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Convective layered flows of a vertically whirling viscous incompressible fluid. Velocity field investigation

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Abstract

This article discusses the solvability of an overdetermined system of heat convection equations in the Boussinesq approximation. The Oberbeck–Boussinesq system of equations, supplemented by an incompressibility equation, is overdetermined. The number of equations exceeds the number of unknown functions, since non-uniform layered flows of a viscous incompressible fluid are studied (one of the components of the velocity vector is identically zero). The solvability of the non-linear system of Oberbeck–Boussinesq equations is investigated. The solvability of the overdetermined system of non-linear Oberbeck–Boussinesq equations in partial derivatives is studied by constructing several particular exact solutions. A new class of exact solutions for describing three-dimensional non-linear layered flows of a vertical swirling viscous incompressible fluid is presented. The vertical component of vorticity in a non-rotating fluid is generated by a non-uniform velocity field at the lower boundary of an infinite horizontal fluid layer. Convection in a viscous incompressible fluid is induced by linear heat sources. The main attention is paid to the study of the properties of the flow velocity field. The dependence of the structure of this field on the magnitude of vertical twist

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is investigated. It is shown that, with nonzero vertical twist, one of the components of the velocity vector allows stratification into five zones through the thickness of the layer under study (four stagnant points). The analysis of the velocity field has shown that the kinetic energy of the fluid can twice take the zero value through the layer thickness.

Keywords: exact solution, layered convection, tangential stress, stagnation point, counterflow, stratification, Oberbeck–Boussinesq equation system, vertical twist.

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Introduction. Mathematical models that describe viscous fluid flow are generally based on the Navier–Stokes equations [1–5]. Assumptions regarding specific mass forces involved in these equations make it possible to distinguish regularities that are imperceptible when these equations are considered in general terms. One of the most well-known and widely used assumptions is the linear temperature dependence of fluid density: $\rho = \rho_0(1 - \beta T)$, where ρ_0 is the average density, β is the coefficient of volumetric expansion. After substituting the expression relating density and temperature into the Navier-Stokes equation, we obtain the equation of the motion of a viscous fluid in the Boussinesq approximation (the Oberbeck–Boussinesq system) [6–10]. In addition to the velocity vector components, the equations of the Oberbeck–Boussinesq system include scalar pressure and temperature fields. The system of equations is not closed. To close the Navier–Stokes equations and the continuity equation, use the energy equation (heat equation) [6, 11].

The difficulty of finding the exact solutions of the system of differential Oberbeck–Boussinesq equations (partial differential equations) stems from its nonlinearity due to the presence of a convective derivative in the equations describing pulse transfer and in the heat equation. The properties of the solution are influenced by the boundary conditions, the physical parameters of the fluid and the environmental characteristics [12–14].

A number of interesting flows arising in technical problems and technological processes, for example, a submerged jet [15–17], trail behind the body [18–20], or the flow of fluid or gas from a hole [2, 21] belong to the class of so-called shear flows [22–26]. Shear flows have the property that one of the three velocity components is assumed to be zero. In this case, the closed system of equations describing the motion of a fluid becomes overdetermined (the number of equations exceeds the number of unknown functions).

One of the approaches that allow one to solve overdetermined systems arising in mathematical physics when considering shear flows is the construction of generalized classes of exact solutions [27–29], the substitution of which into the system of equations under consideration leads to the identical satisfaction of some of the equations from the Oberbeck–Boussinesq system and reduces the initially nonlinear system of partial differential equations to a simpler system.

These solution families differ, among other things, in the way the vorticity calculated for the selected class behaves. A family of exact solutions for a vectorial velocity field generating no vertical twist was discussed in [30–34]. Taking into account vertical twist [35–43] changes the form of the particular solution of the boundary value problem and complicates its structure.

This article discusses the effect of constant vertical twist on the topology of the velocity field of the flow in a boundary value problem describing the flow of a fluid in an infinite horizontal layer, induced by tangential stresses specified on the free surface. A comparison is made with the case when the vertical twist is set equal to zero in the selected velocity class.

1. Problem statement. An exact solution to the Oberbeck–Boussinesq system. The following system of equations of thermal shear convection in the Boussinesq approximation is considered [30, 31, 35, 36, 44, 45]:

$$\begin{gathered} V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu \triangle V_x, \\ V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu \triangle V_y, \\ \frac{\partial P}{\partial z} = g\beta T, \\ V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = \chi \triangle T, \\ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \end{gathered} \tag{1}$$

Here, P is pressure deviation from hydrostatic, divided by constant average fluid density ρ; T is deviation from the average temperature; ν and χ are the coefficients of kinematic viscosity and thermal diffusivity of the fluid, respectively; $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.

The system of equations (1) is overdetermined, since it consists of five equations for the determination of four unknown functions V_x, V_y, P, and T. For the solvability of system (1), it is necessary to make sure that the equations involved in it are compatible and to construct exact solutions that are non-trivial. By choosing a class of generalized solutions of a special type, one can achieve identical satisfaction of "extra" equations. It was shown in [30, 31, 46–48] that, for the velocity field

$$V_x = U(z), \quad V_y = V(z) \tag{2}$$

the incompressibility equation holds identically. The choice of the class (2) allows system (1) to be reduced to the form

$$\begin{gathered} \nu \frac{\partial^2 U}{\partial z^2} = \frac{\partial P}{\partial x}, \quad \nu \frac{\partial^2 V}{\partial z^2} = \frac{\partial P}{\partial y}, \quad \frac{\partial P}{\partial z} = g\beta T, \\ \chi \Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \Big) = U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y}. \end{gathered} \tag{3}$$

In the system of equations (3), the number of equations coincides with the number of unknowns.

Ekman was the first to suggest considering solutions in the form (2) for the description of large-scale flows of rotating fluids [49]. An exact solution of the form (2) generalizes the unidirectional Couette flow [50, 51] and the Birich–Ostroumov flow [52, 53] in the inertial reference system. Note that the exact solution (2) is not the only family, the substitution of which into the incompressibility equation leads to an identity. Velocities of a more general form [54–56],

$$V_x = V_x(y, z), \quad V_y = V_y(x, z), \tag{4}$$

also possess the property under study and allow one to reduce the number of equations in system (1). Substituting class (4) into system (1) also leads to the identical satisfaction of the incompressibility equation:

$$\begin{gathered}
V_y \frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu\Big(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\Big), \\
V_x \frac{\partial V_y}{\partial x} = -\frac{\partial P}{\partial y} + \nu\Big(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2}\Big), \\
\frac{\partial P}{\partial z} = g\beta T, \\
V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = \chi\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\Big).
\end{gathered}$$

The procedure of constructing such classes is considered in detail in [8, 30, 31, 34–36, 45, 57, 58]. For the velocity field (4), it is possible, using a number of transformations, to construct exact solutions to the three-dimensional Oberbeck–Boussinesq system (1). In contrast to class (4), for which all the vorticity components $\mathbf{\Omega} = \operatorname{rot} \mathbf{V}$,

$$\Omega_x = -\frac{\partial V_y}{\partial z}, \quad \Omega_y = \frac{\partial V_x}{\partial z}, \quad \Omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \tag{5}$$

are non-zero in the general case, the vertical component Ω_z of vorticity (5) is always zero for the velocity field (2),

$$\Omega_x = -\frac{\partial V}{\partial z}, \quad \Omega_y = \frac{\partial U}{\partial z}, \quad \Omega_z = 0.$$

In other words, the family of velocities (4) can describe vertical spin in a fluid, which occurs without setting rotation at the boundaries of the region in question. The class of exact solutions (4) for the Oberbeck–Boussinesq equation system allows large-scale flows in the equatorial zone of the World Ocean to be studied with the use of the traditional approximation for the angular velocity vector (one Coriolis parameter is used) [38, 44, 55, 56, 59, 60].

