

Star versions of the Hurewicz basis covering property and strong measure zero spaces

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Abstract: In this paper, we introduced the star version of the Hurewicz basis covering property, studied by Babinkostova, Kočinac, and Scheepers in 2004. The authors obtained the relationship between star-selection principles and the star version of the Hurewicz basis covering property for metrizable spaces. We also consider the star version of Hurewicz measure zero spaces and characterized these new spaces by the star-selection principle related to star-Hurewicz spaces.

Key words: Hurewicz basis property, star-Hurewicz basis property, Hurewicz measure zero spaces, star-Hurewicz measure zero spaces

1. Introduction

The study of topological properties via various changes is not a new idea in topological spaces. The study of selection principles in topology and their relations to game theory and Ramsey theory was started by Scheepers [16] (see also [22]). In the last two decades, it has gained enough importance to become one of the most active areas of set theoretic topology. Thus, the study of covering properties has become an active area for research. In covering properties, the Hurewicz property is one of the most important properties. A number of results in the literature show that many topological properties can be described and characterized in terms of star covering properties (see [3, 9, 13, 17, 18]). The method of stars has been used to study the problem of metrization of topological spaces, and for definitions of several important classical topological notions. We use here such a method in investigation of selection principles for topological spaces.

In 1925, Hurewicz [20] (see also [21]) introduced the Hurewicz property in topological spaces and studied it. This property is stronger than Lindelöf and weaker than σ -compactness. In 2004, Bonanzinga et al. [14] introduced the star version of the Hurewicz property. Continuing these investigations, we study the Hurewicz basis covering property using star and also study strong measure zero spaces using star.

This paper is organized as follows. In Section 2, the definitions of the terms used in this paper are provided. In Section 3, we studied the star-Hurewicz basis property and star-Hurewicz measure zeroness for metrizable spaces.

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2. Preliminaries

Let (X, τ) or X be a topological space. We will denote by $Cl(A)$ and $Int(A)$ the closure of A and the interior of A , for a subset A of X , respectively. Throughout this paper, X stands for a Hausdorff topological space and the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. For the terms and symbols that we do not define, follow [19]. The basic definitions are given.

Let \mathcal{A} and \mathcal{B} be collections of open covers of a topological space X .

The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of \mathcal{A} there exists a sequence $\langle U_n : n \in \omega \rangle$ such that for each n , $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\} \in \mathcal{B}$ [16].

The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of \mathcal{A} there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigcup_{n \in \omega} \mathcal{V}_n$ is an element of \mathcal{B} [16].

In [7], Kočinac introduced star selection principles in the following way.

The symbol $S_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of \mathcal{A} there exists a sequence $\langle U_n : n \in \omega \rangle$ such that for each n , $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

The symbol $S_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of \mathcal{A} there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigcup_{n \in \omega} \{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of \mathcal{B} .

The symbol $U_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of \mathcal{A} there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$ or there is some n such that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = X$.

Here, as usual, for a subset A of a space X and a collection \mathcal{P} of subsets of X , $St(A, \mathcal{P})$ denotes the star of A with respect to \mathcal{P} , that is the set $\bigcup \{P \in \mathcal{P} : A \cap P \neq \emptyset\}$; for $A = \{x\}$, $x \in X$, we write $St(x, \mathcal{P})$ instead of $St(\{x\}, \mathcal{P})$.

In [8] it was explained that selection principles in uniform spaces are actually a kind of star selection principles.

In this paper \mathcal{A} and \mathcal{B} will be collections of the following open covers of a space X :

\mathcal{O} : the collection of all open covers of X .

Ω : the collection of ω -covers of X . An open cover \mathcal{U} of X is an ω -cover [4] if X does not belong to \mathcal{U} and every finite subset of X is contained in an element of \mathcal{U} .

Γ : the collection of γ -covers of X . An open cover \mathcal{U} of X is a γ -cover [4] if it is infinite and each $x \in X$ belongs to all but finitely many elements of \mathcal{U} .

\mathcal{O}^{gp} : the collection of groupable open covers. An open cover \mathcal{U} of X is groupable [11] if it can be expressed as a countable union of finite, pairwise disjoint subfamilies \mathcal{U}_n , such that each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many n .

Definition 2.1 [20] *A space X is said to have the Hurewicz property if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n .*

Definition 2.2 [14] *A space X is said to have the star-Hurewicz property if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and each $x \in X$ belongs to $St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n .*

3. Star-Hurewicz basis property and Star-Hurewicz measure zero property in metrizable spaces

In 1924, Menger defined the following basis property:

A metric space (X, d) is said to have the *Menger basis covering property* [20] if for each basis \mathcal{B} of metric space (X, d) there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of \mathcal{B} such that $\{U_n : n \in \omega\}$ is a cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$.

