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Demand Functions in Dynamic Games ⋆

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Abstract: The paper is devoted to construction of solutions in dynamic bimatrix games. In the model, the payoffs are presented by discounted integrals on the infinite time horizon. The dynamics of the game is subject to the system of the A. N. Kolmogorov type differential equations. The problem of construction of equilibrium trajectories is analyzed in the framework of the minimax approach proposed by N. N. Krasovskii and A. I. Subbotin in the differential games theory. The concept of dynamic Nash equilibrium developed by A. F. Kleimenov is applied to design the structure of the game solution. For obtaining constructive control strategies of players, the maximum principle of L. S. Pontryagin is used in conjunction with the generalized method of characteristics for Hamilton–Jacobi equations. The impact of the discount index is indicated for equilibrium strategies of the game and demand functions in the dynamic bimatrix game are constructed.

Keywords: Optimal control, differential games, guaranteed strategies, equilibrium trajectories, demand functions.

1. INTRODUCTION

We consider the dynamic bimatrix game with discounted integral payoff functionals on the infinite horizon. Usually the discount parameter appears to be very uncertain value which reflects subjective components in economic and financial models. In this case, models with discounted indices require an implementation of sensitivity analysis for solutions with respect to changing of the discount parameter. In the paper, optimal control strategies are designed on the basis of the Krasovskii minimax approach (see Krasovskii and Krasovskii (1995); Krasovskii and Subbotin (1988)), using constructions of the Pontryagin maximum principle (see Pontryagin et al. (1962)) and the Subbotin technique of the method of characteristics for generalized (minimax) solutions of Hamilton-Jacobi equations (see Subbotin (1991); Subbotin and Tarasyev (1985)). Basing on constructed optimal control strategies, equilibrium trajectories for dynamic bimatrix game are simulated in the framework of the approach proposed by A. F. Kleimenov (see Kleimenov (1993)). It is important to note that in the considered statement we can obtain analytical solutions for control strategies depending explicitly on an uncertain discount parameter. That allows to implement the sensitivity analysis for equilibrium trajectories with respect to changes of the discount parameter and determine the asymptotic behavior of solutions when the discount parameter converges to zero. We show that control strategies and equilibrium solutions asymptotically converge to the solution of the dynamic bimatrix game with the average integral payoff functional considered in papers by V. I. Arnold (see Arnold (2002)).

Let us note that we use dynamic constructions and methods of evolutionary games analysis proposed in the paper (see Kryazhimskii and Osipov (1995)). To explain the dynamics of players' interaction we use elements of evolutionary games (see Basar and Olsder (1982); Vorobyev (1985)). For the analysis of shifting equilibrium trajectories from a competitive static Nash equilibrium to points of the cooperative Pareto maximum we consider ideas and constructions of cooperative dynamic games (see Petrosjan and Zenkevich (2015)). The dynamics of the bimatrix game can be interpreted as a generalization of Kolmogorov's equations for probabilities of states (see Kolmogorov (1938)), which are widely used in Markov processes, stochastic models of mathematical economics and queuing theory.

The solution of dynamic bimatrix games is based on construction of positional strategies that maximize own payoffs, which means "guaranteeing" strategies (see Krasovskii and Krasovskii (1995); Krasovskii and Subbotin (1988); Kurzhanskii (1977)). Elements of the Pontryagin maximum principle (see Pontryagin et al. (1962)) are considered in the aggregation with the method of characteristics for Hamilton-Jacobi equations (see Krasovskii and Taras'ev (2007); Subbotin (1991)). The optimal trajectory in each time interval is constructed from pieces of characteristics while switching moments from one characteristic to another are determined by the maximum principle. Let us note that analogous methods for construction of positional strategies are used in papers (see Klaassen et al. (2004); Krasovskii and Tarasyev (2015, 2016)).

In the framework of proposed approach we consider the model of competition on financial markets which is described by dynamic bimatrix game. For this game we construct switching curves for optimal control strategies and synthesize equilibrium trajectories for various values of the discount parameter. Results of the sensitivity analysis

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are demonstrated for the obtained solutions. This analysis shows that switching curves of optimal control strategies for the series of the discount parameter values have the convergence property by the parameter. We construct demand functions and provide qualitative analysis for trends of demand curves.