We set the task to analyze how the consideration of vertical twist affects the behavior of the flow. For convenience and clarity, we choose a family of exact solutions [35, 36, 44, 61]

$$V_x = U(z) + u(z)y, \quad V_y = V(z), \tag{6}$$

which is a special case of class (4). The vorticity vector components calculated for it take the form

$$\Omega_x = -\frac{\partial V}{\partial z}, \quad \Omega_y = \frac{\partial U}{\partial z}, \quad \Omega_z = -u.$$

Class (6) differs from class (2) in that the additional term $u(z)y$ in the expression for the velocity V_x is taken into account. Note that, when $u = 0$, class (6) degenerates into family (2).

The substitution of class (6) into the system of Oberbeck–Boussinesq equations brings the system (1) to the form

$$\nu\Big(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 u}{\partial z^2}y\Big) = \frac{\partial P}{\partial x} + uV,$$
$$\nu\frac{\partial^2 V}{\partial z^2} = \frac{\partial P}{\partial y}, \quad \frac{\partial P}{\partial z} = g\beta T, \tag{7}$$
$$(U + uy)\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \chi\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\Big).$$

It follows from the second equation of system (7) that the horizontal pressure gradient $\partial P/\partial y$ depends only on the transverse (vertical) coordinate z; therefore, the pressure P can be represented as

$$P = P_1(x, z) + P_2(z)y.$$

Substituting the partial derivative $\partial P/\partial x = \partial P_1/\partial x$ into the first equation of system (7), we obtain that $\partial P_1/\partial x$ depends only on z, i.e. the pressure P proves to be linear in the coordinate x. Finally, we arrive at the form

$$P = P_0(z) + P_1(z)\,x + P_2(z)\,y. \tag{8}$$

At the end, we substitute (8) into the third equation of system (7),

$$\frac{\partial P_0}{\partial z} + \frac{\partial P_1}{\partial z}x + \frac{\partial P_2}{\partial z}y = g\beta T$$

and we find that the temperature T is a linear function of the horizontal coordinates, i.e.

$$T = T_0(z) + T_1(z)\,x + T_2(z)\,y. \tag{9}$$

In view of the chosen structure of the temperature and pressure fields (8), (9), the equations of system (1) take the form

$$\nu\Big(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 u}{\partial z^2}y\Big) = P_1 + uV, \quad \nu\frac{\partial^2 V}{\partial z^2} = P_2,$$
$$\frac{\partial P_0}{\partial z} + \frac{\partial P_1}{\partial z}x + \frac{\partial P_2}{\partial z}y = g\beta(T_0 + T_1 x + T_2 y), \tag{10}$$
$$UT_1 + VT_2 + uT_1 y = \chi\Big(\frac{\partial^2 T_0}{\partial z^2} + \frac{\partial^2 T_1}{\partial z^2}x + \frac{\partial^2 T_2}{\partial z^2}y\Big).$$

The equations of system (10) are equalities of the form

$$a_k(z) + b_k(z)x + c_k(z)y = 0. \tag{11}$$

Applying the method of undetermined coefficients, we equate to zero the coefficients at the independent variables x, y and the free terms in the polynomial expressions (11). Inasmuch as all the required functions depend only on z, we denote the derivatives with respect to the coordinate z by a prime. As a result, we obtain the following system of equations to determine the unknown components of the hydrodynamic fields (the equations in the system are written in the order of integration):

$$\begin{gathered} u'' = 0, \quad T_1'' = 0, \quad P_1' = g\beta T_1, \quad \chi T_2'' = uT_1, \quad P_2' = g\beta T_2, \\ \nu V'' = P_2, \quad \nu U'' = Vu + P_1, \quad \chi T_0'' = UT_1 + VT_2, \quad P_0' = g\beta T_0. \end{gathered} \tag{12}$$

Note that only the first two equations in system (12) are isolated. After integrating the differential equations in order to determine the functions u and T_1, the exact solutions for which are the linear functions

$$u = c_1 z + c_2, \quad T_1 = c_3 z + c_4,$$

we arrive at a solution for the remaining functions involved in system (1). Hereinafter, we discuss the case of constant vertical twist, setting $u = \Omega = \text{const}$.

2. Boundary value problem. As boundary conditions for the horizontal temperature gradients T_1 and T_2, the horizontal pressure gradients P_1 and P_2, the background temperature T_0, the background pressure P_0 and the velocities U and V, we select the conditions described in [30, 31]. The absolutely solid bottom surface $z = 0$ of the infinite horizontal layer under study is the reference level of temperature measurement. Without loss of generality, we assume the reference temperature to be zero,

$$T(x, y, 0) = 0.$$

The velocity of the lower boundary $z = 0$ is set as

$$V_x(0) = \Omega y, \quad V_y(0) = 0.$$

On the upper undeformed (free) boundary $z = h$, a constant atmospheric pressure is set and, by analogy with temperature setting, it is measured from zero,

$$P(x, y, h) = 0.$$

We also assume that a homogeneous field of tangential stresses is specified on the upper boundary as

$$\eta \frac{\partial V_x}{\partial z} = \eta \frac{\partial U}{\partial z} = \xi_1, \quad \eta \frac{\partial V_y}{\partial z} = \eta \frac{\partial V}{\partial z} = \xi_2.$$

Here, η is the dynamic viscosity coefficient. Note that, due to the structure of the velocity field $\mathbf{V}$, the resulting tangential stress field, as well as in [30, 31], is homogeneous. In addition, on both boundaries of the fluid layer, heat sources are specified as

$$T(x, y, 0) = Ax + By, \quad T(x, y, h) = \vartheta + Cx + Dy.$$

Taking into account the class of generalized solutions (6), (8), and (9), we write the selected boundary conditions as follows:

$$U(0) = V(0) = 0,$$

$$\begin{gathered}\eta U'(h) = \xi_1, \quad \eta V'(h) = \xi_2, \\ T_0(0) = 0, \quad T_1(0) = A, \quad T_2(0) = B, \\ T_0(h) = \vartheta, \quad T_1(h) = C, \quad T_2(h) = D, \\ P_0(h) = P_1(h) = P_2(h) = 0.\end{gathered} \tag{13}$$

