinearity is both of theoretical and practical importance mentioned above. Here the emphasis is done on the derivation of equations describing the dynamics of external ellipsoidal estimates of reachable sets of control systems of the class under consideration. These features describe the importance and the novelty of the problem formulation and of related results presented in the paper.

The rest of the paper is structured as follows. In section 2, we describe the basic structure of the model, give necessary definitions and formulate the problem. In section 3, we discuss some preliminary constructions which are necessary to formulate and prove the main result and illustrate these ideas and results by an example. In section 4, we formulate and prove the main theorem, the illustrative example is also included. Section 5 concludes the paper.

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2. PROBLEM FORMULATION

2.1 Basic Notations

We need first to introduce the following basic notations.

Let $\mathbb{R}^n$ be the n–dimensional Euclidean space, $\mathbb{R}^{n\times n}$ be the set of all $n\times n$–matrices and $x'y$ be the usual inner product of $x, y \in \mathbb{R}^n$ with prime as a transpose. The Euclidian norm in $\mathbb{R}^n$ will be denoted as $\|x\| = (x'x)^{1/2}$, $x \in \mathbb{R}^n$. Let $\operatorname{comp}\mathbb{R}^n$ denote the set of all compact subsets of $\mathbb{R}^n$ with $h(A, B)$ being the Hausdorff distance between sets $A, B \in \operatorname{comp}\mathbb{R}^n$.

We denote as $B(a, r)$ the ball in $\mathbb{R}^n$ with a center a and of radius $r > 0$, $B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \le r\}$. Let I be the identity $n \times n$-matrix.

Denote as $E(a, Q)$ the ellipsoid in $\mathbb{R}^n$,

$$E(a, Q) = \{x \in \mathbb{R}^n : \ (Q^{-1}(x - a), (x - a)) \le 1\}$$

with a center $a \in \mathbb{R}^n$ and with a symmetric positive definite $n \times n$–matrix Q.

We use also the notation for the trace of a square $n \times n$ matrix A, denoted as $Tr(A)$,

$$Tr(A) = \sum_{i=1}^{i=n} a_{ii}, \quad A = \{a_{ij}\} \in \mathbb{R}^{n\times n}.$$

2.2 Dynamical System

Consider the following nonlinear control system

$$\begin{aligned} &\dot{x} = A(t)x + f(x)d + u(t), \\ &x_0 \in X_0, \quad t_0 \le t \le T, \end{aligned} \tag{1}$$

here $x, d \in \mathbb{R}^n$, $\|x\| \le K$ $(K > 0)$. We assume here that the $n \times n$-matrix function $A(t)$ in (1) has the form

$$A(t) = A^0 + A^1(t), \tag{2}$$

where the $n \times n$-matrix A^0 is given and the measurable $n \times n$-matrix $A^1(t)$ is unknown but bounded, $A^1(t) \in \mathcal{A}^1$ $(t \in [t_0, T])$,

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1. \tag{3}$$

Here

$$\begin{aligned} \mathcal{A}^1 = \{ \ & A{=}\{a_{ij}\}{\in}R^{n\times n} : a_{ij} = 0 \ \text{ for } i \ne j, \\ & a_{ii} = a_i, \quad i = 1, \ldots, n, \\ & a = (a_1, \ldots, a_n), \quad a'Da \le 1 \ \}, \end{aligned} \tag{4}$$

where $D \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix.

It is assumed here that $f(x) = x'Bx$ is a scalar function, with a symmetric positive definite $n \times n$–matrix B.

We assume that the controls $u(t)$ in (1) are defined as functions which are Lebesgue measurable on $[t_0, T]$ and satisfy the constraint

$$u(t) \in U, \quad U \in \operatorname{comp}\mathbb{R}^m \quad \text{for a.e. } t \in [t_0, T]. \tag{5}$$

We denote the class of admissible controls as $\mathcal{U}$. Assume that X_0 in (1) is an ellipsoid, $X_0 = E(a_0, Q_0)$, with a symmetric and positive definite matrix Q_0 and with a center a_0.

