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Traversing Target Points Under Lack of Information: A Game-Theoretical Approach ⋆

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Abstract: A dynamical object controlled under conditions of unknown disturbances or counter-actions is considered. The goal is to bring the object to given target points at given times regardless of the disturbance actions. To solve this problem, the quality index is introduced that evaluates the distance between the object and target points at the indicated times, and a zero-sum differential game is considered in which the control actions minimize this quality index while the disturbance actions maximize it. The initial problem is reduced to calculating the game value and constructing a saddle point of the game. The corresponding resolving procedure is proposed that is based on the upper convex hulls method. An example is considered. Results of numerical simulations are presented.

Keywords: Control problem, disturbances, traversal of target points, non-terminal quality index, differential game, numerical method.

1. INTRODUCTION

We consider a dynamical object controlled under conditions of unknown disturbances or counter-actions. A motion of the object is described by ordinary differential equations. The control and disturbance actions are restricted by geometric constraints. The goal is to bring the object to given target points at given times regardless of the disturbance actions. To solve this problem, we introduce the quality index that is the Euclidian norm of the deviations of a motion at the indicated times from the corresponding target points. We consider a zero-sum differential game in which the first player minimizes this quality index by means of the control actions while the second player maximizes it by choosing the disturbance actions. We formalize the game within the positional approach (see, e.g., Krasovskii (1985); Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1995)) in the classes of feedback strategies of the first player and feedback counter-strategies of the second player. The connection between the initial problem and differential game is the following. The value of the game is equal to zero if and only if it is possible to guarantee that the target points are traversed whatever the disturbance actions. Moreover, the optimal control strategy of the first player brings the object to the target points as close as possible. To calculate the value of the game, we apply the upper convex hulls method (see, e.g., Krasovskii and Krasovskii (1995); Krasovskii and Reshetova (1988); Lukoyanov (1998); Gomoyunov and Kornev (2016)) and its numerical implementation given in Kornev (2012); Gomoyunov, Kornev and Lukoyanov (2015). The optimal strategy of the first player and counter-strategy of the second player are constructed by the method of extremal shift to accompanying points (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)).

The paper is organized as follows. In Section 2, we formulate the control problem under consideration. In Section 3, we formalize this problem as a zero-sum differential game, and, in particular, we describe admissible feedback control schemes of the players. A method for calculating the value of the game and constructing the optimal feedback control schemes is given in Section 4. In Section 5, an example is considered and results of numerical simulations are presented. The conclusion is given in Section 6.

2. DESCRIPTION OF THE PROBLEM

Let us consider a controlled object which state at the current time t is described by the d-dimensional vector $q(t)$. A motion of the object is considered on the finite time interval $[t_0, \vartheta]$ and is described by the following differential equation:

$$\begin{gathered} \ddot{q}(t) = A_1(t)q(t) + A_2(t)\dot{q}(t) + F(t, u(t), v(t)), \\ q(t) \in \mathbb{R}^d,\ u(t) \in P,\ v(t) \in Q,\ t \in [t_0, \vartheta], \end{gathered} \tag{1}$$

under the initial condition

$$q(t_0) = q_0,\ \dot{q}(t_0) = \dot{q}_0. \tag{2}$$

Here the upper dots denote the derivatives with respect to t; $u(t)$ is the current control action, $v(t)$ is the current action of unknown disturbances; $P \subset \mathbb{R}^r$ and $Q \subset \mathbb{R}^s$ are

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known compact sets. The elements of the matrices $A_1(t)$, $A_2(t)$ and the vector $F(t,u,v)$ are continuous on $[t_0,\vartheta]$ and $[t_0,\vartheta]\times P\times Q$, respectively.