In what follows, we will study the velocity field in detail; therefore, the exact polynomial solution of the boundary problem (12), (13) is not completely given. The expressions for the functions T_0 and P_0 are cumbersome; however, they can be easily obtained by integrating the corresponding equations of system (12). The exact solution of the boundary value problem (12), (13) has the form

$$u = \Omega; \quad T_1 = A + \frac{C-A}{h} z; \quad P_1 = \frac{g\beta}{2!\,h}\left((C-A)z^2 + 2Ahz - (C+A)h^2\right);$$
$$T_2 = B + \frac{D-B}{h} z - \frac{\Omega}{3!\,h\chi}(h-z)z(2Ah + Ch - Az + Cz);$$

$$P_2 = -\frac{g\beta}{2!\,h}(h-z)(Bh + Dh - Bz + Dz) + \\ + \frac{g\beta\Omega}{4!\,h\chi}(h-z)^2(Ah^2 + Ch^2 + 2Ahz + 2Chz - Az^2 + Cz^2);$$

$$V = \frac{\xi_2 z}{\eta} + \frac{g\beta z}{4!\,h\nu}\big[B(4h^3 - 6h^2 z + 4hz^2 - z^3) + D(8h^3 - 6h^2 z + z^3)\big] - \\ - \frac{g\beta\Omega z}{6!\,h\nu\chi}\big[A(14h^5 - 15h^4 z + 10h^2 z^3 - 6hz^4 + z^5) + \\ + C(16h^5 - 15h^4 z + 5h^2 z^3 - z^5)\big]; \tag{14}$$

$$U = \frac{g\beta z}{6!\,h\nu}\big[A(4h^3 - 6h^2 z + 4hz^2 - z^3) + C(8h^3 - 6h^2 z + z^3)\big] - \\ - \frac{g\beta\Omega z}{4!\,h\nu^2}\big[B(24h^5 - 20h^3 z^2 + 15h^2 z^3 - 6hz^4 + z^5) + \\ + D(66h^5 - 40h^3 z^2 + 15h^2 z^3 - z^5)\big] + \\ + \frac{g\beta\Omega^2 z}{3\cdot 8!\,h\nu^2\chi}\big[A(528h^7 - 392h^5 z^2 + 210h^4 z^3 - 56h^2 z^5 + 24hz^6 - 3z^7) + \\ + C(648h^7 - 448h^5 z^2 + 210h^4 z^3 - 28h^2 z^5 + 3z^7)\big] - \\ - \frac{\Omega z}{3!\,\eta\nu}(3h^2\xi_2 - \xi_2 z^2) + \frac{\xi_1 z}{\eta}.$$

Note that, in view of the exact solution (14), the condition $u = 0$ determining the degeneracy of class (6) into class (2), is equivalent to the condition $\Omega = 0$. Therefore, the effect of the parameter Ω on the topology of the velocity field will be further studied in detail.

3. Velocity field analysis. We set the horizontal temperature gradients $B = D = 0$ in the boundary conditions (13) and the exact solution (14). The flow induced by this heating of the boundaries is a generalization of the unidirectional Birich convective flow [52].

As we pass to the dimensionless coordinate $Z = z/h \in [0, 1]$, the expressions for the velocities U and V assume the form

$$V = \frac{\xi_2 h}{\eta} Z - \frac{g\beta\Omega h^5}{6!\,\nu\chi} Z\big[A(14 - 15Z + 10Z^3 - 6Z^4 + Z^5) + \\ + C(16 - 15Z + 5Z^3 - Z^5)\big], \quad (15)$$

$$U = \frac{g\beta h^3}{6!\,\nu} Z\big[A(4 - 6Z + 4Z^2 - Z^3) + C(8 - 6Z + Z^3)\big] + \\ + \frac{g\beta\Omega^2 h^7}{3\cdot 8!\,\nu^2\chi} Z\big[A(528 - 392Z^2 + 210Z^3 - 56Z^5 + 24Z^6 - 3Z^7) + \\ + C(648 - 448Z^2 + 210Z^3 - 28Z^5 + 3Z^7)\big] - \\ - \frac{\Omega\xi_2 h^3}{3!\,\eta\nu} Z(3 - Z^2) + \frac{\xi_1 h}{\eta} Z. \quad (16)$$

The velocity field (15), (16) describes the convective flow of a viscous incompressible fluid, which cannot be reduced to unidirectional flow at $\Omega \neq 0$. Thus, the boundary value problem (12), (13) is essentially non-one-dimensional.

Denote by U^0 and V^0 the velocities (15), (16) calculated in the absence of vertical twist ($\Omega = 0$),

$$V^0 = \frac{\xi_2 h}{\eta} Z,$$

$$U^0 = \frac{g\beta h^3}{6!\,\nu} Z\big[A(4 - 6Z + 4Z^2 - Z^3) + C(8 - 6Z + Z^3)\big] + \frac{\xi_1 h}{\eta} Z. \quad (17)$$

Let us now study how the inclusion of the terms containing spatial acceleration Ω in Eqs. (15), (16) changes the structure of the flow velocities in comparison with the velocity field V^0, U^0 when different values of the temperature gradients A and C are specified. We start with the simplest case, namely, the case of a uniform heat source ($A = B = C = D = 0$).

When a uniform heat source $T_1 = T_2 = 0$, $T_0 = \vartheta Z$ is set, the velocities U and V are determined by linear functions as

$$U = U^0 = \frac{\xi_1 h}{\eta} Z, \quad V = V^0 = \frac{\xi_2 h}{\eta} Z.$$

Thus, in the direction of both longitudinal axes, the flow is reduced to a combination of unidirectional flows of the Couette type [50], which correspond to a constant field of tangential stresses

$$\tau_{xz} = \eta \frac{\partial U}{\partial z} = \frac{\eta}{h}\frac{\partial U}{\partial Z} = \xi_1, \quad \tau_{yz} = \eta \frac{\partial V}{\partial z} = \frac{\eta}{h}\frac{\partial V}{\partial Z} = \xi_2.$$

Moreover, the direction of the vortex $\mathbf{\Omega}$ remains unchanged everywhere inside the layer. Hereinafter, it is assumed that the flow is convective, i.e. that $A^2 + C^2 \neq 0$.

We start by analyzing the velocity V (15), since the structure of Eq. (15) is simpler than Eq. (16) determining the velocity U. According to Eq. (15), the velocity V is determined by the superposition of the flow caused by setting the tangential stresses at the upper boundary and two convective flows induced by setting the heat sources. Note that the contribution of each of these flows not only is determined by the values of the parameters A, C, ξ_2, Ω, but also depends on the thickness of the layer h. By choosing h, one can make the linear term $\frac{\xi_2 h}{\eta} Z$ prevail over the non-linear terms in the velocity expression (15). Let us consider two limiting cases, $A = 0$ and $C = 0$, which allow us to reduce the number of streams contributing to the resultant flow.

Assume that $A = 0$, then the expression (15) for the velocity V becomes

$$V_1 = \frac{\xi_2 h}{\eta} Z - \frac{Cg\beta\Omega h^5}{6!\,\nu\chi} Z(16 - 15Z + 5Z^3 - Z^5) = \\ = \frac{Cg\beta\Omega h^5}{6!\,\nu\chi} Z\Big[Z^5 - 5Z^3 + 15Z - 16 + \frac{6!\,\nu\chi\xi_2}{Cg\beta\eta\Omega h^4}\Big],$$

wherefrom it follows that the velocity V_1 can have stagnant points only if the polynomial equation

$$Z^5 - 5Z^3 + 15Z + a_1 = 0,$$

with some

$$a_1 = -16 + \frac{6!\nu\chi\xi_2}{Cg\beta\eta\Omega h^4},$$

has roots within the interval $(0, 1)$. The analysis of the solvability of the equation shows that such a root in the $(0, 1)$ interval is unique and that it exists only when the control parameters of the problem satisfy the condition

$$\frac{1}{144} \leqslant \frac{\nu\chi\xi_2}{Cg\beta\eta\Omega h^4} \leqslant \frac{1}{45}.$$

The dependence of the position of the stagnant point of the velocity V_1 on the value of the parameter a_1 is shown in Fig. 1 (curve *1*).

