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Target problem for mean field type differential game ⋆

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Abstract: The paper is concerned with the mean field type differential game that describes the behavior of the large number of similar agents governed by the unique decision maker and influenced by disturbances. It is assumed that the decision maker wishes to bring the distribution of agents onto the target set in the space of probabilities within the state constraints. The solution of this problem is obtained based on the notions u- and v-stability first introduced for the finite dimensional differential games.

Keywords: Differential games, feedback control, stable sets, distributed feedback, mean field type differential game.

1. INTRODUCTION

In this note we examine the mean field type control system governed by two decision makers with the opposite goals. Such systems can be called mean field type differential games. The consider the target problem. That means that one player (for definiteness, the first player) wishes to bring the system to the target set within state constraint; whereas the second player tries to prevent this. Notice that for the mean field type differential games the target set is a subset of the product of the time interval and the space of probabilities.

Previously, the target problem was studied for the case of finite-dimensional differential games. For this type of games under some conditions it is proved that either first player can solve target problem within the feedback strategies or the second player can prevent the approach using also feedback strategies see Krasovskii and Subbotin (1988). Notice that the most important condition necessary for this result is so called Isaacs' condition which is satisfied for example when the dynamics of the system is split to the sum of two dynamics governed by the first and the second players independently.

The mean field type control system describes the cooperative behavior of the large number of similar agents with mean field interaction. It was first consider in Ahmed and Ding (2001). Nowadays, the mean field field control system are well studied. The approach going back to the maximum principle was considered in Andersson and Djehiche (2011), Buckdahn et al. (2011), Carmona and Delarue (2013); the dynamic programming for mean field type control system was developed in Bayraktar et al. (2018), Bensoussan et al. (2015), Laurière and Pironneau (2014)). Case when the dynamics is given by deterministic evolution was considered in Cavagnari and Marigonda (2015), Cavagnari et al. (2017), Pogodaev (2016).

The mean field type differential game appears naturally when we examine the mean field type control system under assumption that the agents are influenced by the exogenous disturbance. They were studied in Djehiche and Hamadène (2016), Cosso and Pham (2018), Averboukh (2018). In Cosso and Pham (2018) the mean field type differential game with the dynamics described by SDE were considered in the class of nonanticipative strategies. In Averboukh (2018) the feedback approach for the case of deterministic evolution was studied. In that paper it is assumed that the quality of control is described by the payoff functional, whereas in the present paper we study the target problem.

2. PROBLEM STATEMENT

We examine the target problem on the finite time interval for the mean field type control system governed by two players. It is assumed that, given the flow of probabilities describing the distribution of agents $m(t)$, the motion of each agent is given by the controlled differential equation

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u, v), \qquad t \in [0, T], \quad x(t) \in \mathbb{R}^d, \quad u \in U, \quad v \in V. \tag{1}$$

Here u (respectively, v) is a control of the first (respectively, second) player. The sets U and V are control spaces for the first and the second players respectively. Integrating formally (1) we get that the motion of the flow of probabilities $m(\cdot)$ obeys the following equation:

$$\frac{d}{dt}m(t) = \langle f(t, \cdot, m(t), u, v), \nabla\rangle m(t).$$

In this equation, for each $t \in [0, T]$, $m(t)$ should be considered as functional from $C_b^1(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$.

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We assume that the first player wishes to aim the system to the set M subject to the constrains given by the set N. The goal of the second player is opposite. We assume that both M and N lie in the product of $[0,T]$ and the set of probabilities on phase space. Moreover, it is natural to assume that $M \subset N$.