Let target points $g_i\in\mathbb{R}^d$, $i=\overline{1,N}$, be given in the object state space, and times $\vartheta_i\in(t_0,\vartheta]$, $i=\overline{1,N}$, be specified such that $\vartheta_i<\vartheta_{i+1}$, $i=\overline{1,N-1}$. The goal of the control is, by choosing the actions $u(t)$, to bring the object to the target point g_i at the time ϑ_i for any $i=\overline{1,N}$, i.e., to ensure the equalities

$$q(\vartheta_i)=g_i,\quad i=\overline{1,N}. \tag{3}$$

We assume that, at a time t, the information about the current state $q(t)$ and velocity $\dot q(t)$ of the object is available to determine the current control action $u(t)$, i.e., feedback control schemes are considered.

The disturbance actions are unknown, can take any admissible values and, in particular, may counteract the control actions.

We consider the following two problems:

Problem 1. Determine whether or not it is possible to ensure equalities (3) regardless of the disturbance actions.

Problem 2. Design a feedback control scheme that ensures equalities (3) if it possible or brings the object as close as possible to the target points g_i at the times ϑ_i.

To solve these problems, we introduce the quality index

$$\gamma=\big(\mu_1\|q(\vartheta_1)-g_1\|^2+\ldots+\mu_N\|q(\vartheta_N)-g_N\|^2\big)^{1/2} \tag{4}$$

and consider a two-person zero-sum differential game in which the first player, by choosing the actions $u(t)$, minimizes γ and the second player, by choosing the actions $v(t)$, maximizes γ. Here the symbol $\|\cdot\|$ denotes the Euclidian norm of a vector, and the weights $\mu_i>0$, $i=\overline{1,N}$, can be used for ranging the target points.

3. DIFFERENTIAL GAME

We formalize the game within the positional approach (see, e.g., Krasovskii (1985); Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1995)).

3.1 Dynamical System and Quality Index

Equation (1) can be rewritten in the normal form

$$\begin{aligned}&\dot x(t)=A(t)x(t)+f(t,u(t),v(t)),\\&x(t)\in\mathbb{R}^n,\ u(t)\in P,\ v(t)\in Q,\ t\in[t_0,\vartheta],\end{aligned} \tag{5}$$

where $n=2d$ and

$$x(t)=\begin{pmatrix}q(t)\\ \dot q(t)\end{pmatrix},\ A(t)=\begin{pmatrix}A_1(t)&0\\0&A_2(t)\end{pmatrix},$$

$$f(t,u,v)=\begin{pmatrix}0\\F(t,u,v)\end{pmatrix}.$$

Respectively, initial condition (2) and quality index (4) take the following form:

$$x(t_0)=x_0, \tag{6}$$

and

$$\gamma=\big(\|D_1(x(\vartheta_1)-c_1)\|^2+\ldots+\|D_N(x(\vartheta_N)-c_N)\|^2\big)^{1/2}, \tag{7}$$

where

$$x_0=\begin{pmatrix}q_0\\ \dot q_0\end{pmatrix},\ D_i=(\mu_i I_d\ 0),\ c_i=\begin{pmatrix}g_i\\0\end{pmatrix}.$$

Here I_d denotes the identity $(d\times d)$-matrix.

By admissible control realizations of the first and the second players, we mean Borel measurable functions

$$u(\cdot)=\big\{u(t)\in P,\ t_0\le t<\vartheta\big\},$$

$$v(\cdot)=\big\{v(t)\in Q,\ t_0\le t<\vartheta\big\}.$$

Such realizations uniquely generate the motion

$$x(\cdot)=\big\{x(t)\in\mathbb{R}^n,\ t_0\le t\le\vartheta\big\}$$

of system (5), which is an absolutely continuous function that satisfies initial condition (6) and, together with $u(\cdot)$ and $v(\cdot)$, satisfies equation (5) for almost all $t\in[t_0,\vartheta]$. Thus, each pair of realizations $u(\cdot)$ and $v(\cdot)$ uniquely determines the corresponding value of quality index (7).