Note that in the case under consideration, for $A = 0$, there is such a value Z_1 of the vertical coordinate Z that the tangential stress

$$\tau^1_{yz} = \frac{\eta}{h}\frac{dV_1}{dZ} = \frac{Cg\beta\eta\Omega h^4}{6!\,\nu\chi}[6Z^5 - 20Z^3 + 30Z + a_1] \qquad (18)$$

vanishes. In this case, the stress changes its type (from tensile to compressive). Such Z_1 exists only for $a_1 \in [-16, 0]$, i.e. when

$$0 \leqslant \frac{\nu\chi\xi_2}{Cg\beta\eta\Omega h^4} \leqslant \frac{1}{45}.$$

The dependences of the coordinate Z_1 (the zeros of the polynomial (18)) on the parameter a_1 are shown in Fig. 1 (curve *2*).

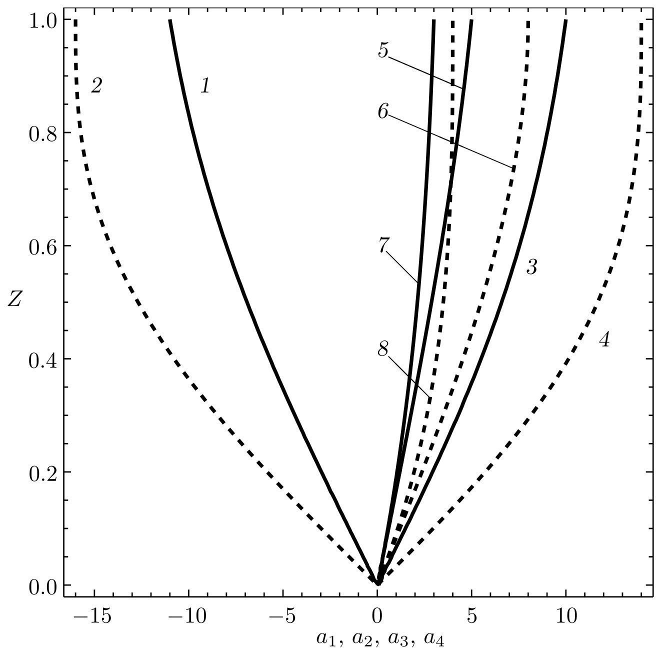

Figure 1. Dependencies of the position of the stagnant points of the velocities V_1, V_2, U_1^0, U_2^0 and the critical point of the corresponding stresses τ_{yz}^1, τ_{yz}^2, τ_{xz}^{01}, τ_{xz}^{01} on the parameters a_1, a_2, a_3, a_4: curve *1* — the set of points satisfying the condition $V_1 = 0$; curve *2* — the set of points satisfying the condition $\tau_{yz}^1 = 0$; curve *3* — the set of points satisfying the condition $V_2 = 0$; curve *4* — the set of points satisfying the condition $\tau_{yz}^2 = 0$; curve *5* — the set of points satisfying the condition $U_1^0 = 0$; curve *6* — the set of points satisfying the condition $\tau_{xz}^{01} = 0$; curve *7* — the set of points satisfying the condition $U_2^0 = 0$; curve *8* — the set of points satisfying the condition $\tau_{xz}^{02} = 0$

Consider another limiting case, assuming that $C = 0$ in (15). Then the velocity $V_2 = V\big|_{C=0}$ takes the form

$$V_2 = \frac{\xi_2 h}{\eta} Z - \frac{Ag\beta\Omega h^5}{6!\,\nu\chi} Z(14 - 15Z + 10Z^3 - 6Z^4 + Z^5) = \\ = -\frac{Ag\beta\Omega h^5}{6!\,\nu\chi} Z\Big[Z^5 - 6Z^4 + 10Z^3 - 15Z + 14 - \frac{6!\,\nu\chi\xi_2}{Ag\beta\eta\Omega h^4}\Big].$$

Obviously, the velocity V_2 can have stagnant points only if the equation

$$Z^5 - 6Z^4 + 10Z^3 - 15Z + a_2 = 0$$

has solutions inside the layer $(0, 1)$ for some

$$a_2 = 14 - \frac{6!\,\nu\chi\xi_2}{Ag\beta\eta\Omega h^4}.$$

The tangential stress

$$\tau_{yz}^2 = \frac{\eta}{h}\frac{dV_2}{dZ} = -\frac{Ag\beta\eta\Omega h^4}{6!\,\nu\chi}[6Z^5 - 30Z^4 + 40Z^3 - 30Z + a_2]$$

corresponding to the velocity V_2 can also change its type. The corresponding dependencies are shown in Fig. 1 (curves *3* and *4*). Similarly, one can obtain estimates for the control parameters of the boundary value problem, at which the velocity V_2 and the tangential stress τ_{yz}^2 vanish. We have the following inequality for the velocity V_2:

$$\frac{1}{180} \leqslant \frac{\nu\chi\xi_2}{Ag\beta\eta\Omega h^4} \leqslant \frac{7}{360}.$$

For the tangential stress τ_{yz}^2, it is written as follows:

$$0 \leqslant \frac{\nu\chi\xi_2}{Ag\beta\eta\Omega h^4} \leqslant \frac{7}{360}.$$

Note that, even in these limiting cases, the velocity V can have one stagnant point.

Next, we assume in (15) that $A \neq 0$, $C \neq 0$,

$$V = \frac{Cg\beta\Omega h^5}{6!\,\nu\chi} Z\big[Z^5 - 5Z^3 + 15Z + a_1 - a(Z^5 - 6Z^4 + 10Z^3 - 15Z + 14)\big].$$

Here, $a = A/C$ is a dimensionless parameter. The velocity V can have two stagnant points (Fig. 2).

Thus, in the chosen case of heating of the boundaries ($B = D = 0$), the velocity V (15), when the vertical twist ($\Omega \neq 0$) is taken into account, can have up to two zero points.