2.3 Reachable Sets and Trajectory Tubes

Let the function $x(t) = x\big(t; u(\cdot), A(\cdot), x_0\big)$ be a solution to dynamical system (1)–(3) with initial state $x_0 \in \mathcal{X}_0$, with admissible control $u(\cdot)$ and with a matrix $A(\cdot)$ satisfying (2)–(4). We assume that all these solutions $\{x(t)\}$ are extendable up to the instant T and are bounded $\|x(t)\| \le K$ (with some $K > 0$) (see e.g. Filippova and Berezina (2007) for a detailed discussion of related assumptions).

The reachable set $X(t)$ at time t $(t_0 < t \le T)$ of system (1)–(3) is defined as the following set

$$\begin{aligned} X(t) = \{x \in \mathbb{R}^n : \exists\, x_0{\in}X_0, \exists\, u(\cdot){\in}\mathcal{U}, \exists\, A(\cdot){\in}\mathcal{A}, \\ \text{such that} \ \ x = x(t) = x\big(t; u(\cdot), A(\cdot), x_0\big)\}. \end{aligned} \tag{6}$$

The *trajectory tube* $X(\cdot) = X(\cdot; \mathcal{U}, \mathcal{A}, X_0)$ of the system of system (1)–(3) is the set whose time cross-sections $X(t)$ coincide with reachable sets (6) of the system (1)–(3) (Kurzhanski and Filippova (1993)).

2.4 Estimation Problem

Analyzing the special type of nonlinear control systems with uncertain initial data we consider here the techniques which allow us to find the external ellipsoidal estimate $E(a^+(t), Q^+(t))$ of the reachable set $X(t)$ $(t_0 < t \le T)$ and to study the dynamics of the proposed estimating sets.

The approach given in this paper develops and refines previous researches in this area. In particular, here we continue the study initiated in Filippova (2018) where the discrete scheme was proposed for construction the external estimate of reachable set of control system of type (1)–(3).

We present the limit procedure and we prove new results on deriving the differential equations which describe the trajectory tube of the system (1)–(3). We underline that the novelty of the result is connected with the consideration of a new class of constraints on the unknown elements of the matrix in the right-hand sides of the system (1)–(3), namely, the constraints (4) have a quadratic type, in contrast to the previously investigated bilinear-quadratic case with constraints on the modulus of unknown matrix variables (Filippova (2016)). Mention here also earlier results Filippova (2010) in this field but of a special type when a bilinearity was not assumed in the control system and the case of quadratic nonlinearity in the system state velocities together with the uncertainty in initial states was under study there.

Illustrative examples related to procedures of set-valued approximations of trajectory tubes and reachable sets are given further. The applications of the problems studied in this paper are in modeling the set-valued dynamics for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

3. PRELIMINARY CONSTRUCTIONS

We will need some auxiliary results which will be used in the following.

3.1 System Dynamics Nonlinearity Defined by a Positive Definite Quadratic Form

Consider the following control system

$$\begin{aligned} &\dot{x} = A^0 x + f(x)d + u(t), \\ &x_0 \in X_0 = E(a_0, Q_0), \quad t_0 \le t \le T. \end{aligned} \tag{7}$$

Here $x \in \mathbb{R}^n$, $\|x\| \le K$ $(K > 0)$, $A^0 \in \mathbb{R}^{n\times n}$ is a given matrix, $u(t) \in U = E(\hat{a}, \hat{Q})$. Vectors $d, a_0, \hat{a} \in \mathbb{R}^n$ are given, $f(x) = x'Bx$, B, Q_0, $\hat{Q}$ are symmetric and positive definite $n \times n$-matrices.

Denote the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$ by k^2, in this case k^2 is the smallest number for which the inclusion $X_0 \subseteq E(a_0, k^2B^{-1})$ is true (Filippova (2010)).

The dynamics of the external ellipsoidal estimates of reachable sets $X(t) = X(t; t_0, X_0)$ $(t_0 \le t \le T)$ for the system (7) is described by the following theorem.