3.2 Strategies of the First Player

A control strategy of the first player is a function

$$U=U(t,x,\varepsilon)\in P,\ t\in[t_0,\vartheta],\ x\in\mathbb{R}^n,\ \varepsilon>0,$$

where ε is an auxiliary parameter related to the accuracy of achieving of the guaranteed result of the strategy (see (9) below).

A strategy U acts onto system (5) in the discrete time scheme on the basis of a partition

$$\Delta_\delta=\big\{\tau_j:\tau_1=t_0,0<\tau_{j+1}-\tau_j\le\delta,j=\overline{1,k},\tau_{k+1}=\vartheta\big\} \tag{8}$$

of the control interval $[t_0,\vartheta]$. A triple $\{U,\varepsilon,\Delta_\delta\}$ determines a feedback control law of the first player, that forms a piecewise constant realization $u(\cdot)$ as follows:

$$u(t)=U(\tau_j,x(\tau_j),\varepsilon),\ t\in[\tau_j,\tau_{j+1}),\ j=\overline{1,k}.$$

The value of the guaranteed result of a strategy U is defined by

$$\Gamma_u(U)=\limsup_{\varepsilon\downarrow0}\ \lim_{\delta\downarrow0}\ \sup_{\Delta_\delta}\ \sup_{v(\cdot)}\gamma, \tag{9}$$

where γ is the value of quality index (7) that corresponds to the motion of system (5), (6) that is uniquely determined by the law $\{U,\varepsilon,\Delta_\delta\}$ together with the control realization $v(\cdot)$ of the second player.

The optimal guaranteed result of the first player is

$$\Gamma_u^0=\inf_U\Gamma_u(U). \tag{10}$$

A strategy of the first player U^0 is optimal if

$$\Gamma_u(U^0)=\Gamma_u^0.$$

According to definition (9), it means that, for any number $\zeta>0$, there exist a number $\varepsilon^0>0$ and a function $\delta^0(\varepsilon)>0$, $\varepsilon\in(0,\varepsilon^0]$, such that, for any value of the parameter $\varepsilon\in(0,\varepsilon^0]$ and any partition Δ_δ with $\delta\le\delta^0(\varepsilon)$, the control law $\{U^0,\varepsilon,\Delta_\delta\}$ of the first player guarantees for quality index (7) the inequality

$$\gamma\le\Gamma_u^0+\zeta \tag{11}$$

regardless of a control realization $v(\cdot)$ of the second player.

It follows from the results of Krasovskii (1985); Krasovskii and Krasovskii (1995) that the value Γ_u^0 coincides with the value of the considered differential game when the second player uses counter-strategies.

3.3 Counter-Strategies of the Second Player

A counter-strategy of the second player is a function
$$V = V(t, x, u, \varepsilon) \in Q,\ t \in [t_0, \vartheta],\ x \in \mathbb{R}^n,\ u \in P,\ \varepsilon > 0,$$
which is Borel measurable in u for any fixed t, x and ε.

A triple $\{V, \varepsilon, \Delta_\delta\}$ is called a feedback control law of the second player. This law together with a control realization $u(\cdot)$ of the first player forms a realization $v(\cdot)$ as follows:
$$v(t) = V(\tau_j, x(\tau_j), u(t), \varepsilon),\ t \in [\tau_j, \tau_{j+1}),\ j = \overline{1,k},$$
where $u(t)$ is the current value of the control actions of the first player. Note that, since V is Borel measurable in u, the obtained realization $v(\cdot)$ is admissible.