We now study in a similar way the velocity U (16) of fluid flow along the Ox axis. The expression for the velocity U, in contrast to the velocity V (15), is determined by the superposition of six streams: two streams caused by setting the tangential stresses ξ_1 and ξ_2 and four streams induced by the temperature gradients A and C. Besides, some of these streams are caused by the presence of a vertical twist in the fluid, characterized by the Ω parameter. In the same way as

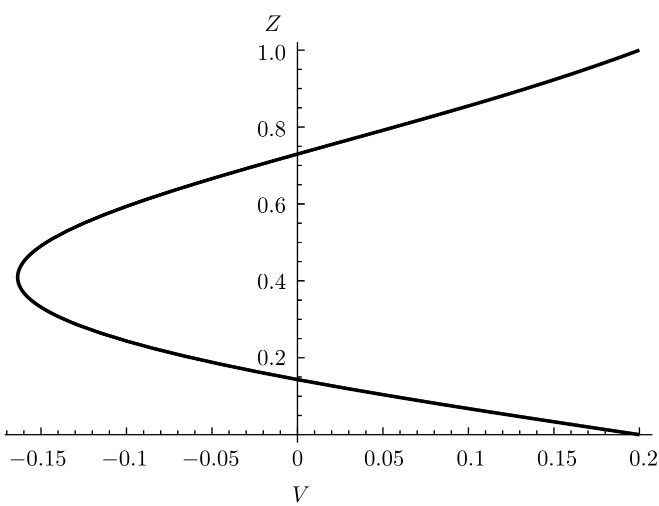

Figure 2. The profile of the velocity V for $a_1 = -15.2, a = -1.1$

in the analysis of the velocity V (15), we start with the limiting cases, when one of the longitudinal temperature gradients appears to be zero. The vanishing of each temperature gradient reduces the number of flows forming the velocity (16) by two.

Note that, when there is no vertical twist (when $\Omega = 0$ and the velocity U is determined by the expression (17)), the type of velocities

$$U_1^0 = U^0\big|_{A=0} = \frac{Cg\beta h^3}{6!\,\nu} Z(8 - 6Z + Z^3) + \frac{\xi_1 h}{\eta} Z = \\ = \frac{Cg\beta h^3}{6!\,\nu} Z\Big(Z^3 - 6Z + 8 + \frac{6!\nu\xi_1}{Cg\beta h^2 \eta}\Big),$$

$$U_2^0 = U^0\big|_{C=0} = \frac{Ag\beta h^3}{6!\nu} Z(4 - 6Z + 4Z^2 - Z^3) + \frac{\xi_1 h}{\eta} Z = \\ = \frac{Ag\beta h^3}{6!\,\nu} Z\Big(-Z^3 + 4Z^2 - 6Z + 4 + \frac{6!\,\nu\xi_1}{Ag\beta h^2 \eta}\Big)$$

is similar to the form of the velocities V_1 and V_2 in the sense that any of them is determined by the interaction of two streams, linear and nonlinear. By analogy, it can be shown that the velocities U_1^0 and U_2^0, as well as the corresponding tangential stresses $\tau_{xz}^{01} = \tau_{xz}^0\big|_{A=0}$, and $\tau_{xz}^{02} = \tau_{xz}^0\big|_{C=0}$, can vanish inside the layer. The location of the stagnation points depends on the combination of the parameters $\frac{\nu\xi_1}{Cg\beta h^2\eta}$ and $\frac{\nu\xi_1}{Ag\beta h^2\eta}$ (Fig. 1, curves *5* to *8*).

Let us now study the effect of the vertical twist Ω on the behavior of the velocity U (16) in the limiting cases $A = 0$ and $C = 0$. If $A = 0$, then the velocity $U_1 = U|_{A=0}$ takes the form

$$U_1 = Z\Big[\frac{Cg\beta h^3}{6!\,\nu}(8 - 6Z + Z^3) - \frac{\Omega\xi_2 h^3}{3!\,\eta\nu}(3 - Z^2) + \frac{\xi_1 h}{\eta} + \\ + \frac{Cg\beta\Omega^2 h^7}{3\cdot 8!\,\nu^2\chi}(648 - 448Z^2 + 210Z^3 - 28Z^5 + 3Z^7)\Big] = \\ = \frac{Cg\beta\Omega^2 h^7}{3\cdot 8!\,\nu^2\chi} Z\Big[3Z^7 - 28Z^5 + 210Z^3 - 448Z^2 + 648 + \frac{3\cdot 8!\,\nu^2\chi\xi_1}{Cg\beta\Omega^2\eta h^6} + \\ + \frac{18\nu\chi}{\Omega^2 h^4}(8 - 6Z + Z^3) - \frac{4\cdot 7!\,\nu\chi\xi_2}{Cg\beta\Omega\eta h^4}(3 - Z^2)\Big]$$

by virtue of (16).

The polynomials $3Z^7 - 28Z^5 + 210Z^3 - 448Z^2 + 648$, $8 - 6Z + Z^3$, and $3 - Z^2$ involved in the expression of the velocity U_1 are strictly monotonic on the domain of definition $[0, 1]$, and each coefficient in front of these polynomials contains at least one independent control parameter. The profile of the velocity U_1 with three stagnant points is shown in Fig. 3.

When $C = 0$, it follows from (16) that the velocity $U_2 = U|_{C=0}$ can be transformed in the same way

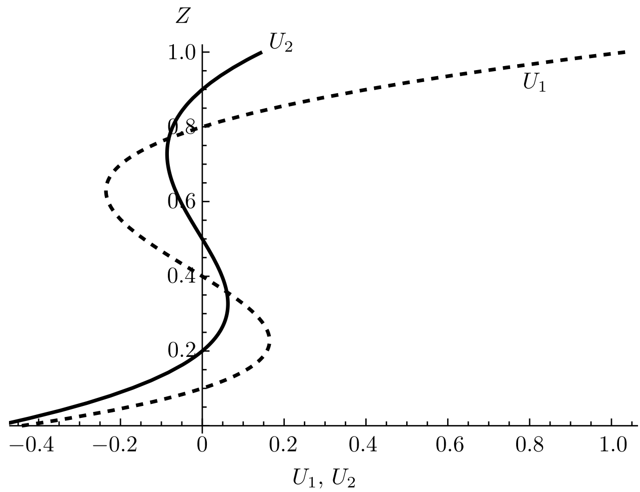

Figure 3. The profiles of the velocities U_1 and U_2 when they have three stagnant points

$$U_2 = Z\Big[\frac{Ag\beta h^3}{6!\,\nu}(4-6Z+4Z^2-Z^3) - \frac{\Omega\xi_2 h^3}{3!\,\eta\nu}(3-Z^2) + \frac{\xi_1 h}{\eta} + \\ + \frac{Ag\beta\Omega^2 h^7}{3\cdot 8!\,\nu^2\chi}(528-392Z^2+210Z^3-56Z^5+24Z^6-3Z^7)\Big] = \\ = \frac{Ag\beta\Omega^2 h^7}{3\cdot 8!\,\nu^2\chi}Z\Big[-3Z^7+24Z^6-56Z^5+210Z^3-392Z^2+528+ \\ + \frac{168\nu\chi}{\Omega^2 h^4}(4-6Z+4Z^2-Z^3) - \frac{4\cdot 7!\,\nu\chi\xi_2}{\eta Ag\beta\Omega h^4}(3-Z^2) + \frac{3\cdot 8!\,\nu^2\chi\xi_1 h}{Ag\beta\Omega^2 h^7\eta}\Big],$$

whence it follows that U_2 also can have up to three stagnant points (Fig. 3).

Thus, in the considered limiting cases ($A = 0$ and $C = 0$) with $\Omega = 0$, the velocity U can have no more than one critical point, and when the vertical twist ($\Omega \neq 0$) is taken into account, their number can reach three.