3. GENERAL NOTATION AND STANDING ASSUMPTIONS

If (Ω', Σ') and (Ω'', Σ'') are measurable spaces, m is a probability on Σ', $h : \Omega' \to \Omega''$ is measurable, then denote by $h_{\#}m$ the probability on Σ'' given by the following rule: for $\Upsilon \in \Sigma''$,

$$(h_{\#}m)(\Upsilon) \triangleq m(h^{-1}(\Upsilon)).$$

Further, we restrict our attention to the Borel probabilities defined on the separable metric space satisfying the Radon property (X, ρ_X). Denote the set of probabilities on (X, ρ_X) by $\mathcal{P}(X)$ the set of Borel probabilities on (X, ρ_X). t is natural to endow $\mathcal{P}(X)$ by the topology of the narrow convergence. It is metrizable. Further, let $\mathcal{P}^2(X)$ denote the set of probabilities $m \in \mathcal{P}(X)$ such that, for some (and, thus, any) $x_* \in X$,

$$\int_X \rho_X(x, x_*)^2 m(dx) < \infty.$$

We consider on $\mathcal{P}^2(X)$ the 2-Wasserstein metric defined by the rule: if $m_1, m_2 \in \mathcal{P}(X)$, then

$$W_2(m_1, m_2) \triangleq \inf_{\pi \in \Pi(m_1,m_2)} \left[\int_{X\times X} \rho_X(x_1, x_2)^2 \pi(dx_1, dx_2) \right]^{1/2}.$$

Here $\Pi(m_1, m_2)$ the sets of probabilities on $X \times X$ with the marginal distributions equal to m_1, m_2. Note that $\mathcal{P}^2(X)$ is a Polish space when X is Polish. Additionally, if X is compact, then the space $\mathcal{P}(X)$ is also compact; the sets $\mathcal{P}(X)$ and $\mathcal{P}^2(X)$ coincide and W_2 metricize the narrow convergence and $\mathcal{P}^2(X)$ is compact.

If (X, ρ_X), (Y, ρ_Y) are separable metric spaces satisfying the Radon property, then denote by $\mathrm{WM}(X, Y)$ the set of weakly measurable functions from X to $\mathcal{P}(Y)$. If $b \in \mathrm{WM}(X, Y)$, then let $m \star b$ be a measure on $X \times Y$ defined by the rule: for $\varphi \in C_b(X \times Y)$,

$$\int_{X\times Y} \varphi(x, y)(m \star b)(dx, dy) \triangleq \int_X \int_Y \varphi(x, y) b(x, dy) m(dx).$$

Hereinafter, we write $b(x, dy)$ instead of $b(x)(dy)$. If (Z, ρ_Z) is also a separable metric space satisfying the Radon property, $\xi \in \mathcal{P}(Y)$, $\zeta \in \mathcal{P}(Z)$, then denote by $\xi\zeta$ the product of probabilities, i.e., $\xi\zeta$ is a probability on $Y \times Z$ defined by the rule: for $\phi \in C_b(Y \times Z)$,

$$\int_{Y\times Z} \phi(y, z)(\xi\zeta)(d(y, z)) \triangleq \int_Y \int_Z \phi(y, z)\xi(dy)\zeta(dz).$$

Further, if $b \in \mathrm{WM}(X, Y)$, $c \in \mathrm{WM}(X, Z)$, then denote by bc the element of $\mathrm{WM}(X, Y \times Y)$ defined by

$$(bc)(x, d(y, z)) \triangleq b(x, dy)c(x, dz).$$

Further, if m is a finite measure on X, then $\Lambda(X, m, Y)$ is a quotient space of $\mathrm{WM}(X, Y)$ under relation given by m-a.e. coincidence. We assume the narrow convergence on $\Lambda(X, m, Y)$ i.e. $b_n \to b$ iff $m \star b_n \to m \star b$ in the narrow sense. If X and Y are compact, then $\Lambda(X, m, Y)$ is also compact.

If $\pi \in \mathcal{P}(X \times Y)$, then denote by $\pi(\cdot|x)$ (respectively, $\pi(\cdot|y)$) its disintegration with respect to marginal distribution on X (respectively, Y).

To simplify notations put $\mathcal{C} \triangleq C([0, T], \mathbb{R}^d)$. Let e_t stand for the evaluation operator defined by the rule: for $x(\cdot) \in \mathcal{C}$,

$$e_t(x(\cdot)) \triangleq x(t).$$

Obviously, if $\chi_1, \chi_2 \in \mathcal{P}^2(\mathcal{C})$, then

$$W_2(e_{t\#}\chi_1, e_{t\#}\chi_2) \le W_2(\chi_1, \chi_2). \quad (2)$$

Further, let $\mathcal{M}$ denote the set of all continuous functions defined on $[0, T]$ with values in $\mathcal{P}^2(\mathbb{R}^d)$.