2. DYNAMICS OF THE MODEL

The system of differential equations which defines the dynamics of behavior for two players is investigated

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t), \quad x(t_0) = x_0, \\ \dot{y}(t) &= -y(t) + v(t), \quad y(t_0) = y_0. \end{aligned} \tag{1}$$

The parameter $x = x(t)$, $0 \leq x \leq 1$, means the probability that first player holds to the first strategy (respectively, $(1 - x)$ is the probability that he holds to the second strategy). The parameter $y = y(t)$, $0 \leq y \leq 1$, is the probability of choosing the first strategy by the second player (respectively, $(1-y)$ is the probability that he holds to the second strategy). Control parameters $u = u(t)$ and $v = v(t)$ satisfy conditions $0 \leq u \leq 1$, $0 \leq v \leq 1$, and can be interpreted as signals, that recommend change of strategies by players. For example, value $u = 0$ ($v = 0$) corresponds to the signal: "change the first strategy to the second one". The value $u = 1$ ($v = 1$) corresponds to the signal: "change the second strategy to the first one". The value $u = x$ ($v = y$) corresponds to the signal: "keep the previous strategy".

It is worth to note, that the basis for the dynamics (1) and its properties were examined in papers (see Kryazhimskii and Osipov (1995)). This dynamics generalizes Kolmogorov's differential equations for probabilities of states (see Kolmogorov (1938)). Such generalization assumes that coefficients of incoming and outgoing streams inside coalitions of players are not fixed a priori and can be constructed as positional strategies in the controlled process.

3. LOCAL PAYOFF FUNCTIONS

Let us assume that the payoff of the first player is described my the matrix $A = a_{ij}$, and the payoff of the second player is described by the matrix $B = b_{ij}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Local payoff functions of the players in the time period t, $t \in [t_0, +\infty)$ are determined by the mathematical expectation of payoffs, given by corresponding matrices A and B in the bimatrix game, and can be interpreted as "local" interests of the players

$$g_A(x(t), y(t)) = C_A x(t) y(t) - \alpha_1 x(t) - \alpha_2 y(t) + a_{22},$$

Here parameters C_A, α_1, α_2 are determined according to the classical theory of bimatrix games (see Vorobyev (1985))

$$C_A = a_{11} - a_{12} - a_{21} + a_{22}, \quad D_A = a_{11}a_{22} - a_{12}a_{21},$$
$$\alpha_1 = a_{22} - a_{12}, \quad \alpha_2 = a_{22} - a_{21},$$

Payoff g_B and parameters C_B, β_1, β_2 are determined analogously.

4. DYNAMIC NASH EQUILIBRIUM

In this section we consider the non-zero sum differential game for two players with discounted payoff functionals $JD_A^{\infty} = [JD_A^-, JD_A^+]$

$$\begin{aligned} JD_A^- &= \liminf_{T \to \infty} \int_{t_0}^{T} e^{-\lambda(t-t_0)} g_A(x(t), y(t))\, dt, \\ JD_A^+ &= \limsup_{T \to \infty} \int_{t_0}^{T} e^{-\lambda(t-t_0)} g_A(x(t), y(t))\, dt, \end{aligned} \tag{2}$$

defined on the trajectories $(x(\cdot), y(\cdot))$ of the system (1).

Payoff functionals of the second player JD_B^{∞}, JD_B^-, JD_B^+ are determined analogously by replacement of the function $g_A(x, y)$ by the function $g_B(x, y)$.

Let us introduce the notion of dynamic Nash equilibrium for the evolutionary game with the dynamics (1) and discounted payoff functionals (2) in the context of constructions of non-antagonistic positional differential games (see Kleimenov (1993); Krasovskii and Subbotin (1988); Kryazhimskii and Osipov (1995)). Let us define the dynamic Nash equilibrium in the class of positional strategies (feedbacks) $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$.

Definition 1. The dynamic Nash equilibria (U^0, V^0), $U^0 = u^0(t, x, y, \varepsilon)$, $V^0 = v^0(t, x, y, \varepsilon)$ from the class of controls by the feedback principle $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ for the given problem is determined by inequalities

$$JD_A^-(x^0(\cdot), y^0(\cdot)) \geq JD_A^+(x_1(\cdot), y_1(\cdot)) - \varepsilon,$$
$$JD_B^-(x^0(\cdot), y^0(\cdot)) \geq JD_B^+(x_2(\cdot), y_2(\cdot)) - \varepsilon,$$
$$(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U^0, V^0),$$
$$(x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V^0),$$
$$(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U^0, V).$$

Here symbol X stands for the set of trajectories, starting from initial point and generated by corresponding postional strategies is the sense of the paper (see Krasovskii and Subbotin (1988)).