In 1925 [21], Hurewicz characterized this property by the Menger property for metrizable spaces.

In 2004 [6], Babinkostova et al. defined the following basis property :

A metric space (X, d) is said to have the *Hurewicz basis covering property* [6] if for each basis \mathcal{B} of metric space (X, d) there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of \mathcal{B} such that $\{U_n : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$.

Theorem 3.1 [6] *For a metric space (X, d) with no isolated points, X has the Hurewicz property if and only if it has the Hurewicz basis property.*

Since the Hurewicz property and star-Hurewicz property are equivalent in metrizable spaces. It can be noted that for a metric space (X, d) with no isolated points, X has the star-Hurewicz property if and only if it has the Hurewicz basis property. Now we consider corresponding basis properties to observe their relation.

First we define the star-Hurewicz basis property for metric spaces.

Definition 3.2 *A metric space (X, d) is said to have the star-Hurewicz basis property if for each basis \mathcal{B} of metric space (X, d) , there is a sequence $\langle V_n : n \in \omega \rangle$ of elements of \mathcal{B} such that $\{St(V_n, \mathcal{B}) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(V_n) = 0$.*

As the Hurewicz property implies the star-Hurewicz property but the Hurewicz basis property may not imply the star-Hurewicz basis property. For the implication, we need the following definition.

Definition 3.3 [12] *Let \mathcal{A} and \mathcal{B} be families of subsets of the infinite set S . Then $CDR_{subfin}^*(\mathcal{A}, \mathcal{B})$ denotes the statement that for each sequence $\langle A_n : n \in \omega \rangle$ of elements of \mathcal{A} there is a sequence $\langle B_n : n \in \omega \rangle$ such that for each n , $B_n \subseteq A_n$ and for $m \neq n$ and for each finite subset $C_n \subseteq B_n$ and $C_m \subseteq B_m$, $\{St(B, A_m) : B \in C_m\} \cap \{St(B, A_n) : B \in C_n\} = \emptyset$, and each B_n is a member of \mathcal{B} .*

For the next theorem, \mathcal{B} denotes the collection of all the basis of metric space (X, d) .

Theorem 3.4 *For a metric space (X, d) if X has the star Hurewicz property and $CDR_{subfin}^*(\mathcal{B}, \mathcal{B})$ holds, then X has the star-Hurewicz basis property.*

Proof Let X be having the star-Hurewicz property and \mathcal{B} be a basis of X . By $CDR_{subfin}^*(\mathcal{B}, \mathcal{B})$, there is a sequence $\langle \mathcal{B}'_n (\subseteq \mathcal{B}) : n \in \omega \rangle$ of basis such that for $m \neq n$ and for each finite subsets $\mathcal{V}_n \subseteq \mathcal{B}'_n$ and $\mathcal{V}_m \subseteq \mathcal{B}'_m$, $\{St(V, \mathcal{B}) : V \in \mathcal{V}_n\} \cap \{St(V, \mathcal{B}) : V \in \mathcal{V}_m\} = \emptyset$. Now define

$$\mathcal{U}_n = \{U \in \mathcal{B}'_n : \text{diam}_d(U) < 1/(n + 1)\}.$$

Then for each n , \mathcal{U}_n is an open cover. Then by the star-Hurewicz property, there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and for each $x \in X$, $x \in \text{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n .

Since $\mathcal{U}_n \subseteq \mathcal{B}'_n \subseteq \mathcal{B}$,

$$\text{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) \subseteq \text{St}(\bigcup \mathcal{V}_n, \mathcal{B}'_n) \subseteq \text{St}(\bigcup \mathcal{V}_n, \mathcal{B}).$$

Then for each $x \in X$, $x \in \text{St}(\bigcup \mathcal{V}_n, \mathcal{B})$ for all but finitely many n . As

$$\text{St}(\bigcup \mathcal{V}_n, \mathcal{B}) = \bigcup \{ \text{St}(V, \mathcal{B}) : V \in \mathcal{V}_n \}$$

and

$$\{ \text{St}(V, \mathcal{B}) : V \in \mathcal{V}_n \} \cap \{ \text{St}(V, \mathcal{B}) : V \in \mathcal{V}_m \} = \emptyset \text{ for } m \neq n.$$

Then $\{ \text{St}(V_n, \mathcal{B}) : n \in \omega \}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} \text{diam}_d(V_n) = 0$. □

Corollary 3.5 *For a metric space (X, d) with no isolated points, if X has the Hurewicz basis property and $CDR^*_{\text{subfin}}(\mathcal{B}, \mathcal{B})$ holds, then X has the star-Hurewicz basis property.*