If $\mathcal{W}$ is a subset of $[0, T] \times \mathcal{P}^2(\mathbb{R}^d)$, $t \in [0, T]$, then denote

$$\mathcal{W}[t] \triangleq \{m \in \mathcal{P}^2(\mathbb{R}^d) : (t, m) \in \mathcal{W}\}.$$

Below, let dist stand for the distance between the point and the set.

We assume that

- the sets U and V are compact subsets of a metric space;
- the function f is continuous;
- the function f is bounded i.e. there exists $R > 0$ such that, for any $t \in [0, T]$, $x \in \mathbb{R}^d$, $u \in U$, $v \in V$,
$$\|f(t, x, u, v)\| \le R;$$
- the function f is Lipschitz continuous w.r.t. x and m i.e. there exists a constant L such that, for any $t \in [0, T]$, $x', x'' \in \mathbb{R}^d$, $m', m'' \in \mathcal{P}^2(\mathbb{R}^d)$, $u \in U$, $v \in V$,
$$\|f(t, x', m', u, v) - f(t, x'', m'', u, v)\| \le L\|x' - x''\| + LW_2(m', m'');$$
- there exists an even function $\varpi : \mathbb{R} \to [0, +\infty)$ continuous and vanishing at 0 such that
$$\|f(t', x, m, u, v) - f(t', x, m, u, v)\| \le \varpi(t' - t'');$$
- (Isaacs' condition) for any $t \in [0, T]$, $x, w \in \mathbb{R}^d$, $m \in \mathcal{P}^2(\mathbb{R}^d)$,
$$\min_{u\in U} \max_{v\in V} \langle w, f(t, x, m, u, v)\rangle = \max_{v\in V} \min_{u\in U} \langle w, f(t, x, m, u, v)\rangle.$$
- the sets M and N are compact;
- there exists a compact set $E \subset \mathbb{R}^d$ such that
$$M \subset N \subset [0, T] \times \mathcal{P}^2(E).$$

4. STRATEGIES AND MOTIONS

Let $\mathcal{U}^0$ (respectively, $\mathcal{V}^0$) denote the set of measurable functions from $[0, T]$ to U (respectively, V). The elements of the sets $\mathcal{U}^0$ and $\mathcal{V}^0$ are (usual) control of the first and the second players respectively.

It is convenient to introduce the relaxed controls. Set

$$\mathcal{U} \triangleq \Lambda([0, T], \lambda, U), \quad \mathcal{V} \triangleq \Lambda([0, T], \lambda, V).$$

Hereinafter λ denotes the Lebesgue measure. An element of $\mathcal{U}$ (respectively, $\mathcal{V}$) is a function from $[0, T]$ to $\mathcal{P}(U)$ (respectively, $\mathcal{P}(V)$).

Without loss of generality, one can assume that

$$U \subset \mathcal{U}^0 \subset \mathcal{U}, \quad V \subset \mathcal{V}^0 \subset \mathcal{V}.$$

If $s \in [0,T]$, $y \in \mathbb{R}^d$, $m(\cdot)$ $\xi \in \mathcal{U}$, $\zeta \in \mathcal{V}$, then denote by $x(\cdot, s, y, m(\cdot), \xi, \zeta)$ the solution of the initial value problem

$$\frac{d}{dt}x(t) = \int_U \int_V f(t, x, m(t), u, v)\xi(t, du)\zeta(t, dv),$$
$$x(s) = y.$$

It is not difficult to prove the uniqueness and existence theorem for this problem.

Further, let $\mathrm{traj}^s_{m(\cdot)}$ stand for the operator which assigns to y, ξ and ζ the motion $x(\cdot, s, y, m(\cdot), \xi, \zeta)$. Notice that traj maps $\mathbb{R}^d \times \mathcal{U} \times \mathcal{V}$ to $\mathcal{C}$.