5. AUXILIARY ZERO-SUM GAMES

For the construction of desired equilibrium feedbacks U^0, V^0 we use the approach (see Kleimenov (1993)). In accordance with this approach we construct the equilibrium using optimal feedbacks for differential games $\Gamma_A = \Gamma_A^- \cup \Gamma_A^+$ and $\Gamma_B = \Gamma_B^- \cup \Gamma_B^+$ with payoffs JD_A^{∞} JD_B^{∞} (2). In the gamed Γ_A the first player maximizes the functional $JD_A^-(x(\cdot), y(\cdot))$ with the guarantee using the feedback $U = u(t, x, y, \varepsilon)$, and the second player oppositely provides the minimization of the functional $JD_A^+(x(\cdot), y(\cdot))$ using the feedback $V = v(t, x, y, \varepsilon)$. Vice versa, in the game Γ_B the second player maximizes the functional $JD_B^-(x(\cdot), y(\cdot))$ with the guarantee, and the first player maximizes the functional $JD_B^+(x(\cdot), y(\cdot))$.

Let us introduce following notations. By $u_A^0 = u_A^0(t, x, y, \varepsilon)$ and $v_B^0 = v_B^0(t, x, y, \varepsilon)$ we denote feedbacks that solve the problem of guaranteed maximization for payoff functionals JD_A^- and JD_B^- respectively. Let us note, that these feedbacks represent the guaranteed maximization of players' payoffs in the long run and can be named "positive". By $u_B^0 = u_B^0(t,x,y,\varepsilon)$ and $v_A^0 = v_A^0(t,x,y,\varepsilon)$ we denote feedbacks mostly favorable for opposite players, namely, those, that minimize payoff functionals JD_B^+, JD_A^+ of the opposite players. Let us call them "punishing".

Let us note, that inflexible solutions of selected problems can be obtained in the framework of the classical bimatrix games theory. Let us propose for definiteness, (this proposition is given for illustration without loss of generality for the solution), that the following relations corresponding to the almost antagonistic structure of bimatrix game hold for the parameters of matrices A and B, $C_A > 0$, $C_B < 0$,

$$0 < x_A = \alpha_2/C_A < 1, \quad 0 < x_B = \beta_2/C_B < 1,$$
$$0 < y_A = \alpha_1/C_A < 1, \quad 0 < y_B = \beta_1/C_B < 1.$$

The following proposition is valid.

Lemma 1. Pairs of differential games Γ_A^-, Γ_A^+ and Γ_B^-, Γ_B^+ have equal values $w_A^- = w_A^+ = w_A = D_A/C_A$, $w_B^- = w_B^+ = w_B = D_B/C_B$.

The proof of this proposition can me obtained by the direct substitution of static strategies related to switching lines $x = x_A$, $y = y_A$ and $x = x_B$, $y = y_B$ to corresponding payoff functionals (2).

Remark 1. Values of payoff functions $g_A(x,y)$, $g_B(x,y)$ coincide at points (x_A, y_B), (x_B, y_A)

$$g_A(x_A, y_B) = g_A(x_B, y_A) = w_A,$$
$$g_B(x_A, y_B) = g_B(x_B, y_A) = w_B.$$

The point $NE = (x_B, y_A)$ is the "mutually punishing" Nash equilibrium, and the point (x_A, y_B) does not possess equilibrium properties in the corresponding static game.

6. CONSTRUCTION OF DYNAMIC EQUILIBRIUM

Let us construct the pair of feedbacks, which consist the Nash equilibrium. For this let us combine "positive" feedbacks u_A^0, v_B^0 and "punishing" feedbacks u_B^0, v_A^0.

Let us choose the initial position $(x_0, y_0) \in [0,1] \times [0,1]$ and accuracy parameter $\varepsilon > 0$. Let us choose the trajectory $(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U_A^0(\cdot), v_B^0(\cdot))$, generated by "positive" $u_A^0 = U_A^0(t,x,y,\varepsilon)$ and $v_B^0 = v_B^0(t,x,y,\varepsilon)$. Let us choose $T_\varepsilon > 0$ such that for $t \in [T_\varepsilon, +\infty]$

$$g_A(x^0(t), y^0(t)) > JD_A^- - \varepsilon, \quad g_B(x^0(t), y^0(t)) > JD_B^- - \varepsilon.$$

Let us denote by $u_A^\varepsilon(t)$: $[0, T_\varepsilon) \to [0,1]$, $v_B^\varepsilon(t)$: $[0, T_\varepsilon) \to [0,1]$ step-by-step implementation of strategies u_A^0, v_B^0 such that the corresponding step-by-step trajectory $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ satisfies the condition

$$\max_{t \in [0, T_\varepsilon]} \|(x^0(t), y^0(t)) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon.$$

From the results of the paper Kleimenov (1993) the next proposition follows.