Theorem 1. (Filippova (2010)). The inclusion is true for any $t \in [t_0, T]$

$$X(t; t_0, X_0) \subseteq E(a^+(t), r^+(t)B^{-1}), \tag{8}$$

where functions $a^+(t)$, $r^+(t)$ are the solutions of the following system of ordinary differential equations

$$\begin{aligned} &\dot{a}^+(t) = A^0a^+(t) + ((a^+(t))'Ba^+(t) + \\ &r^+(t))d + \hat{a}, \quad t_0 \le t \le T, \\ &\dot{r}^+(t) = \max_{\|l\|=1} \Big\{ l'\big(2r^+(t)B^{1/2}(A^0 + \\ &2d(a^+(t))'B)B^{-1/2} + \\ &q^{-1}(r^+(t))B^{1/2}\hat{Q}B^{1/2})\big)l \Big\} + q(r^+(t))r^+(t), \\ &q(r) = ((nr)^{-1}\, Tr(B\hat{Q}))^{1/2}, \end{aligned} \tag{9}$$

with initial state

$$a^+(t_0) = a_0, \quad r^+(t_0) = k^2.$$

This result is convenient for constructing estimating ellipsoids, since the formulas of Theorem 1 are fairly simple and can be programmed using related computational packages (for example, MatLab), this was done for a number of examples in Filippova (2010). Moreover Theorem 1 may be reformulated also for the more general case when instead of a constant matrix A^0 in (7) we have a given continuous matrix $A(t) \in \mathbb{R}^{n\times n}$ $(t_0 \le t \le T)$.

Unfortunately, in the case under study this result can not be apply directly for estimating the trajectory tubes $X(\cdot)$ of the system (1)–(3), since there is an assumption about the uncertainty of the matrix coefficients $A(t)$ in the system (1)–(3) which is not covered by Theorem 1.

3.2 Systems with Unknown Matrix

We will assume further that $0 \in X_0$ and $0 \in U$. In this case the reachable sets $X(t)$ of the system (1)–(3) are star-shaped (Filippova (2018)) and the following equality is true for Minkowski (gauge) functional $h_M(z)$,

$$h_M(z) = \inf\{t > 0 : z \in tM, x \in \mathbb{R}^n\}.$$

Theorem 2. (Filippova (2018)). Assume that

$$X_0 = E(a_0, k^2B^{-1}), \quad k \ne 0.$$

Then for all $\sigma > 0$ the following external estimate is true,

$$\begin{aligned} &X(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0, 1), \\ &\lim_{\sigma\to+0} \sigma^{-1}o(\sigma) = 0, \end{aligned} \tag{10}$$

where

$$a^+(\sigma) = a_0 + \sigma(A^0a_0 + \hat{a} + k^2d + a_0'Ba_0 \cdot d),$$

$$Q^+(\sigma) = (p^{-1} + 1)Q_1(\sigma) + (p + 1)\sigma^2\hat{Q}^*,$$

$$Q_1(\sigma) = \mathrm{diag}\{\, (p_*^{-1} + 1)\sigma^2a_{0i}^2 +$$

$$(p_* + 1)r^2(\sigma) \mid i = 1, \dots, n\,\},$$

$$r(\sigma) = \max_z \|z\| \cdot (h_{(I+\sigma\mathcal{A}^1)*X_0}(z))^{-1},$$

$$(I + \sigma\mathcal{A}^1) * X_0 = \bigcup_{A\in\mathcal{A}^1} (I + \sigma A)X_0$$

here numbers p_*, p are the unique positive roots of the related equations (respectively)

$$\sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p + 1)}$$

with $\alpha_i = \alpha_i(\sigma) \ge 0$ $(i = 1, ..., n)$ satisfying for p_* the equations

$$\prod_{i=1}^{n} (\sigma^2(a_i^0)^2 - \alpha r^2(\sigma)) = 0$$

and for p, the equation

$$|Q_1(\sigma) - \alpha\sigma^2\hat{Q}^*| = 0,$$

here also $E(0, \hat{Q}^*)$ denotes the ellipsoid with minimal volume such that

$$E(0, \hat{Q}) + (2d \cdot a_0'B + A^0)E(0, k^2B^{-1}) \subseteq E(0, \hat{Q}^*).$$

Theorem 2 opens the way to construct external estimates of reachable sets in the general case considered here by stepwise approximations (see the algorithm and the example in the next Sections). However the above formulas of Theorem 2 look rather complicated, especially in contrast to the previous Theorem 1. This is also the motivation for further researches and for attempts to derive the limit procedures for external estimates in the case under study.