The guaranteed result of a counter-strategy V and the optimal guaranteed result of the second player are defined as follows:
$$\Gamma_v(V) = \liminf_{\varepsilon \downarrow 0} \liminf_{\delta \downarrow 0} \inf_{\Delta_\delta} \inf_{u(\cdot)} \gamma, \tag{12}$$
$$\Gamma_v^0 = \sup_V \Gamma_v(V). \tag{13}$$
A counter-strategy of the second player V^0 is optimal if
$$\Gamma_v(V^0) = \Gamma_v^0.$$
According to definition (12), it means that, for any $\zeta > 0$, a control law $\{V^0, \varepsilon, \Delta_\delta\}$ of the second player under the sufficiently small $\varepsilon > 0$ and $\delta > 0$ guarantees the inequality
$$\gamma \geq \Gamma_v^0 - \zeta \tag{14}$$
for any control realization $u(\cdot)$ of the first player.

3.4 Game Value and Saddle Point

Note that, from definitions (9), (10) and (12), (13) it follows that the players' optimal guaranteed results satisfy the inequality
$$\Gamma_u^0 \geq \Gamma_v^0. \tag{15}$$
Moreover, under the considered conditions, according to Krasovskii (1985); Krasovskii and Krasovskii (1995), the equality holds in (15), i.e., the differential game (5)–(7) has the value
$$\Gamma^0 = \Gamma_u^0 = \Gamma_v^0. \tag{16}$$
In addition, the game has a saddle point $\{U^0, V^0\}$ consisting of optimal strategies U^0 and V^0.

In terms of the original control problem, in which the object (1) should traverse the target points g_i, $i = \overline{1,N}$, (see Section 2), relations (11), (14) and (16) mean the following. Concerning Problem 1, we derive that equalities (3) can be ensured (regardless of the disturbance actions) if and only if the equality $\Gamma^0 = 0$ is valid. Moreover, a control law of the first player that ensures inequality (11) solves Problem 2. For instance, one can choose the control law $\{U^0, \varepsilon, \Delta_\delta\}$ on the basis of the optimal strategy U^0. Thus, Problems 1 and 2 are reduced to calculating the game value Γ^0 and constructing a control law of the first player that ensures inequality (11).

4. SOLVING PROCEDURE

For solving the differential game (5)–(7), let us apply the upper convex hulls method (see, e.g., Krasovskii and Krasovskii (1995); Lukoyanov (1998); Gomoyunov and Kornev (2016)).

Let Δ_δ be a partition (8) that contains all the times ϑ_i from (7), i.e., the inclusions
$$\vartheta_i \in \Delta_\delta,\ i = \overline{1,N}, \tag{17}$$
are valid. Basing on the partition Δ_δ, we define the sets
$$G_j^{\pm} \subset \mathbb{R}^n$$
and the functions
$$\varphi_j^{\pm}(m) \in \mathbb{R},\ m \in G_j^{\pm},\ j = \overline{1,k+1},$$
according to the following recurrent procedure.

For $j = k+1$, we put
$$G_{k+1}^{+} = \{m = 0\},\ \varphi_{k+1}^{+}(m) = 0,$$
$$G_{k+1}^{-} = \big\{m \in \mathbb{R}^n :\ m = D_N^T l,\ l \in \mathbb{R}^d,\ \|l\| \leq 1\big\}, \tag{18}$$
$$\varphi_{k+1}^{-}(m) = -\langle m, c_N \rangle,$$
where the upper symbol T denotes transposition.

For $j = \overline{1,k}$, firstly, we define
$$G_j^{+} = G_{j+1}^{-},\ \psi_j(m) = \Delta\psi_j(m) + \varphi_{j+1}^{-}(m),$$
$$\varphi_j^{+}(\cdot) = \big\{\psi_j(\cdot)\big\}^*_{G_j^{+}}, \tag{19}$$
where
$$\Delta\psi_j(m) = \int_{\tau_j}^{\tau_{j+1}} \min_{u \in P} \max_{v \in Q} \langle m, X(\vartheta, \tau) f(\tau, u, v) \rangle d\tau, \tag{20}$$
and the symbol $\big\{\psi_j(\cdot)\big\}^*_G$ denotes the upper convex hull of the function $\psi_j(\cdot)$ on the set G, i.e., $\varphi_j(\cdot)$ is the minimal concave function that majorizes $\psi_j(\cdot)$ for $m \in G$. Here $X(\vartheta, \tau)$ is the fundamental solution matrix of the equation $\dot{x}(t) = A(t)x(t)$ such that $X(\tau, \tau) = I_n$.