Lemma 2. The pair of feedbacks $U^0 = u^0(t,x,y,\varepsilon)$, $V^0 = v^0(t,x,y,\varepsilon)$, combines together "positive" feedbacks u_A^0, v_B^0 and "punishing" feedbacks u_B^0, v_A^0 according to relations

$$U^0 = u^0(t,x,y,\varepsilon) = \begin{cases} u_A^\varepsilon(t), & \|(x,y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon, \\ u_B^0(x,y), & \textit{otherwise}, \end{cases}$$

$$V^0 = v^0(t,x,y,\varepsilon) = \begin{cases} v_B^\varepsilon(t), & \|(x,y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon, \\ v_A^0(x,y), & \textit{otherwise} \end{cases}$$

is the dynamic ε-Nash equilibrium.

Below we construct flexible "positive" feedbacks that generate trajectories $(x^{fl}(\cdot), y^{fl}(\cdot))$, which reduce to "better" positions than the inflexible dynamic equilibrium (x_B, y_A), (x_A, y_B) by both criteria $JD_A^\infty(x^{fl}(\cdot), y^{fl}(\cdot)) \geq v_A$, $JD_B^\infty(x^{fl}(\cdot), y^{fl}(\cdot)) \geq v_B$.

7. TWO-STEP OPTIMAL CONTROL PROBLEMS

For the construction of "positive" feedbacks $u_A^0 = u_A^{fl}(x,y)$, $v_B^0 = v_B^{fl}(x,y)$ we consider in this section the auxiliary two-step optimal control problem with discounted payoff functional for the first player in the situation, when actions of the second player are most unfavorable. Namely, let us analyze the optimal control problem for the dynamic system (1) with the payoff functional

$$JD_A^f = \int_0^{T_f} e^{-\lambda t} g_A(x(t), y(t))\, dt. \tag{3}$$

Here without loss of generality let us consider that $t_0 = 0$, $T = T_f$, and terminal moment of time $T_f = T_f(x_0, y_0)$ we determine later. Without loss of generality, we assume that the value of the static game equals to zero, $w_A = D_A/C_A = 0$.

Let us consider the case when initial conditions (x_0, y_0) of the system (1) satisfy relations $x_0 = x_A$, $y_0 > y_A$. Let us assume that actions of the second player are mostly unfavorable for the first player. For trajectories of the system (1), which start from initial positions (x_0, y_0), these actions $v_A^0 = v_A^{cl}(x,y)$ are determined by the relation $v_A^{cl}(x,y) \equiv 0$.

Optimal actions $u_A^0 = u_A^{fl}(x,y)$ of the first player according to the payoff functional JD_A^f (3) in this situation can be presented as the two-step impulse control: it equals one from the initial time moment $t_0 = 0$ till the moment of optimal switch s and then equals to zero till the final time moment T_f

$$u_A^0(t) = u_A^{fl}(x(t), y(t)) = \begin{cases} 1, & t_0 \leq t < s, \\ 0, & s \leq t < T_f. \end{cases}$$

Here the parameter s is the optimization parameter. The final time moment T_f is determined by the following condition. The trajectory $(x(\cdot), y(\cdot))$ of the system (1), which starts from the line where $x(t_0) = x_A$, returns to this line when $x(T_f) = x_A$.

Let us consider two aggregates of characteristics. The first one is described by the system of differential equations with the value on the control parameter $u = 1$

$$\dot{x}(t) = -x(t) + 1, \quad \dot{y}(t) = -y(t), \tag{4}$$

solutions of which are determined by the Cauchy formula

$$x(t) = (x_0 - 1)e^{-t} + 1, \quad y(t) = y_0 e^{-t}, \quad 0 \leq t < s. \tag{5}$$

The second aggregate of characteristics is given by the system of differential equations with the value of the control parameter $u = 0$

$$\dot{x}(t) = -x(t), \quad \dot{y}(t) = -y(t), \tag{6}$$

solutions of which are determined by the Cauchy formula

$$x(t) = x_1 e^{-t}, \quad y(t) = y_1 e^{-t}. \tag{7}$$

Here initial positions $(x_1, y_1) = (x_1(s), y_1(s))$ are determined by relations

$$x_1 = (x_0 - 1)e^{-s} + 1, \quad y_1 = y_0 e^{-s}, \quad 0 \le t < p \tag{8}$$

Here the final time moment $p = p(s)$ and the final position $(x_2, y_2) = (x_2(s), y_2(s))$ of the whole trajectory $(x(\cdot), y(\cdot))$ is given by formulas

$x_1 e^{-p} = x_A, p = p(s) = \ln x_1(s)/x_A, x_2 = x_A, y_2 = y_1 e^{-p}$.