We assume that the player can influence on each agent and his/her strategy is a function of time and current distribution of agents. Since the agents are similar it is reasonable to assume that the strategy is a distribution of controls on $\mathbb{R}^d$. We will consider the case when the player chooses the constant control on a short time interval. Denote the set of distributions of the first player's constant controls by $\mathcal{A}^c$ and the set distributions of the second player's constant controls by $\mathcal{A}^c$ by $\mathcal{B}^c$ i.e.

$$\mathcal{A}^c \triangleq \mathrm{WM}(\mathbb{R}^d, U), \quad \mathcal{B}^c \triangleq \mathrm{WM}(\mathbb{R}^d, V).$$

Further, we will consider the case when the players choose the distribution of relaxed controls. Put

$$\mathcal{A} \triangleq \mathrm{WM}(\mathbb{R}^d, \mathcal{U}), \quad \mathcal{B} \triangleq \mathrm{WM}(\mathbb{R}^d, \mathcal{V}).$$

Notice that $\mathcal{A}$ (respective, $\mathcal{B}$) is the set of distribution of relaxed control of the first (respectively, second) player.

In the following the distributions of pairs of control will play the crucial role. Set $\mathcal{D} \triangleq \mathrm{WM}(\mathbb{T}^d, \mathcal{U} \times \mathcal{V})$. Further, we will consider the distributions of players' control consistent with the distribution of controls of the first (respectively, second) player. Namely, if $\alpha \in \mathcal{A}$, then denote by $\mathcal{D}_1[\alpha]$ the set of distributions of pairs of controls $\varkappa \in \mathcal{D}$ such that, for each $x \in \mathbb{T}^d$, marginal distribution of $\varkappa(x)$ on $\mathcal{U}$ is $\alpha(x)$. Further, denote by $\mathcal{D}_1^0[\alpha]$ the set of distributions $\varkappa \in \mathcal{D}_1[\alpha]$ such that $\varkappa(x)$ is concentrated on $\mathcal{U} \times \mathcal{V}^0$. Analogously, given $\beta \in \mathcal{B}$, let $\mathcal{D}_1[\beta]$ (respectively, $\mathcal{D}_1^0[\beta]$) stand for the distribution of players' controls $\varkappa \in \mathcal{D} = \mathrm{WM}(\mathbb{T}^d, \mathcal{U} \times \mathcal{V})$ (respectively, $\varkappa \in \mathrm{WM}(\mathbb{T}^d, \mathcal{U}^0 \times \mathcal{V})$) such that, for any $x \in \mathbb{T}^d$, the projection of $\varkappa(x)$ on $\mathcal{V}$ is $\beta(x)$.

Let $t_0 \in [0,T]$, $m_0 \in \mathcal{P}^2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$. We say that the flow of probabilities $m(\cdot) \in \mathcal{M}$ is generated by t_0, m_0, and $\varkappa \in \mathcal{D}$ if there exists a probability $\chi \in \mathcal{C}$ such that

- $m(t) = e_{t\#}\chi$;
- $\chi = \mathrm{traj}^{t_0}_{m(\cdot)\#}(m_0 \star \varkappa)$.

Below we denote the flow of probabilities generated by t_0, m_0, α and β by $m(\cdot, t_0, m_0, \varkappa)$.

One can prove the existence and uniqueness theorem for the flow of probabilities $m(\cdot, t_0, m_0, \varkappa)$.

Now let us introduce the feedback formalization. As it was mentioned above we assume that the feedback strategy of the first player is a distribution of constant controls i.e. we say that a function $\mathfrak{u}$ from $[0,T] \times \mathcal{P}^2(\mathbb{R}^d)$ to $\mathcal{A}^c$ is a feedback strategy of the first player. It is reasonable to assume that if the first player uses a feedback strategy, then the second player can use a distribution of measurable controls. We also assume that the first player corrects his/her control in the finite number of time instants. Thus, we say that the motion $m^1(\cdot)$ is generated by the initial time t_0, initial distribution of agents m_0, strategy of the first player $\mathfrak{u}$ and a partition of time interval $\Delta = \{t_i\}_{i=0}^n$ if there exist $\varkappa_i$, $i = 1, \ldots, n-1$ such that

- $m(t_0) = m_0$;
- $\varkappa_i \in \mathcal{D}_1^0[\mathfrak{u}[t_i, m(t_i)]]$;
- for $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$,
$$m(t) = m(t, t_{i-1}, m_{i-1}, \alpha_{i-1}, \varkappa_{i-1}).$$

Let $\mathcal{X}_0^1(t_0, m_0, \mathfrak{u}, \Delta)$ stand for the set of step-by-step flows of probabilities generated by t_0, m_0, $\mathfrak{u}$ and Δ.