In the case $A \neq 0, C \neq 0$, the structure of the velocity U^0 (17) is similar to the structure of the expression (15) for the velocity V; namely, the expression of velocity U^0, as well as the velocity V, is determined by the superposition of three flows: two non-linear flows, caused by temperature factors, and one linear flow induced by tangential stresses specified on the upper boundary $z = h$. Consequently, the velocity U^0 (17) can have up to two stagnant points.

Let us now analyze how the velocity U determined by the expression (16) is affected by the contribution of the terms caused by the presence of a vertical twist. We write (16) as follows:

$$U = \frac{g\beta h^3}{6!\,\nu}Z\big[A(4-6Z+4Z^2-Z^3)+C(8-6Z+Z^3)\big] + \\ + \frac{g\beta\Omega^2 h^7}{3\cdot 8!\,\nu^2\chi}Z\big[A(528-392Z^2+210Z^3-56Z^5+24Z^6-3Z^7)+ \\ + C(648-448Z^2+210Z^3-28Z^5+3Z^7)\big] - \frac{\Omega\xi_2 h^3}{3!\,\eta\nu}Z(3-Z^2) + \frac{\xi_1 h}{\eta}Z =$$

$$= Z\Big[\frac{g\beta h^3}{720\nu}\big(Ag_1(Z)+Cg_2(Z)\big)+\frac{g\beta\Omega^2h^7}{120960\nu^2\chi}\big(Ag_3(Z)+Cg_4(Z)\big)- \\ -\frac{\Omega\xi_2h^3}{6\eta\nu}g_5(Z)+\frac{\xi_1h}{\eta}\Big]. \quad (19)$$

All the polynomials $g_i(Z)$ in (19) are strictly monotonic. The study of the spectral properties of the polynomial (19) shows that the velocity (19) can have no more than four stagnant points. The coefficient in front of g_5 and the free term are independent due to the arbitrariness of the choice of the control parameters ξ_1 and ξ_2; it can be seen from (19) that the coefficients in front of the polynomials g_1, g_2, g_3, and g_4 are related,

$$\frac{g\beta\Omega^2h^7}{3\cdot 8!\,\nu^2\chi}=\frac{g\beta h^3}{6!\,\nu}\cdot\frac{\Omega^2h^4}{168\nu\chi},$$

i.e. we have only three independent parameters for all these four coefficients, namely, A, C, Ω, and this imposes additional restrictions on the behavior of the function U if we consider the properties of the flow of a particular fluid in a horizontal layer of a fixed thickness h.

Thus, in the case $AC\neq 0$, the number of critical points of the velocity U, as in the cases $A=0$ and $C=0$, increases by two when the vertical twist characterized by the parameter Ω is taken into account. The qualitatively different profiles of the velocity U are shown in Fig. 4.

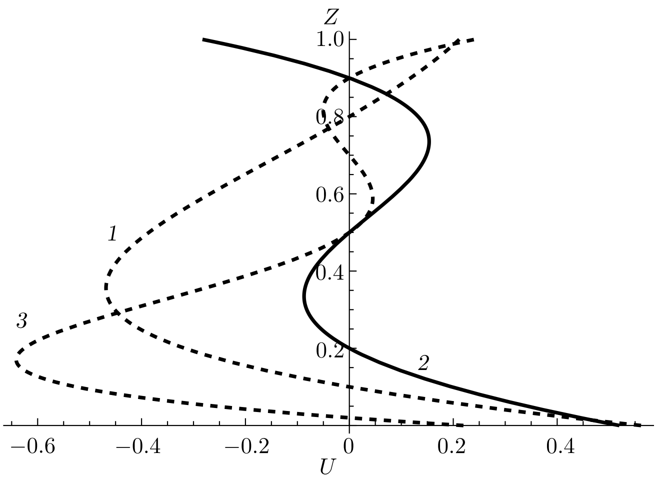

Figure 4. The profiles of the velocity U with different numbers of stagnant points: curve *1* — two stagnant points; curve *2* — three stagnant points; curve *3* — four stagnant points

Conclusion. This article provides a new exact solution for the Oberbeck–Boussinesq equations describing the shear convection of a vertically swirling fluid. The resulting exact solution allows you to resolve this overdetermined system. Fluid motion is induced by specifying heat sources at both boundaries of an infinite horizontal layer and taking into account external friction at the free boundary (specifying tangential stresses). It has been demonstrated that no more than two stagnant points can exist in a fluid, although one of the components of the velocity vector can vanish up to four times through the layer thickness.

Competing interests. We declare that we have no conflicts of interest in the authorship or publication of this contribution.

Authors' contributions and responsibilities. We are fully responsible for submitting the final manuscript in print. Each of us has approved the final version of the manuscript.

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References

1. Navier C. L. M. H. Mémoire sur les lois du mouvement des fluides, *Mémoires de l'Académie des sciences de l'Institut de France*, 1827, vol. 6, pp. 389–416.
2. Kochin N. E., Kibel I. A., Roze N. V. *Theoretical Hydromechanics*. New York, Wiley Interscience, 1964, v+577 pp.
3. Poisson S.-D. Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides, *Journal de l'École Polytechnique*, 1831, vol. 13, pp. 139–186.
4. de Saint-Venant B. Note à joindre au Mémoire sur la dynamique des fluides, *Comptes rendus*, 1843, vol. 17, pp. 1240–1244.
5. Stokes G. G. On the theories of internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, In: *Mathematical and Physical Papers*, vol. 1. Cambridge, Cambridge University Press, 2009, pp. 75–129. doi: 10.1017/CBO9780511702242.005; From the *Transactions of the Cambridge Philosophical Society*, 1880, vol. 8, pp. 287–319, https://archive.org/details/transactionsofca08camb.
6. Landau L. D., Lifshits E. M. *Course of Theoretical Physics*, vol. 6, Fluid Mechanics. New York, Pergamon Press, 1987, xiii+539 pp.
7. Gershuni G. Z., Zhukhovitskii E. M. *Convective Stability of Incompressible Fluids*. Jerusalem, Keter Publ. House, 1976, 330 pp.
8. Lin C. C. Note on a class of exact solutions in magneto-hydrodynamics, *Arch. Ration. Mech. Anal.*, 1858, vol. 1, pp. 391–395.
9. Drazin P. G. *Introduction to hydrodynamic stability*, Cambridge Texts in Applied Mathematics. Cambridge, Cambridge Univ. Press, 2002, xvii+258 pp. doi: 10.1017/cbo9780511809064.
10. Chandrasekhar S. *Hydrodynamic and hydromagnetic stability*, International Series of Monographs on Physics. Oxford, Clarendon Press, 1961, xix+652 pp.
11. Falkovich G. *Fluid mechanics. A short course for physicists*. Cambridge, Cambridge Univ. Press, 2011, xii+167 pp. doi: 10.1017/cbo9780511794353.
12. Grigoreva P. M. , Vilchevskaya E. N. Influence of diffusion models on chemical reaction front kinetics, *Diagnostics, Resource and Mechanics of materials and structures*, 2018, no. 6, pp. 59–82 (In Russian). doi: 10.17804/2410-9908.2018.6.059-082.
13. Nefedova O. A., Vykhodets V. B. A procedure for online investigation of deuterium diffusion in materials, *Diagnostics, Resource and Mechanics of materials and structures*, 2018, no. 5, pp. 57–63 (In Russian). doi: 10.17804/2410-9908.2018.5.057-063.
14. Kazakov A. L., Spevak L. F., Nefedova O. A. On the numerical-analytical approaches to solving a nonlinear heat conduction equation with a singularity, *Diagnostics, Resource and Mechanics of materials and structures*, 2018, no. 6, pp. 100-116 (In Russian). doi: 10.17804/2410-9908.2018.6.100-116.
15. Litvinenko M. V., Litvinenko Yu. A., Kozlov G. V., Vohirev V. V. Experimental investigation of a free round jet with Dean vortices, *Vestn. Novosib. Gos. Un-ta. Ser. Fizika*, 2014, vol. 9, no. 2, pp. 128–135 (In Russian).
16. Shtertser A. A., Grinberg B. E. Impact of a hydroabrasive jet on material: Hydroabrasive wear, *J. Appl. Mech. Tech. Phys.*, 2013, vol. 54, no. 2, pp. 508–516. doi: 10.1134/S002189441303022X.