The optimal control problem is to find such moment of time s and the corresponding switching point $(x_1, y_1) = (x_1(s), y_1(s))$ on the trajectory $(x(\cdot), y(\cdot))$, where the integral $I = I(s) = I_1(s) + I_2(s)$ reaches the maximum value

$$I_1(s) = \int_0^s e^{-\lambda t}(C_A((x_0 - 1)e^{-t} + 1)y_0 e^{-t} - \alpha_1((x_0 - 1)e^{-t} + 1) - \alpha_2 y_0 e^{-t} + a_{22})\, dt,$$

$$I_2(s) = e^{-\lambda s} \int_0^{p(s)} e^{-\lambda t}(C_A x_1(s) y_1(s) e^{-2t} - \alpha_1 x_1(s) e^{-t} - \alpha_2 y_1(s) e^{-t} + a_{22})\, dt. \tag{9}$$

8. SOLUTION OF OPTIMAL CONTROL PROBLEM

We obtain the solution of the two-step optimal control problem (4)-(9), by calculating the derivative dI/ds, presenting it as the function of optimal switching points $(x, y) = (x_1, y_1)$, equating this derivative to zero $dI/ds = 0$ and finding the equation $F(x, y) = 0$ for the curve, that consist from optimal switching points (x, y).

Sufficient maximum conditions in such construction are obtained from the fact that the integral $I(s)$ has the property of monotonic increase by the variable s in the initial period, because the integrand $g_A(x, y)$ is positive, $g_A(x, y) > w_A = 0$, in the domain $x > x_A$, $y > y_A$. In the finite period the integral $I(s)$ strictly monotonically decreases by the variable s, because the integrand $g_A(x, y)$ is negative, $g_A(x, y) < w_A = 0$, when $x > x_A$, $y < y_A$.

Let us calculate integrals I_1, I_2, derivatives $dI_1/ds, dI_2/ds$, summarize derivatives dI_1/ds and dI_2/ds, equalize the expression to zero, express y by x and obtain the final expression for the switching curve $M_A^1(\lambda)$:

$$M_A^1(\lambda) = \frac{(\lambda + 2)\left(x^{(\lambda+1)} - x_A^{(\lambda+1)}\right) y_A x}{(\lambda + 1)\left(x^{(\lambda+2)} - x_A^{(\lambda+2)}\right)}. \tag{10}$$

To construct the final switching curve $M_A(\lambda)$ for the optimal strategy of the first player in the game with the discounted functional in the case $C_A > 0$, we add to the curve $M_A^1(\lambda)$ the similar curve $M_A^2(\lambda)$ in the domain, where $x \le y_A \;\; y \le y_A$.

The curve $M_A(\lambda)$ divides the unit square $[0, 1] \times [0, 1]$ into two parts: the upper part $D_A^u \supset \{(x, y) : x = x_A, y > y_A\}$, and the lower part $D_A^l \supset \{(x, y) : x = x_A, y < y_A\}$.

The "positive" feedback u_A^{fl} has the following structure

$$u_A^{fl} = \begin{cases} \max\{0, -sgn(C_A)\}, & if\ (x, y) \in D_A^u, \\ \max\{0, sgn(C_A)\}, & if\ (x, y) \in D_A^l, \\ [0, 1], & if\ (x, y) \in M_A(\lambda). \end{cases} \tag{11}$$

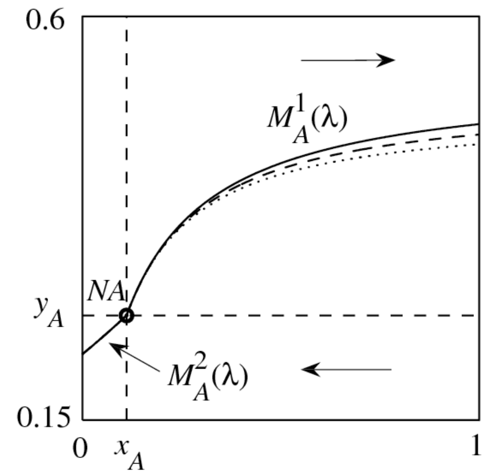

Fig. 1. Switching curves $M_A^1(\lambda)$, $M_A^2(\lambda)$.

In the case when $C_A < 0$, curves $M_A(\lambda)$, $M_A^1(\lambda)$ and $M_A^2(\lambda)$ are constructed analogously

On Fig. 1 we show switching curves $M_A^1(\lambda)$, $M_A^2(\lambda)$ for the first player. Directions of velocities $\dot{x}$ are depicted by horizontal (left and right) arrows.