Further, if $\tau_j \neq \vartheta_i$ for any $i = \overline{1,N-1}$, then we set
$$G_j^{-} = G_j^{+},\ \varphi_j^{-}(m) = \varphi_j^{+}(m). \tag{21}$$
Otherwise, if $\tau_j = \vartheta_i$ for some $i = \overline{1,N-1}$, then we put
$$G_j^{-} = \Big\{m \in \mathbb{R}^n :\ m = \nu m_* + X^T(\vartheta_i, \vartheta) D_i^T l,$$
$$\nu \geq 0,\ m_* \in G_j^{+},\ l \in \mathbb{R}^d,\ \|l\|^2 + \nu^2 \leq 1\Big\}, \tag{22}$$
$$\varphi_j^{-}(m) = \max_{(\nu, m_*, l)} \big(\nu \varphi_j^{+}(m_*) - \langle l, D_i c_i \rangle\big), \tag{23}$$
where the maximum is calculated over all the triples $(\nu, m_*, l) \in \mathbb{R} \times G_j^{+} \times \mathbb{R}^d$ that correspond to the vector $m \in G_j^{-}$ according to (22).

Let us denote
$$e_j^{\pm}(x) = \max_{m \in G_j^{\pm}} \big(\langle m, X(\vartheta, \tau_j) x \rangle + \varphi_j^{\pm}(m)\big),$$
$$x \in \mathbb{R}^n,\ j = \overline{1,k+1}. \tag{24}$$

The following proposition is valid (see Gomoyunov and Kornev (2016); Lukoyanov (1998)).

Proposition 1. For any number $\xi > 0$, there exists a number $\delta > 0$ such that, for any partition Δ_δ (8), (17), the inequality below holds:
$$|\Gamma^0 - e_1^{-}(x_0)| \leq \xi,$$
where Γ^0 is the value of the differential game (5)–(7), and $e_1^{-}(x_0)$ is the value (24) constructed on the basis of the partition Δ_δ.

Let us consider a strategy U_* and a counter-strategy V_* that at the times τ_j of the partition Δ_δ are defined by the method of extremal shift to accompanying points (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)) chosen by the values $e_j^+(\cdot)$. We obtain the following formulas (see Krasovskii and Krasovskii (1995); Krasovskii and Reshetova (1988) and also Kornev (2012)):

$$\begin{aligned} U_*(\tau_j, x, \varepsilon) &\in \operatorname*{argmin}_{u\in P} \max_{v\in Q} \langle m_j^-, X(\vartheta, \tau_j) f(\tau_j, u, v)\rangle, \\ V_*(\tau_j, x, u, \varepsilon) &\in \operatorname*{argmax}_{v\in Q} \langle m_j^+, X(\vartheta, \tau_j) f(\tau_j, u, v)\rangle, \end{aligned} \tag{25}$$

where

$$m_j^{\mp} \in \operatorname*{argmax}_{m\in G_j^+} \Big(\langle m, X(\vartheta, \tau_j) x\rangle + \varphi_j^+(m) \mp r(\tau_j, \varepsilon)\sqrt{1 + \|X^T(\vartheta, \tau_j) m\|^2}\Big),$$

and

$$r(t, \varepsilon) = \sqrt{\varepsilon + (t - t_0)\varepsilon}\, e^{\lambda(t-t_0)},$$
$$\lambda = \max_{t\in[t_0,\vartheta]} \max_{\|x\|\le 1} \|A(t)x\|.$$

Basing on Proposition 1 and the properties of the values $e_j^{\pm}(x)$ (see Gomoyunov and Kornev (2016); Lukoyanov (1998)), the following result can be proved.