3.3 Numerical Algorithm

The following algorithm is based on Theorem 2 and produce the external ellipsoidal estimates for the reachable sets $X(t)$ of the system (1)–(3) in a step-by-step mode.

Algorithm 1. Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$, where $t_i = t_0 + ih$ $(i = 1, \dots, m)$, $h = (T - t_0)/m$.

- Take $\sigma = h$ and for given $X_0 = E(a_0, Q_0)$ define the smallest $k_0 > 0$ such that $E(a_0, Q_0) \subseteq E(a_0, k_0^2B^{-1})$ (k_0^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$ (Bellman (1997), see also related comments in Filippova (2018)).
- For $X_0 = E(a_0, k_0^2B^{-1})$ as an initial set define by Theorem 2 the upper estimate $X_1 = E(a^+(\sigma), Q^+(\sigma))$ of the set $X(t_0 + \sigma, t_0, X_0)$.
- Consider the system on the next time segment $[t_1, t_2]$ with the initial ellipsoid $E(a_1, k_1^2B^{-1})$ found as above for $X_1 = E(a^+(\sigma), Q^+(\sigma))$ (now for the current interval $[t_1, t_2]$ the set X_1 is taken as the starting ellipsoid instead of $X_0 = E(a_0, k_0^2B^{-1})$ at the previous step).

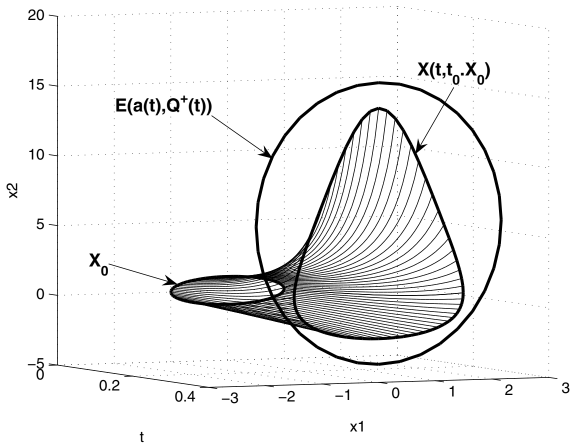

Fig. 1. External ellipsoidal estimate $E(a(t), Q^+(t))$ for $X(t)$.

- The next step repeats the previous iteration beginning with new initial data.

At the end of the process we will get the external estimate tube $E(a^+(t), Q^+(t))$ of the reachable sets $X(t)$ ($t_0 \le t \le T$) of the system (1)–(3).

3.4 Example

Example 1. Consider the following nonlinear control system:

$$\begin{cases} \dot{x_1} = x_1 + u_1, \\ \dot{x_2} = 3x_2 + x_2^2 + u_2. \end{cases} \quad (11)$$

Here $t_0 = 0$, $T = 0.4$, $X_0 = B(0, 1)$, $U = B(0, 0.01)$. The trajectory tube $X(t)$ and its external ellipsoidal estimate $E(a(t), Q^+(t))$ found by the Algorithm described above are shown in Fig. 1.

4. MAIN RESULTS

We will begin this section with a brief description of the place and novelty of the result considered in this paper and formulated below.

Earlier some approaches had been proposed to obtain differential equations describing dynamics of external ellipsoidal estimates for reachable sets of control system under uncertainty, e.g., in Chernousko and Rokityanskii (2000) the authors studied estimation problems for systems with uncertain matrices in dynamical equations, but additional nonlinear terms in dynamics were not considered there. In Filippova (2010) differential equations of ellipsoidal estimates for reachable sets of a nonlinear dynamical control system were derived for the case when system state velocities contain quadratic forms but in that case the uncertainty in matrix coefficients was not assumed.