For the second player one can get similar switching curves $M_B(\lambda)$ for the optimal control problem with the discounted functional, corresponding to the matrix B.

Remark 2. Let us note that in papers by V. I. Arnold Arnold (2002) average integral payoff functionals were considered

$$\frac{1}{(T - t_0)} \int_{t_0}^{T} g_A(x(t), y(t))\, dt. \tag{12}$$

In the paper (see Krasovskii and Tarasyev (2016)) switching curves for optimal control strategies of players in the game with average integral functionals were obtained. For example, for the first player in the case when $C_A > 0$ the switching curve in the domain $x \ge x_A$, $y \ge y_A$ is described by relations

$$y = 2\alpha_1 x/(C_A x + \alpha_2). \tag{13}$$

The asymptotical analysis of solutions (10) for the game with discounted payoff functionals shows, that according to L'Hospital's rule, when the discount parameter λ tends to zero, the relation for switching curves (10) of the control strategy for the first player converges to switching curves (13) in the game with average payoff functionals (12).

9. GUARANTEED VALUES OF PAYOFFS

Let us formulate the proposition, which confirms, that the "positive" optimal control by the feedback principle $u_A^{fl}(x, y)$ (11) with the switching curve M_A, defined by formulas (10), guarantee that the value of discounted payoff functionals is more or equal than the value w_A of the static matrix game.

Theorem 1. For any initial position $(x_0, y_0) \in [0, 1] \times [0, 1]$ and for any trajectory $(x^{fl}(\cdot), y^{fl}(\cdot)) \in X(x_0, y_0, u_A^{fl})$, $x^{fl}(t_0) = x_0$, $y^{fl}(t_0) = y_0$, $t_0 = 0$, generated by the optimal control by the feedback principle $u_A^{fl} = u_A^{fl}(x, y)$ there exists the final moment of time $t_* \in [0, T_A]$ such that in this moment of time the trajectory $(x^{fl}(\cdot), y^{fl}(\cdot))$ reaches the line where $x = x_A$, namely $x^{fl}(t_*) = x_A$. Then, according to the construction of the optimal control, that maximizes the integral (9) by the feedback principle u_A^{fl}, the following estimate holds $\forall T \geq t_*$

$$\int_{t_*}^{T} e^{-\lambda(t-t_*)} g_A(x(t), y(t))\, dt \geq \frac{w_A}{\lambda}\big(1 - e^{-\lambda(T-t_*)}\big). \quad (14)$$

In particular, this inequality remains valid when time T tends to infinity

$$\liminf_{T\to+\infty} \lambda \int_{t_*}^{T} e^{-\lambda(t-t_*)} g_A(x^{fl}(t), y^{fl}(t))\, dt \geq w_A. \quad (15)$$

Inequalities (14), (15) mean, that the value of the discounted functional is not worse, than the value w_A of the static matrix game.

The analogous result is fair for trajectories, which generated by the optimal control v_B^{fl}, that corresponds to the switching curve M_B.

Proof. The result of the theorem follows from the fact that the value of the payoff functional (3) is maximum on the constructed broken line. In particular, it is more or equal, than the value of this functional on the trajectory which stays on the segment $x = x_A$ with the control $u(t) = x_A$. The value of the payoff functional on such trajectory is following

$$\int_{t_*}^{T} e^{-\lambda(t-t_*)} w_A\, dt = \frac{w_A}{\lambda}\big(1 - e^{-\lambda(T-t_*)}\big).$$

These arguments imply the required relation (14), which in the limit transition provides the relation (15). □

Remark 3. Let us consider the acceptable trajectory $(x_{AB}^{fl}(\cdot), y_{AB}^{fl}(\cdot))$, generated by "positive" feedbacks u_A^{fl}, v_B^{fl}. Then according to the Theorem 1, the inequalities for payoffs of both players analogous to relation (15) take place. Hence, the acceptable trajectory $(x_{AB}^{fl}(\cdot), y_{AB}^{fl}(\cdot))$ provides the better result for both players, than trajectories, convergent to points of the static Nash equilibrium, in which payoffs are equal to values w_A and w_B.