Proposition 2. For any number $\zeta > 0$, there exist a number $\varepsilon_* > 0$ and a function $\delta_*(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_*]$, such that the following statement is valid. Let $\varepsilon \in (0, \varepsilon_*]$ and Δ_δ be a partition (8), (17) with $\delta \le \delta_*(\varepsilon)$. Let the strategy U_* and the counter-strategy V_* be defined by (25) on the basis of the partition Δ_δ. Then the control law $\{U_*, \varepsilon, \Delta_\delta\}$ of the first player guarantees inequality (11) for any control realization $v(\cdot)$ of the second player, and the control law $\{V_*, \varepsilon, \Delta_\delta\}$ of the second player guarantees inequality (14) for any control realization $u(\cdot)$ of the first player.

Thus, the problem is reduced to recurrent construction (18)–(23) of the functions $\varphi_j^{\pm}(m)$, $m \in G_j^{\pm}$. Note that this procedure can be realized analytically only in rare cases and numerical methods should be applied. The stability of the resolving constructions (18)–(25) with respect to computational and informational errors is proved in Gomoyunov and Lukoyanov (2015). In the example below, we use the numerical method given in Kornev (2012) for solving the differential games of the considered type. The method is based on a "pixel" approximation for the compact sets $G_j^{\pm}$ and the approximate construction of the upper convex hull $\varphi_j^+(\cdot)$ of the function $\psi_j(\cdot)$ as the envelope of a finite family of supporting hyperplanes to the subgraph of $\psi_j(\cdot)$. The convergence of this numerical method is proved in Gomoyunov, Kornev and Lukoyanov (2015).

5. EXAMPLE

Let us consider a motion of a material point of unit mass in the plane. By $q = (q_1, q_2)$ we denote the radius vector of the point. There are two forces acting on the point. The first one is the friction force that is proportional to the velocity $\dot q = (\dot q_1, \dot q_2)$ with the coefficient $\alpha \ge 0$. The second one is the control force. It has the constant value $\beta \ge 0$, and we can choose its direction $u = (u_1, u_2)$ from the four possible variants: forward, backward, to the left, to the right. On the other hand, there are disturbances that can rotate the direction u by the angle $v \in [-\omega, \omega]$, where $\omega \ge 0$ is given. The control process is considered during the finite time interval $[t_0, \vartheta]$. Thus, a motion of the material point is described by the equations

$$\begin{cases} \ddot q_1(t) = -\alpha \dot q_1(t) + u_1(t)\cos v(t) - u_2(t)\sin v(t), \\ \ddot q_2(t) = -\alpha \dot q_2(t) + u_1(t)\sin v(t) + u_2(t)\cos v(t), \end{cases}$$
$$u(t) = (u_1(t), u_2(t)) \in \big\{(0,\beta), (0,-\beta), (\beta,0), (-\beta,0)\big\},$$
$$v(t) \in [-\omega, \omega],\ t \in [t_0, \vartheta].$$

For the radius vector and the velocity vector, the initial values are given:

$$q_1(t_0) = q_{10},\ q_2(t_0) = q_{20},\ \dot q_1(t_0) = \dot q_{10},\ \dot q_2(t_0) = \dot q_{20}.$$

Let points $g_1 = (g_{11}, g_{12})$ and $g_2 = (g_{12}, g_{22})$ in the plane and times $\vartheta_1 \in (t_0, \vartheta)$ and $\vartheta_2 = \vartheta$ be specified. The goal of the control is to bring the material point as close as possible to the point g_1 at the time ϑ_1 and to the point g_2 at the time ϑ_2. We consider this goal as the minimization problem for the quality index

$$\gamma = \big(\|q(\vartheta_1) - g_1\|^2 + \|q(\vartheta_2) - g_2\|^2\big)^{1/2}.$$