Below we denote by $\mathcal{X}^1(t_0, m_0, \mathfrak{u})$ the set of limit motions $m(\cdot)$ such that there exist a sequence of partitions of time interval $\{\Delta_r\}$ and a sequence of flows of probabilities $\{m_r(\cdot)\}$ such that

- $d(\Delta_r) \to 0$ as $r \to \infty$;
- $m_r(\cdot) \in \mathcal{X}_0^1(t_0, m_0, \mathfrak{u}, \Delta)$;
- $$\lim_{r\to\infty} \sup_{t\in[t_0,T]} W_2(m(t), m_r(t)) = 0.$$

The first player wins at (t_0, m_0) is there exists a strategy $\mathfrak{u}$ such that for any $m(\cdot) \in \mathcal{X}^1(t_0, m_0, \mathfrak{u})$ one can find a time instant τ satisfying

- $m(\tau) \in M[\tau]$
- for any $t \in [t_0, \tau]$, $m(t) \in N[t]$.

Below we denote the set of (t_0, m_0) where the first player wins by $\mathcal{W}_*^1$.

If we consider the problem of the second player, then his/her feedback strategy is a function $\mathfrak{v}$ from $[0,T] \times \mathcal{P}^2(\mathbb{R}^d)$ to $\mathcal{B}^c$. Given an initial time t_0, an initial distribution of agents m_0, the second players strategy $\mathfrak{v}$, a partition Δ one can define the set of corresponding flows of probabilities $\mathcal{X}_0^2(\cdot, t_0, m_0, \mathfrak{v}, \Delta)$. Further, let $\mathcal{X}^2(t_0, m_0, \mathfrak{v})$ denote the set of limit motions.

The second player wins at (t_0, m_0), if there exists a strategy $\mathfrak{v}$ such that for any $m(\cdot) \in \mathcal{X}^2(t_0, m_0, \mathfrak{v})$ and some $\tau \in [t_0, T]$ the following conditions hold:

- either $\tau = T$ or $m(\tau) \notin N[\tau]$;
- for any $t \in [0, \tau)$, $m(t) \notin M[t]$.

Let $\mathcal{W}_*^2$ denote the set of (t_0, m_0) at those the second player wins.

5. STABLE SETS

The following definitions extend to the case of mean field type differential games the notions of u- and v-stability first introduced in Krasovskii and Subbotin (1988) for the finite dimensional differential games.

We say that the set $\mathcal{W} \subset N \subset [0,T] \times \mathcal{P}^2(\mathbb{R}^d)$ is u-stable, if for any $(t_*, m_*) \in \mathcal{W}$, $t_+ \in [t_*, T]$, $\beta \in \mathcal{B}^c$ there exists a distribution of pairs of controls $\varkappa \in \mathcal{D}_2[\beta]$ and a time τ such that, for $m(\cdot) = m(\cdot, t_*, m_*, \varkappa)$,

- either $m(\tau) \in M[\tau]$ or $\tau = t_+$;

- $m(t) \in \mathcal{W}[t]$ when $t \in [t_*, \tau]$.