In Filippova (2016) differential equations for external ellipsoidal estimates of reachable sets of a control system with nonlinearity and with uncertain matrix were derived under assumptions of another type of constraints on unknown matrix than we consider here. Namely, it was assumed in Filippova (2016) that all elements (a_{ij}) of the unknown matrix $A(t)$ are bounded in modulus. Here we study the different (and maybe more complicated) case when we assume that we have the quadratic-type restrictions on elements (a_{ij}) of unknown matrix $A(t)$.

4.1 Differential Equations for Upper Ellipsoidal Estimates

As before the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$ is denoted by k^2, therefore k^2 is the smallest positive number for which the inclusion

$$X_0 \subseteq E(a_0, k^2B^{-1})$$

is satisfied. The following result describes the dynamics of the external ellipsoidal estimates of the reachable set $X(t) = X(t; t_0, X_0)$ ($t_0 \le t \le T$) of the system (1)–(3).

Theorem 3. The inclusion is true for any $t \in [t_0, T]$

$$X(t; t_0, X_0) \subseteq E(a^+(t), Q^+(t)), \quad (12)$$

with

$$Q^+(t) = r^+(t)B^{-1}, \quad (13)$$

where functions $a^+(t)$, $r^+(t)$ are the solutions of the following system of ordinary differential equations

$$\begin{aligned} \dot{a}^+(t) &= A^0a^+(t) + ((a^+(t))'Ba^+(t) + \\ &\quad r^+(t))d + \hat{a}, \quad t_0 \le t \le T, \end{aligned} \quad (14)$$

$$\begin{aligned} \dot{r}^+(t) &= \max_{\|l\|=1} \Big\{ l'\big(2r^+(t)B^{1/2}(A^0 + \\ &\quad 2d(a^+(t))'B)B^{-1/2} + \\ &\quad q^{-1}(r^+(t))B^{1/2}\hat{Q}^*B^{1/2}\big)l \Big\} + q(r^+(t))r^+(t), \end{aligned} \quad (15)$$

$$q(r) = ((nr)^{-1}\, Tr(B\hat{Q}^*))^{1/2}, \quad (16)$$

where the positive definite matrix $\hat{Q}^*$ is such that

$$\mathcal{A}^1a_0 + E(0, \hat{Q}) + kD^{1/2}B^{1/2}B(0, 1) \subset E(0, \hat{Q}^*), \quad (17)$$

and initial states for $a^+(\cdot)$ and $r^+(\cdot)$ are

$$a^+(t_0) = a_0, \quad r^+(t_0) = k^2.$$

Proof. The proof of the theorem is based on the limiting procedure in the relations of Theorem 2 and is carried out according to the scheme of Theorem 1 with necessary corrections due to the more complicated structure of constraints on the initial data. We note that the differential equations (14)–(16) are similar in form to (9) of Theorem 1, but differ in essence, since the matrix $Q^+(t)$ in (13) and therefore the equations (14)–(16) for $a^+(t)$ and $r^+(t)$ contain a matrix $\hat{Q}^*$ found in accordance with the inclusion (17) which takes into account the specifics of more general constraints of the problem studied here. □

Remark 1. Note that the matrix $\hat{Q}^*$ (and therefore the estimating ellipsoid $E(0, \hat{Q}^*)$) in (17) depends on a_0, it is a new feature appeared here due to more complicated bilinear structure of uncertainties in the system dynamics.

Remark 2. The numerical scheme and related algorithm for constructing upper estimates of reachable sets of the system under consideration may be also formulated similar to Algorithm 1.