Denoting

$$x_1(t) = q_1(t),\ x_2(t) = q_2(t),\ x_3(t) = \dot q_1(t),\ x_4(t) = \dot q_2(t),$$

we consider the differential game for the conflict-controlled dynamical system in the normal form

$$\begin{cases} \dot x_1(t) = x_3(t), \\ \dot x_2(t) = x_4(t), \\ \dot x_3(t) = -\alpha x_3(t) + u_1(t)\cos v(t) - u_2(t)\sin v(t), \\ \dot x_4(t) = -\alpha x_4(t) + u_1(t)\sin v(t) + u_2(t)\cos v(t), \end{cases} \tag{26}$$
$$u(t) = (u_1(t), u_2(t)) \in \big\{(0,\beta), (0,-\beta), (\beta,0), (-\beta,0)\big\},$$
$$v(t) \in [-\omega, \omega],\ t \in [t_0, \vartheta],$$

under the initial condition

$$x_1(t_0) = q_{10},\ x_2(t_0) = q_{20},\ x_3(t_0) = \dot q_{10},\ x_4(t_0) = \dot q_{20}, \tag{27}$$

and with the quality index

$$\gamma = \Big(\big(x_1(\vartheta_1) - g_{11}\big)^2 + \big(x_2(\vartheta_1) - g_{12}\big)^2 + \big(x_1(\vartheta_2) - g_{21}\big)^2 + \big(x_2(\vartheta_2) - g_{22}\big)^2\Big)^{1/2}. \tag{28}$$

The results of the computer simulations given below were obtained with the help of the resolving constructions (18)–(25). The following values of the parameters were chosen:

$$\alpha = 0.1,\ \beta = 3,\ \omega = 0.5,\ t_0 = 0,$$
$$q_{10} = 0,\ q_{20} = -0.5,\ \dot q_{10} = -0.5,\ \dot q_{20} = 1,$$
$$g_1 = (0.5, 0.5),\ \vartheta_1 = 1,\ g_2 = (0, 0),\ \vartheta_2 = \vartheta = 2.$$

The procedure (18)–(23) was realized on the basis of the partition Δ_δ with the constant step $\delta = 0.01$. The value (24) that, according to Proposition 1, approximates the value of the differential game (26)–(28) is

$$\Gamma^0 \approx e_1^-(0, -0.5, -0.5, 1) \approx 0.522.$$

For constructing the strategy U_* and the counter-strategy V_* according to (25), the value of the accuracy parameter $\varepsilon = 0.05$ was chosen.

We considered the following cases. In the first case, the control and disturbance actions $u(t)$ and $v(t)$ are formed respectively by the optimal control law $\{U_*, \varepsilon, \Delta_\delta\}$ and the

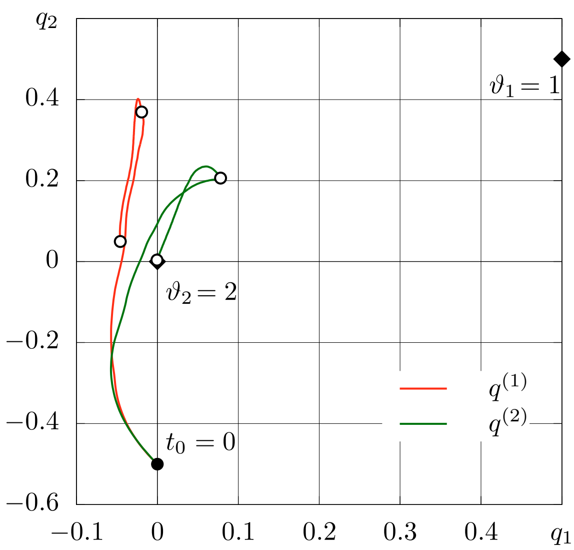

Fig. 1. The realized trajectories $q^{(1)}(\cdot)$ and $q^{(2)}(\cdot)$.