The notion of v-stability is defined slightly non-symmetric. First, recall that, if $F \subset [0,T] \times \mathcal{P}^2(\mathbb{R}^d)$, then an open set $\mathcal{O}(F)$ containing F is called a neighborhood of F. Below we assume that $F \subset [0,T] \times \mathcal{P}^2(E)$. Thus,

$$\sup_{(t',m')\in\partial F,(t'',m'')\in\partial\mathcal{O}(F)} [|t-t''| + W_2(m',m'')] < \infty.$$

We say that the set $\mathcal{W} \subset [0,T] \times \mathcal{P}^2(\mathbb{R}^d)$ is v-stable, if, there exist neighborhoods of M and N $\mathcal{O}(M)$ and $\mathcal{O}(N)$ satisfying the following: $\mathcal{W} \cap \mathcal{O}(M) = \varnothing$ and for any $(t_*, m_*) \in \mathcal{W}$, $t_+ \in [t_*, T]$, $\alpha \in \mathcal{A}^c$ one can find a distribution of the pairs of controls $\varkappa \in \mathcal{D}_1[\alpha]$ and a time τ such that, for $m(\cdot) = m(\cdot, t_*, m_*, \varkappa)$,

- either $m(\tau) \notin \mathcal{O}(N)[\tau]$ or $\tau = t_+$;
- $m(t) \in \mathcal{W}[t]$ when $t \in [t_*, \tau]$.

Without loss of generality, one can assume that u- and v-stable sets are closed.

Theorem 1. Let $\mathcal{W}$ be a u-stable set. Furthermore, assume that $\mathcal{W}[T] \subset M[T]$. Then there exists a strategy $\mathfrak{u}$ such that, for any $(t_0, x_0) \in \mathcal{W}$ and any $m(\cdot) \in \mathcal{X}^1(t_0, m_0, \mathfrak{u})$, one can find a time instant τ satisfying

- $m(\tau) \in M[\tau]$
- for any $t \in [t_0, \tau]$, $m(t) \in N[t]$.

In particular, Theorem 1 states that if $\mathcal{W}$ is u-stable, then

$$\mathcal{W} \subset \mathcal{W}_*^1.$$

The key idea of the proof of Theorem 1 is an extension of Krasovskii-Subbotin extremal shift rule Krasovskii and Subbotin (1988) to the case of mean field type differential games.

We construct the strategy $\mathfrak{u}$ in the following way. First, given $s \in [0,T]$, $m_* \in \mathcal{P}^2(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$, put

$$\hat{u}(s,x,y,m_*) \triangleq \operatorname*{argmin}_{u\in U} \max_{v\in V} \langle x-y, f(s,x,m_*,u,v)\rangle.$$

Analogously, set

$$\hat{v}(s,x,y,m_*) \triangleq \operatorname*{argmax}_{v\in V} \min_{u\in U} \langle x-y, f(s,x,m_*,u,v)\rangle.$$

Without loss of generality, one can assume that the functions $(x,y) \mapsto \hat{u}(s,x,y,m_*)$ and $(x,y) \mapsto \hat{v}(s,x,y,m_*)$ are measurable.

Now, let $s \in [0,T]$, $m_* \in \mathcal{P}^2(\mathbb{R}^d)$. If $(s, m_*) \in \mathcal{W}$, then we put $\mathfrak{u}(s, m_*)$ to be equal to an arbitrary distribution of the first player's controls $\alpha \in \mathcal{A}^c$.

If $(s, m_*) \notin \mathcal{W}$, then let $\nu_* \in \mathcal{P}^2(\mathbb{R}^d)$ be such that

- $\nu_* \in \mathcal{W}(s)$;
- $W_2(m_*, \nu_*) = \min\{W_2(m_*, m) : m \in \mathcal{W}[s]\}$.

Further, pick $\pi \in \Pi(m_*, \nu_*)$ to be an optimal plan between m_* and ν_*. Let $\pi(\cdot|x)$ be an disintegration of π along m_* i.e., for any $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,y)\pi(dx,dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x,y)\pi(dy|x)m_*(dx).$$

Now set, for $(s, m_*) \notin \mathcal{W}$

$$\mathfrak{u}[s,m_*](x) \triangleq \hat{u}(s,m_*,x,\cdot)_{\#}\pi(\cdot|x).$$

The proof of optimality of the strategy $\mathfrak{u}$ is based on the following lemmas.