10. EQUILIBRIUM TRAJECTORIES

Let us consider payoff matrices of players on the financial market, which reflect the data of investigated markets of stocks and bonds in the USA. The matrix A corresponds to the behavior of traders, which play on increase of the course and are called "bulls". The matrix B corresponds to the behavior of traders, which play on the depreciation of the course and are called "bears". Parameters of matrices represent rate of return for stocks and bonds, expressed in the form of interest rates,

$$A = \begin{pmatrix} 10 & 0 \\ 1.75 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 3 \\ 10 & 0.5 \end{pmatrix}. \quad (16)$$

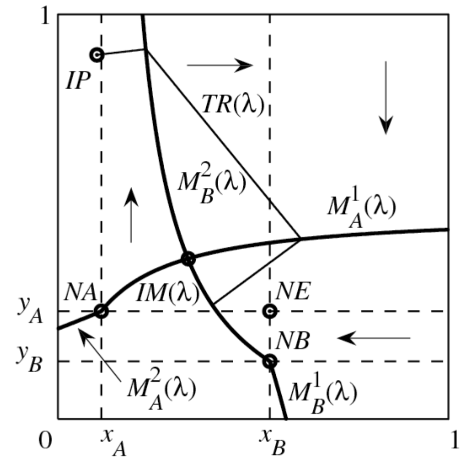

Fig. 2. The equilibrium trajectory in the dynamic game.

Characteristic parameters of static games are given at the following levels (see Vorobyev (1985)):

$$C_A = 11.25, \quad \alpha_1 = 3, \quad \alpha_2 = 1.25,$$
$$x_A = \alpha_2/C_A = 0.11, \quad y_A = \alpha_1/C_A = 0.27;$$
$$C_B = -17.5, \quad \beta_1 = -2.5, \quad \beta_2 = -9.5,$$
$$x_B = \beta_2/C_B = 0.54, \quad y_B = \beta_1/C_B = 0.14.$$

Let us note, that players of the coalition of "bulls" gain in the case of upward trend of markets, when players of both coalitions invest in the same market. And players of the coalition of "bears" make profit from investments in the case of downward trend of markets when players of the coalition of "bulls" move their investments from one market to another.

For the game of coalitions of "bulls" and "bears" we construct switching curves $M_A(\lambda)$, $M_B(\lambda)$ and provide calculations of equilibrium trajectories of the market dynamics with the value of the discount parameter $\lambda = 0.1$.

This calculations are presented on Fig. 2. Here we show saddle points NA, NB in static antagonistic games, the point of the Nash equilibrium NE in the static bimatrix game, switching lines for players' controls $M_A(\lambda) = M_A^1(\lambda) \bigcup M_A^2(\lambda)$ and $M_B(\lambda) = M_B^1(\lambda) \bigcup M_B^2(\lambda)$ in the dynamic bimatrix game with discounted payoff functionals for matrices A, B (16). The field of velocities of players is depicted by arrows.

The field of directions generates equilibrium trajectories, one of which is presented on Fig. 2. This trajectory $TR(\lambda) = (x_{AB}^{fl}(\cdot), y_{AB}^{fl}(\cdot))$ starts from the initial position $IP = (0.1, 0.9)$ and moves along the characteristic in the direction of the vertex $(1, 1)$ of the unit square $[0, 1] \times [0, 1]$ with control signals $u = 1$, $v = 1$. Then it crosses the switching line $M_B(\lambda)$, and the second coalition switches the control v from 1 to 0. Then, the trajectory $TR(\lambda)$ moves in the direction of the vertex $(1, 0)$ until it reaches the switching line $M_A(\lambda)$. Here players of the first coalition change the control signal u from 1 to 0. After that the movement of the trajectory is directed along the characteristic to the the vertex $(0, 0)$. Then the trajectory crosses the line $M_B(\lambda)$, on which the sliding mode arises, during which the switch of controls of the second coalition occurs, and the trajectory $TR(\lambda)$ converge to the point

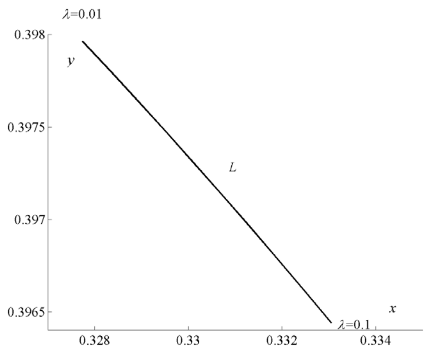

Fig. 3. Demand curve L of equilibrium points.

$IM(\lambda) = M_A(\lambda) \bigcap M_B(\lambda)$ of the intersection of switching lines $M_A(\lambda)$, $M_B(\lambda)$.

11. DEMAND FUNCTIONS

Demand functions play an important role in the research of sensitivity of mathematical models in economics. They determine an equilibrium value of the demand for stocks and bonds, depending on the value of the discount parameter λ (capital borrowing price). These functions are also necessary to study the coefficients of elasticity of demand, the classification of assets and to determine the macroeconomic price equilibrium.