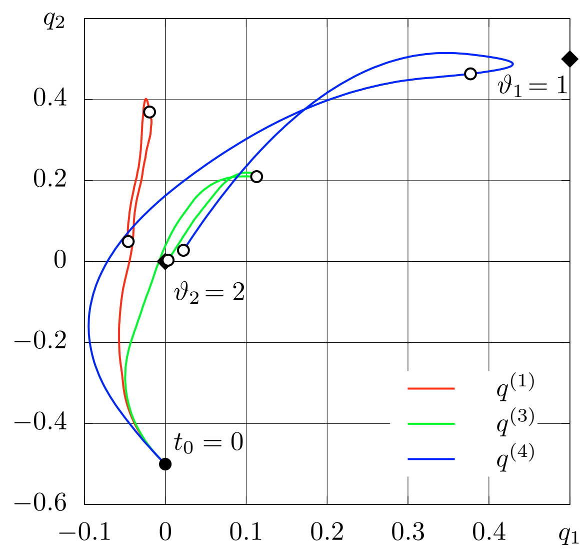

Fig. 2. The realized trajectories $q^{(1)}(\cdot)$, $q^{(3)}(\cdot)$ and $q^{(4)}(\cdot)$.

counter-optimal control law $\{V_*, \varepsilon, \Delta_\delta\}$. The realized value of quality index (28) is

$$\gamma \approx \Big(\big(-0.019 - 0.5\big)^2 + \big(0.369 - 0.5\big)^2 + (-0.046)^2 + 0.05^2\Big)^{1/2} \approx 0.54 \approx \Gamma^0.$$

In the second case, the control actions are formed by the optimal law $\{U_*, \varepsilon, \Delta_\delta\}$ and the disturbance actions are formed on the basis of the following non-optimal in general but reasonable in some sense strategy

$$V = V(t,x) \in \operatorname*{argmax}_{v \in Q} \min_{u \in P} \langle D_i(x - c_i), D_i f(t,u,v)\rangle,$$
$$t \in [\vartheta_{i-1}, \vartheta_i),\ \ i = \overline{1,N},\ \ x \in \mathbb{R}^n,$$

where we denote $\vartheta_0 = t_0$. The obtained result is

$$\gamma \approx \Big(\big(0.078 - 0.5\big)^2 + \big(0.206 - 0.5\big)^2 + (-0.001)^2 + 0.004^2\Big)^{1/2} \approx 0.514 < \Gamma^0.$$

In the third case, the control actions are still formed by the optimal law $\{U_*, \varepsilon, \Delta_\delta\}$ and $v(t) \equiv 0$ (i.e., there are no disturbances, but we do not know about that beforehand). The corresponding result is

$$\gamma \approx \Big(\big(0.113 - 0.5\big)^2 + \big(0.21 - 0.5\big)^2 + 0.004^2 + 0.004^2\Big)^{1/2} \approx 0.484 < \Gamma^0.$$

For comparison, we considered also the fourth case when $\omega = 0$ in (26) (i.e., the disturbances are absent in system (26), and we know about that beforehand). In this case, when we use the corresponding optimal control law, we have

$$\gamma \approx \Big(\big(0.377 - 0.5\big)^2 + \big(0.464 - 0.5\big)^2 + 0.022^2 + 0.028^2\Big)^{1/2} \approx 0.133.$$

The realized trajectories $q^{(i)}(\cdot)$, $i = \overline{1,4}$, of the material point in these four cases are shown in Figures 1 and 2. The target points g_1 and g_2 are marked by black diamonds. The points on the trajectories realized at the times $\vartheta_1 = 1$ and $\vartheta_2 = 2$ are marked by white circles.

6. CONCLUSION

In this paper, we have applied the methods of the differential games theory for solving the problem of traversing the target points under lack of information about the dynamical disturbances and counter-actions. The example has been considered that shows that this approach leads to effective numerical solution.

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