Lemma 2. Let $s \in [0,T]$, $x_*, y_* \in \mathbb{R}^d$, $m(\cdot), \nu(\cdot) \in \mathcal{M}$, $\xi \in \mathcal{U}$, $\zeta \in \mathcal{V}$, $u^* = \hat{u}(s, m(s), x_*, y_*)$, $v^* = \hat{v}(s, m(s), x_*, y_*)$. Denote $x(\cdot) = x(s, x_*, m(\cdot), u^*, \zeta)$, $x(\cdot) = x(s, y_*, \nu(\cdot), \xi, v^*)$. Then, for any $r \in [s,T]$,

$$\begin{aligned}&\|x(r)-y(r)\|^2\\ &\quad\le \|x_*-y_*\|^2(1+3L(r-s))\\ &\qquad + LW_2(m(s),\mu(s)) + \omega_1(r-s)\cdot(r-s).\end{aligned}$$

Here $\omega_1 : \mathbb{R} \to [0, +\infty)$ is a continuous at 0, vanishing at 0 determined only by the function f.

The proof of this Lemma is analogous to the proof of Lemma 1 in Averboukh (2018).

Integrating the result of the Lemma w.r.t. the optimal plan π we get the following.

Lemma 3. Let $s, r \in [0,T]$, $s \le r$, $m_*, \nu_* \in \mathcal{P}^2(\mathbb{T}^d)$, π be an optimal plan between m_* and ν_*, $\pi(\cdot|x)$, $\pi(\cdot|y)$ be its disintegration with respect to m_* and ν_* respectively, $\alpha^*(x) \triangleq \hat{u}(s, m_*, x, \cdot)_{\#}\pi(\cdot|x)$, $\beta^*(y) \triangleq \hat{v}(s, m_*, \cdot, y)_{\#}\pi(\cdot|y)$, $\varkappa \in \mathcal{D}_1[\alpha_*]$, $\vartheta \in \mathcal{D}_2[\beta_*]$, $m(\cdot) = m(\cdot, s, m_*, \varkappa)$, $\nu(\cdot) = m(\cdot, s, \nu_*, \vartheta)$. Then

$$\begin{aligned}W_2(m(r),\nu(r)) &\le W_2(m_*,\nu_*)(1+4L(r-s))\\ &\quad + \varpi_1(r-s)\cdot(r-s).\end{aligned}$$

The proof of this lemma is the same as the proof of Lemma 2 in Averboukh (2018).

Proof of Theorem 1. Let $t_0 \in [0,T]$, $m_0 \in \mathcal{P}^2(\mathbb{R}^d)$ satisfy $(t_0, m_0) \in \mathcal{W}$. We shall prove that for any $\varepsilon > 0$ one can find $\delta > 0$ such that whatever $\Delta = \{t_i\}_{i=0}^n$ and $m(\cdot) \in \mathcal{X}_0^1(t_0, m_0, \mathfrak{u}, \Delta)$ are chosen there exists $\tau \in [t_0, T]$ the following properties hold true:

- $\operatorname{dist}(m(\tau), M[\tau]) \le \varepsilon$;
- $\operatorname{dist}(m(t), \mathcal{W}[t]) \le \varepsilon$ when $t \in [t_0, \tau]$.

Notice that there exist $\{\varkappa_i\}_{i=0}^{n-1}$ such that $\varkappa_i \in \mathcal{D}_1^0[\mathfrak{u}[t_{i-1}, m(t_{i-1})]]$ and, for $t \in [t_{i-1}, t_i]$

$$m(t) = m(t, t_i, m(t_{i-1}), \varkappa_{i-1}).$$

Let j be the greatest number $i = \overline{1,n}$ such that $m(t_i) \notin \mathcal{W}[t_i]$. Recall that $\mathcal{W} \subset [0,T] \times \mathcal{P}^2(E)$, where E is a compact in $\mathbb{R}^d$. Since $m(t_{j-1}) \in \mathcal{W}[t_{j-1}]$ using the u-stability of $\mathcal{W}$ we get that

$$\operatorname{dist}(m(t_j), \mathcal{W}[t_j]) \le 2R(t_j - t_{j-1}) \le 2Rd(\Delta). \quad (3)$$