17. Malikov Z. M., Stasenko A. L. Asymptotics of a submerged jet and transport processes in it, *Proceedings of Moscow Institute of Physics and Technology*, 2013, no. 5(2), pp. 59–68 (In Russian).
18. Moshkin N. P., Fomina A. V., Chernykh G. G. Numerical modelling of dynamics of turbulent wake behind towed body in the linearly stratified medium, *Matem. Mod.*, 2007, vol. 19, no. 1, pp. 29–56 (In Russian).
19. Ozmidov R. V., Nabatov V.N. Hydrophysical model of a turbulent wake behind an underwater mountain, *Izv., Atmos. Ocean. Phys.*, 1992, vol. 28, no. 9, pp. 981–987 (In Russian).
20. Balandina G. N., Papko V. V., Sergeev D. A., Troitskaya Yu. I. Evolution of the far turbulent wake behind a body towed in a stratified fluid with large Reynolds and Froude numbers, *Izv., Atmos. Ocean. Phys.*, 2004, vol. 40, no. 1, pp. 99–113.
21. Fabrikant N. Ya. *Aerodynamics*. Moscow, Nauka, 1964, 816 pp. (In Russian)
22. Kuznetsova Yu. L., Skulskiy O. I. Influence of Flow Regimes on the Stratification of a Shear Fluid Flow with a Non-monotonic Flow Curve, *J. Appl. Mech. Tech. Phys.*, 2019, no. 1, pp. 27–36 (In Russian). doi: 10.15372/PMTF20190101.
23. Georgievsky D. V. Tensor-nonlinear shear flows: Material functions and the diffusion-vortex solutions, *Rus. J. Nonlin. Dyn.*, 2011, vol. 7, no. 2, pp. 451–463. doi: 10.20537/nd1103004.
24. Georgievsky D. V. Generalized Joseph estimates of stability of plane shear flows with scalar nonlinearity, *Bull. Russ. Acad. Sci. Phys.*, 2011, vol. 75, no. 1, pp. 149–152. doi: 10.3103/S1062873810121044.
25. Lyapidevskiy V. Yu. A mixing layer in a homogeneous fluid, *J. Appl. Mech. Tech. Phys.*, 2000, vol. 41, no. 4, pp. 647–657. doi: 10.1007/BF02466862.
26. Taylor G. I. The transport of vorticity and heat through fluids in turbulent motion, *Proc. Roy. Soc. London. Ser. A.*, 1932, vol. 135, no. 828, pp. 685–705. doi: 10.1098/rspa.1932.0061.
27. Pukhnachev V. V. Exact solutions of the hydrodynamic equations derived from partially invariant solutions, *J. Appl. Mech. Tech. Phys.*, 2003, vol. 44, no. 3, pp. 317–323. doi: 10.1023/A:1023472921305.
28. Aristov S. N. A stationary cylindrical vortex in a viscous fluid, *Dokl. Phys.*, 2001, vol. 46, no. 4, pp. 251–253. doi: 10.1134/1.1371043 doi: 000168802500008.
29. Polyanin A. D., Aristov S. N. Systems of hydrodynamic type equations: Exact solutions, transformations, and nonlinear stability, *Dokl. Phys.*, 2009, vol. 54, no. 9, pp. 429–434. doi: 10.1134/S1028335809090079.
30. Burmasheva N. V., Prosviryakov E. Yu. A large-scale layered stationary convection of a incompressible viscous fluid under the action of shear stresses at the upper boundary. Velocity field investigation, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2017, vol. 21, no. 1, pp. 180–196 (In Russian). doi: 10.14498/vsgtu1527.
31. Burmasheva N. V., Prosviryakov E. Yu. A large-scale layered stationary convection of a incompressible viscous fluid under the action of shear stresses at the upper boundary. Temperature and presure field investigation, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2017, vol. 21, no. 4, pp. 736–751 (In Russian). doi: 10.14498/vsgtu1568.
32. Schwarz K. G. Plane-parallel advective flow in a horizontal incompressible fluid layer with rigid boundaries, *Fluid Dyn.*, 2014, vol. 49, no. 4, pp. 438–442. doi: 10.1134/S0015462814040036.
33. Knyazev D. V. Two-dimensional flows of a viscous binary fluid between moving solid boundaries, *J. Appl. Mech. Tech. Phys.*, 2011, vol. 52, no. 2, pp. 212–217. doi: 10.1134/S0021894411020088.
34. Sidorov A. F. Two classes of solutions of the fluid and gas mechanics equations and their connection to traveling wave theory, *J. Appl. Mech. Tech. Phys.*, 1989, vol. 30, no. 2, pp. 197–203. doi: 10.1007/BF00852164.