4.2 Numerical Simulation

Consider the example which shows that in nonlinear case the reachable sets of the control system with nonlinearity and uncertainty of the studied type may lose the convexity property with increasing time $t > t_0$. Nevertheless the

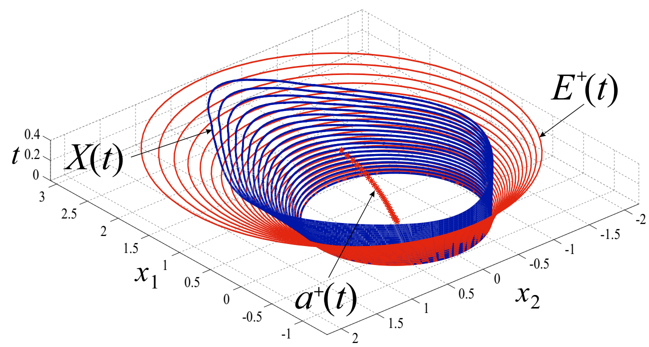

Fig. 2. Trajectory tube $X(\cdot)$ and the upper estimating tube $E^+(\cdot) = E(a^+(\cdot), Q^+(\cdot))$.

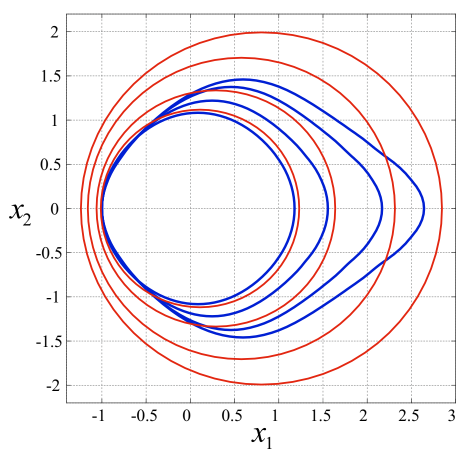

Fig. 3. Reachable sets $X(t)$ and its upper estimates $E(a^+(t), Q^+(t))$.

related external estimates calculated on the base of results of Theorem 3 are ellipsoids (and therefore convex) and they contain the true reachable sets of the studied control system.

Example 2. Consider the following control system

$$\begin{cases} \dot{x}_1 = a_1(t)x_1 + x_1^2 + x_2^2 + u_1, \\ \dot{x}_2 = a_2(t)x_2 + u_2, \end{cases} \tag{18}$$

Here we take $x_0 \in X_0 = B(0,1)$, $0 \le t \le 0.4$ and $U = B(0, 0.1)$. The function $a(t) = (a_1(t), a_2(t))$ is unknown, with the constraint $||a(t)|| \le 0.5$ $(t_0 \le t \le T)$. The reachable sets $X(t)$ and their external ellipsoidal estimates $E^+(t) = E(a^+(t), Q^+(t))$ for different time moments are shown in Figure 2. The central red curve in Figure 2 shows the dynamics of the centers $a^+(t)$ of the estimating ellipsoids $E^+(t) = E(a^+(t), Q^+(t))$.

Figure 3 shows the projections of estimating ellipsoids $E^+(t) = E(a^+(t), Q^+(t))$ (red lines) and the reachable sets $X(t)$ (blue lines) onto the plane of state coordinates for time moments $t = 0.08; 0.2; 0.32; 0.38$.

4.3 Further Investigations and Selected Results

Further studies in obtaining estimates of reachable sets of uncertain systems of the type considered in this paper can be made for control systems with additional state constraints on the states of the system, as well as in cases of a more complex control structure (for example, containing impulse components).

5. CONCLUSIONS

The paper deals with the problems of state estimation for uncertain dynamical control systems for which we assume that the initial system state is unknown but bounded with given constraints.

Basing on the results of ellipsoidal calculus developed earlier we present the new state estimation approaches which use the special quadratic structure of nonlinearity and uncertainty of the control system and allow to construct the external ellipsoidal estimates of reachable sets.

The differential equations which describe the dynamics of the ellipsoidal estimates of reachable sets for nonlinear dynamical control systems with combined nonlinearity of quadratic and bilinear types and with uncertainty in initial states are derived.

Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented.

The applications of the problems studied in this paper are in theoretical and applied problems of guaranteed state estimation for nonlinear control systems with unknown but bounded errors.

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