Now, using the u-stability of $\mathcal{W}$, inclusion $\mathcal{W}[T] \subset M[T]$, and Lemma 3, we construct the number $l \in \{j, \ldots, n-1\}$, time $\theta \in [t_l, t_{l+1}]$, and $\{\nu_i(\cdot)\}_{i=j}^l$ such that, for $\tau_i \triangleq t_i \wedge \theta$,

- $\nu_i(t) \in \mathcal{W}(t)$ when $t \in [\tau_i, \tau_{i+1}]$;
- $\nu_l(\tau) \in M(\tau)$;
- the following inequality holds true: for $t \in [\tau_i, \tau_{i+1}]$,

$$\begin{aligned}&W_2(\nu_i(t),m(t))\\ &\quad\le W_2(\nu_i(\tau_i), m(\tau_i))(1+4L(t-\tau_i))\\ &\qquad + \varpi_1(t-\tau_i)\cdot(t-\tau_i).\end{aligned} \quad (4)$$

Since $\operatorname{dist}(m(t), \mathcal{W}[t]) \le W_2(\nu_i(t), m(t)$, applying (4) sequentially, and taking into account (3) we get

$$\begin{aligned}\operatorname{dist}(m(t),\mathcal{W}[t]) &\le 2Rd(\Delta)\exp(4L(t-t_j))\\ &\quad + \varpi_1(d(\Delta))\exp(4L(t-t_j))\cdot(t-t_j).\end{aligned}$$

The inclusion $\mathcal{W}[t] \subset N[t]$ implies the following inequality, for any $t \in [t_0, \theta]$,

$$\begin{aligned}\mathrm{dist}(m(t), N[t]) \leq 2Rd(\Delta)\exp(4L(t-t_j)) \\ + \varpi_1(d(\Delta))\exp(4L(t-t_j))\cdot(t-t_j).\end{aligned} \quad (5)$$

Furthermore, using the inclusion $\nu_l(\theta) \in M[\theta]$, we conclude that

$$\begin{aligned}\mathrm{dist}(m(\theta), M[\theta]) \leq 2Rd(\Delta)\exp(4L(t-t_j)) \\ + \varpi_1(d(\Delta))\exp(4L(t-t_j))\cdot(t-t_j).\end{aligned} \quad (6)$$

Choosing δ such that

$$2R\delta\exp(4LT) + \varpi_1(\delta)\exp(4LT)\cdot T \leq \varepsilon,$$

and by (5), (6) we obtain the conclusion of the Theorem. □

Theorem 4. Let $\mathcal{W}$ be a v-stable set. Then there exist a neighborhoods of M and N $\mathcal{O}(M)$ and $\mathcal{O}(N)$ and a strategy of the second player $\mathfrak{u}$ such that, for any $(t_0, x_0) \in \mathcal{W}$ and any $m(\cdot) \in \mathcal{X}^2(t_0, m_0, \mathfrak{v})$ one can find a time instant τ satisfying

- either $m(\tau) \notin \mathcal{O}(N)[\tau]$ or $\tau = T$;
- for any $t \in [t_0, \tau]$, $m(t) \notin \mathcal{O}(M)[t]$.

The proof of Theorem 4 is analogous to the proof of Theorem 1.

The Theorems 1 and 4 and the definitions of u- and v-stablity as well as definitions of the set $\mathcal{W}_*^1$ and $\mathcal{W}_*^2$ implies the following.

Theorem 5. $[0,T] \times \mathcal{P}^2(\mathbb{R}^d) = W_*^1 \cup \mathcal{W}_*^2$. The sets $\mathcal{W}_*^1$, $\mathcal{W}_*^2$ are u- and v-stable respectively.

6. CONCLUSION

The paper was consider the mean field type differential game. This problem arises quite naturally when we examine the system of many interacting agents pursuing the common goal under some disturbances. In the paper we study the target problem for the mean field type differential game. Our results are based on Krasovskii-Subbotin approach. In the framework of this approach feedback strategies are used. The nonanticipative strategies for mean field type differential game was developed in Cosso and Pham (2018). In that paper it is assumed that the players wish to minimize/maximize a functional. The proof of equivalence between feedback formalization and the approach based on nonanticipative strategies is the subject of future work.

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