35. Burmasheva N. V., Prosviryakov E. Yu. Investigation of a velocity field for the Marangoni shear convection of a vertically swirling viscous incompressible fluid, *AIP Conference Proceedings*, 2018, vol. 2053, pp. 040011-1–040011-5, 040011. doi: 10.1063/1.5084449.
36. Burmasheva N. V., Prosviryakov E. Yu. Investigation of temperature and pressure fields for the Marangoni shear convection of a vertically swirling viscous incompressible fluid, *AIP Conference Proceedings*, 2018, vol. 2053, pp. 040012-1–040012-6, 040012. doi: 10.1063/1.5084450.
37. Nikulin V. V. Analytical model of motion of turbulent vortex rings in an incompressible fluid, *J. Appl. Mech. Techn. Phys.*, 2014, vol. 55, no. 4, pp. 558-564. doi: 10.1134/S0021894414040026.
38. Aristov S. N., Prosviryakov E. Yu. Stokes waves in vortical fluid, *Rus. J. Nonlin. Dyn.*, 2014, vol. 10, no. 3, pp. 309–318. doi: 10.20537/nd1403005.
39. Brutyan M. A., Kovalev V. E. Vortex Flows of a Micropolar Fluid, *Uchenye zapiski TsAGI*, 2010, vol. 41, no. 4, pp. 52–61 (In Russian).
40. Kovalev V. P., Sizykh G. B. Axisymmetric Helical Flows of an Ideal Fluid, *Proceedings of Moscow Institute of Physics and Technology*, 2016, vol. 8, no. 3, pp. 171–179 (In Russian).
41. Brutyan M. A., Krapivskiy P. L. The exactsolution of the Navier-Stokes equations for the evolution of the vortex structure in a generalized shear-flow, *Comput. Math. Math. Phys.*, 1992, vol. 32, no. 2, pp. 270–272.
42. Grinspen Kh. *Teoriya vraschayuschikhsya zhidkostey* [Theory of Rotating Fluids]. Leningrad, Gidrometeoizdat, 1975, 304 pp. (In Russian)
43. Morozov K. I. Rotation of a droplet in a viscous fluid, *J. Exp. Theor. Phys.*, 1997, vol. 85, no. 4, pp. 728–733. doi: 10.1134/1.558360.
44. Aristov S. N., Prosviryakov E. Yu. Nonuniform convective Couette flow, *Fluid Dyn.*, 2016, vol. 51, no. 5, pp. 581–587. doi: 10.1134/S001546281605001X.
45. Aristov S. N., Prosviryakov E. Yu. A new class of exact solutions for three-dimensional thermal diffusion equations, *Theor. Found. Chem. Eng.*, 2016, vol. 50, no. 3, pp. 286–293. doi: 10.1134/S0040579516030027.
46. Burmasheva N.V., Prosviryakov E. Yu. Exact solution for the layered convection of a viscous incompressible fluid at specified temperature gradients and tangential forces on the free boundary, *AIP Conference Proceedings*, 2017, vol. 1915, UNSP 040005. doi: 10.1063/1.5017353.
47. Chikulaev D. G., Shvarts K. G. Effect of rotation on the stability of advective flow in a horizontal liquid layer with solid boundaries at small Prandtl numbers, *Fluid Dyn.*, 2015, vol. 50, no. 2, pp. 215–222. doi: 10.1134/S0015462815020052.
48. Knutova N. S., Shvarts K. G. A study of behavior and stability of an advective thermocapillary flow in a weakly rotating liquid layer under microgravity, *Fluid Dyn.*, 2015, vol. 50, no. 3, pp. 340–350. doi: 10.1134/S0015462815030047.
49. Ekman V. W. On the influence of the Earth's rotation on ocean currents, *Ark. Mat. Astron. Fys.*, 1905, vol. 2, no. 11, pp. 1–52, http://jhir.library.jhu.edu/handle/1774.2/33989.
50. Couette M. Études sur le frottement des liquides, *Ann. Chim. Phys. Ser. 6*, 1890, vol. 21, pp. 433–510.
51. Kovalev V. P., Prosviryakov E. Yu., Sizykh G. B. Obtaining examples of exact solutions of the Navier–Stokes equations for helical flows by the method of summation of velocities, *Proceedings of Moscow Institute of Physics and Technology*, 2017, vol. 9, no. 1, pp. 71–88 (In Russian).
52. Birikh R. V. Thermocapillary convection in a horizontal layer of liquid, *J. Appl. Mech. Tech. Phys.*, 1966, vol. 7, no. 3, pp. 43–44. doi: 10.1007/BF00914697.
53. Ostroumov G. A. *Free convection under the condition of the internal problem*, NACA Technical Memorandum, vol. 1407. Washington, National Advisory Committee for Aeronautics, 1958, 239 pp., https://ntrs.nasa.gov/search.jsp?R=20030068786.
54. Berker R. A. *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible*, Thèses de l'entre-deux-guerres, no. 185, 1936, 176 pp., http://www.numdam.org/item/THESE_1936__185__1_0/.

55. Aristov S. N., Prosviryakov E. Yu. Unsteady layered vortical fluid flows, *Fluid Dyn.*, 2016, vol. 51, no. 2, pp. 148–154. doi: 10.1134/S0015462816020034.
56. Aristov S. N., Prosviryakov E. Yu. Large-scale flows of viscous incompressible vortical fluid, *Russian Aeronautics*, 2015, vol. 58, no. 4, pp. 413–418. doi: 10.3103/S1068799815040091.
57. Privalova V. V., Prosviryakov E. Yu. Couette–Hiemenz exact solutions for the steady creeping convective flow of a viscous incompressible fluid, with allowance made for heat recovery, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2018, vol. 22, no. 3, pp. 532–548. doi: 10.14498/vsgtu1638.
58. Prosviryakov E. Yu. New class of exact solutions of Navier-Stokes equations with exponential dependence of velocity on two spatial coordinates, *Theor. Found. Chem. Eng.*, 2019, vol. 53, no. 1, pp. 107–114. doi: 10.1134/S0040579518060088.
59. Aristov S. N., Prosviryakov E. Yu. Inhomogeneous Couette flow, *Rus. J. Nonlin. Dyn.*, 2014, vol. 10, no. 2, pp. 177–182. doi: 10.20537/nd1402004.
60. Prosviryakov E. Yu., Spevak L. F. Layered Three-Dimensional Nonuniform Viscous Incompressible Flows, *Theor. Found. Chem. Eng.*, 2018, vol. 52, no. 5, pp. 765–770. doi: 10.1134/S0040579518050391.
61. Burmasheva N. V., Prosviryakov E. Yu. Temperature field investigation in layered flows of a vertically swirling viscous incompressible fluid under two thermocapillar forces at a free boundary, *Diagnostics, Resource and Mechanics of materials and structures*, 2019, no. 1, pp. 6–42 (In Russian). doi: 10.17804/2410-9908.2019.1.006-042.

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Конвективные слоистые течения вертикально завихренной вязкой несжимаемой жидкости. Исследование поля скоростей

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Аннотация

Обсуждается разрешимость переопределенной системы уравнений тепловой конвекции в приближении Буссинеска. Система уравнений Обербека–Буссинеска, дополненная уравнением несжимаемости, является переопределенной. Количество уравнений превосходит количество неизвестных функций, поскольку изучаются неоднородные слоистые потоки вязкой несжимаемой жидкости (одна из компонент вектора скорости тождественно равна нулю). Проведено исследование разрешимости нелинейной системы уравнений Обербека–Буссинеска. Исследование разрешимости переопределенной системы нелинейных уравнений в частных производных Обербека–Буссинеска осуществлялось при помощи построения нескольких частных точных решений. Приведен новый класс точных решений для описания трехмерных нелинейных слоистых течений вертикальной завихренной вязкой несжимаемой жидкости. Вертикальная компонента завихренности в невращающейся жидкости генерируется неоднородным полем скоростей на нижней границе бесконечного горизонтального слоя жидкости. Конвекция в вязкой несжимаемой жидкости индуцируется линейными источниками тепла. Основное внимание уделено исследованию свойств поля скоростей течения. Исследована зависимость структуры этого поля от величины вертикальной закрутки. Показано, что одна из компонент вектора скорости при ненулевой вертикальной закрутке допускает расслоение на пять зон по толщине рассматриваемого слоя (четыре застойные точки). Анализ поля скоростей

Научная статья

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показал, что кинетическая энергия жидкости может дважды принимать нулевой значение по толщине слоя.

Ключевые слова: точное решение, слоистая конвекция, касательное напряжение, застойная точка, противотечение, стратификация, система уравнений Обербека–Буссинеска, вертикальная закрутка.

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