For the converse, we will drop conditions on diameter and need $CDR_{\text{sub}}(\mathcal{A}, \mathcal{B})$. Let \mathcal{A} and \mathcal{B} be families of subsets of the infinite set S . Then $CDR_{\text{subfin}}(\mathcal{A}, \mathcal{B})$ [16] denotes the statement that for each sequence $\langle A_n : n \in \omega \rangle$ of elements of \mathcal{A} there is a sequence $\langle B_n : n \in \omega \rangle$ such that for each n , $B_n \subseteq A_n$ and for $m \neq n$ and for each finite subset $C_n \subseteq B_n$ and $C_m \subseteq B_m$, $\{B : B \in C_n\} \cap \{B : B \in C_m\} = \emptyset$, and each B_n is a member of \mathcal{B} .

In the next theorem, \mathcal{B} denotes the collection of all the basis of metric space (X, d) .

Theorem 3.6 *For a metric space (X, d) if for each basis \mathcal{B} of metric space (X, d) , there is a sequence $\langle V_n : n \in \omega \rangle$ of elements of \mathcal{B} such that $\{ \text{St}(V_n, \mathcal{B}) : n \in \omega \}$ is a groupable cover of X and $CDR_{\text{sub}}(\mathcal{B}, \mathcal{B})$ holds, then there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of \mathcal{B} such that $\{U_n : n \in \omega\}$ is a groupable cover of X .*

Proof Let \mathcal{B} be a basis of X . Since every metric space is paracompact and Hausdorff space, by the Stone characterization of paracompactness [19], \mathcal{B} has an open star-refinement, say \mathcal{B}' . Then $\mathcal{B} \wedge \mathcal{B}'$ is a basis of X : for it, let $x \in X$ and $x \in W$, for some open W in (X, d) . Since \mathcal{B} is a basis for X , there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq W$. As \mathcal{B}' is an open cover, there is $U \in \mathcal{B}'$ such that $x \in U$. Then

$$x \in B \cap U \subseteq B \subseteq W \text{ and } U \cap B \in \mathcal{B}' \wedge \mathcal{B}.$$

Thus, $\mathcal{B}' \wedge \mathcal{B}$ is a basis of (X, d) .

Now by given hypothesis, there is a sequence $\langle V_n : n \in \omega \rangle$ of elements of $\mathcal{B}' \wedge \mathcal{B}$ such that $\{ \text{St}(V_n, \mathcal{B}' \wedge \mathcal{B}) : n \in \omega \}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} \text{diam}_d(V_n) = 0$.

For each $V_n \in \mathcal{B}' \wedge \mathcal{B}$, there is $B_n \in \mathcal{B}$ and $B'_n \in \mathcal{B}'$ such that $V_n = B_n \cap B'_n$. Therefore, for each $B'_n \in \mathcal{B}'$, let $U_{B'_n}$ be a member of \mathcal{B} such that $\text{St}(B'_n, \mathcal{B}') \subseteq U_{B'_n}$. Also $\text{St}(V_n, \mathcal{B}' \wedge \mathcal{B}) \subseteq \text{St}(B'_n, \mathcal{B}') \subseteq U_{B'_n} \in \mathcal{B}$.

For every n , $\mathcal{H}_n = \{U_{B'_n} : B'_n \in \mathcal{B}'\}$ is a finite subset of \mathcal{B} and $\{U_{B'_n} : n \in \omega\}$ is a groupable cover of X . □

If we drop the hypothesis $CDR_{subfin}^*(\mathcal{B}, \mathcal{B})$ in Theorem 3.4, then the star Hurewicz property may not imply the star-Hurewicz basis property. In that case it implies the following basis property.

Theorem 3.7 *For a metric space (X, d) , if X has the star-Hurewicz property, then for each basis \mathcal{B} there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{B} and $\{St(\bigcup \mathcal{V}_n, \mathcal{B}) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$ for each $U_n \in \mathcal{V}_n$.*

Proof Let X have the star-Hurewicz property and \mathcal{B} be a basis of X . Now define

$$\mathcal{U}_n = \{U \in \mathcal{B} : diam_d(U) < 1/(n + 1)\}.$$

Then for each n , \mathcal{U}_n is an open cover. By the star-Hurewicz property of X , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n .

Since $\mathcal{U}_n \subseteq \mathcal{B}$, $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) \subseteq St(\bigcup \mathcal{V}_n, \mathcal{B})$.

Then for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n, \mathcal{B})$ for all but finitely many n . Thus, $\{St(\bigcup \mathcal{V}_n, \mathcal{B}) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(V_n) = 0$ for each $V_n \in \mathcal{V}_n$. □

Now we consider some conditions related to the star-Hurewicz basis property to characterize the star-Hurewicz property.