Demand functions are constructed numerically in the framework of the considered approach. More precisely, for the range of values of the discount parameter λ from 0.01 to 0.1 with the step of 0.01, the control switching curves, equilibrium trajectories, the points of their intersection are constructed and values of payoff functionals in these equilibrium points are calculated.

On Fig. 3 we present the curve L for the dependence of intersection points of equilibrium trajectories (points where the dynamic game comes to equilibrium) on the change of the discount parameter λ. The upper point of this curve corresponds to the value of the discount parameter $\lambda = 0.01$, and the lower point corresponds to the value $\lambda = 0.1$. Calculations show that with the increase of the discount parameter λ, the stocks demand for "bulls" and "bears" has divergent trends.

Table 1. Equilibrium Payoff Values

λ	g_A	g_B
0.0100	2.9867	2.8175
0.0200	2.9871	2.8142
...	...	...
0.0900	2.9903	2.7915
0.1000	2.9907	2.7882

In the Table 1 we list values of payoffs g_A and g_B on the dependence curve of equilibrium solutions. It shows that the value of payoff g_A measured in the interest rate has the growing trend with respect to the discount parameter λ. Vice versa, the value of payoff g_B declines with growth of the discount parameter λ. However, the range of variation of interest rates is rather small and amounts to 0.01 for g_A and 0.03 for g_B. It is interesting to note that all variations are around 2.9885 for g_A and 2.8045 for g_B. It means that at equilibrium interest rates of players, both "bulls" and "bears" are approximately equal and constitute the value of 2.9 of the interest rate, despite the fact that "bulls" and "bears" play qualitatively different strategies on the financial markets of stocks and bonds.

12. CONCLUSION

In the paper, equilibrium solutions for dynamic bimatrix games with discounted integral payoffs are constructed on the infinite time horizon. Equilibrium trajectories describing behavior of two players on financial markets are presented. Demand functions are constructed and qualitative analysis for trends of demand curves is given.

REFERENCES

Arnold, V.I. (2002). Optimization in mean and phase transitions in controlled dynamical systems. *Functional Analysis and Its Applications*, 36–2, 1–11.

Basar, T. and Olsder, G.J. (1982). *Dynamic Noncooperative Game Theory*. Academic Press, London.

Klaassen, G., Kryazhimskii, A.V., and Tarasyev, A.M. (2004). Multiequilibrium game of timing and competition of gas pipeline projects. *Journal of Optimization Theory and Applications*, 120–1, 147–179.

Kleimenov, A. (1993). *Nonantagonistic Positional Differential Games*. Nauka, Yekaterinburg.

Kolmogorov, A.N. (1938). On analytical methods in probability theory. *Uspekhi Mat. Nauk*, 5, 5–41.

Krasovskii, A.A. and Taras'ev, A.M. (2007). Dynamic optimization of investments in the economic growth models. *Automat. Remote Control*, 68–10, 1765–1777.

Krasovskii, A.N. and Krasovskii, N.N. (1995). *Control Under Lack of Information*. Birkhauser, Boston etc.

Krasovskii, N.A. and Tarasyev, A.M. (2015). Decomposition algorithm of searching equilibria in a dynamic game. *Automat. Remote Control*, 76–10, 1865–1893.

Krasovskii, N.A. and Tarasyev, A.M. (2016). Equilibrium trajectories in dynamical bimatrix games with average integral payoff functionals. *Mathematical Game Theory and its Applications*, 8–2, 58–90.

Krasovskii, N.N. and Subbotin, A.I. (1988). *Game–Theoretical Control Problems*. Springer, New York.

Kryazhimskii, A.V. and Osipov, Y.S. (1995). On differential–evolutionary games. *Proceedings of the Steklov Institute of Mathematics*, 211, 234–261.

Kurzhanskii, A.B. (1977). *Control and Observation Under Uncertainty*. Nauka, Moscow.

Petrosjan, L.A. and Zenkevich, N.A. (2015). Conditions for sustainable cooperation. *Automation and Remote Control*, 76–10, 1894–1904.

Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, V.G., and Mischenko, E.F. (1962). *The Mathematical Theory of Optimal Processes*. Intersci. Publishers, New York.

Subbotin, A.I. (1991). *Minimax Inequalities and Hamilton–Jacobi Equations*. Nauka, Moscow.

Subbotin, A.I. and Tarasyev, A.M. (1985). Conjugate derivatives of the value function of a differetnial game. *Doklady AN SSSR*, 283–3, 559–564.

Vorobyev, N.N. (1985). *Game Theory for Economists and System Scientists*. Nauka, Moscow.