Theorem 3.8 *Let (X, d) be a metric space with no isolated points. If X satisfies $U_{fin}^*(\mathcal{O}, \mathcal{O}^{gp})$ or has the star-Hurewicz property, then for each basis \mathcal{B} of metric space (X, d) and for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of (X, d) , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite sets of elements of $\mathcal{B} \wedge \mathcal{U}_n$ such that $\{St(\bigcup \mathcal{V}_n, \mathcal{B} \wedge \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$ for $U_n \in \mathcal{V}_n$.*

Proof Suppose X satisfies $U_{fin}^*(\mathcal{O}, \mathcal{O}^{gp})$. Let \mathcal{B} be a basis of X . Consider any arbitrary sequence $\langle \mathcal{W}_n : n \in \omega \rangle$ of open covers of X . Then

$$\mathcal{W}_n \wedge \mathcal{B} = \{U \cap B : U \in \mathcal{W}_n, B \in \mathcal{B}\}$$

is a basis for each n : For it, let $x \in X$ and $x \in W$, for some open W in (X, d) . Since \mathcal{B} is a basis for X , there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq W$. As \mathcal{W}_n is an open cover for each n , there is $U \in \mathcal{W}_n$ such that $x \in U$. Then $x \in B \cap U \subseteq B \subseteq W$ and $U \cap B \in \mathcal{W}_n \wedge \mathcal{B}$. Therefore, $\mathcal{W}_n \wedge \mathcal{B}$ is a basis of (X, d) .

Now define

$$\mathcal{U}_n = \{U \in \mathcal{W}_n \wedge \mathcal{B} : diam_d(U) < 1/(n + 1)\}.$$

Then for each n , \mathcal{U}_n is an open cover of X . Since X satisfies $U_{fin}^*(\mathcal{O}, \mathcal{O}^{gp})$, there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of $\mathcal{U}_n \subseteq \mathcal{W}_n \wedge \mathcal{B}$ and for each $y \in X$, $y \in \{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ for all but finitely many n . It is clear that for each n ,

$$St(\bigcup \mathcal{V}_n, \mathcal{U}_n) \subseteq St(\bigcup \mathcal{V}_n, \mathcal{W}_n \wedge \mathcal{B}).$$

Then $\{St(\bigcup \mathcal{V}_n, \mathcal{W}_n \wedge \mathcal{B}) : n \in \omega\}$ is a groupable cover of X . Also $\lim_{n \rightarrow \infty} diam_d(V_n) = 0$ for $V_n \in \mathcal{V}_n$. □

The following theorem gives the condition on basis property which proves converse of above theorem.

Theorem 3.9 *Let (X, d) be a metric space with no isolated points. If for each basis \mathcal{B} of metric space (X, d) and for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of (X, d) , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite sets of elements of \mathcal{B} such that $\{St(\bigcup \mathcal{V}_n, \mathcal{B} \wedge \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$ for $U_n \in \mathcal{V}_n$, then X has the star-Hurewicz property.*

Proof Suppose X is a space satisfying the hypothesis. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of X . Now assume that if an open set V is a subset of an element of \mathcal{U}_n , then $V \in \mathcal{U}_n$. For each n define

$$\mathcal{H}_n = \{U_1 \cap U_2 \cap \dots \cap U_n : (\forall i \leq n)(U_i \in \mathcal{U}_i)\} \setminus \{\emptyset\}.$$

Then for each n , \mathcal{H}_n is an open cover of X and has the property that if an open set V is a subset of an element of \mathcal{H}_n , then $V \in \mathcal{H}_n$.

Now let

$$\mathcal{U} = \{U \cup V : (\exists n)(U, V \in \mathcal{H}_n \text{ and } diam_d(U \cup V) > 1/n)\}.$$

First we show that \mathcal{U} is a basis for X . For it, let W be an open subset containing a point x . Since (X, d) does not have isolated points, x is not an isolated point of X and we can choose $y \in W \setminus \{x\}$ and $n > 1$ with $d(x, y) > 1/n$. Since \mathcal{H}_n is an open cover of X , there are $U', V' \in \mathcal{H}_n$ such that $x \in U'$ and $y \in V'$.

Now put

$$U = U' \cap W \setminus \{y\} \text{ and } V = V' \cap W \setminus \{x\}.$$

Then $U, V \in \mathcal{H}_n$ since \mathcal{H}_n has the property that if an open set V is a subset of an element of \mathcal{U}_n , then $V \in \mathcal{U}_n$. Also $U \cup V \subseteq W$ and $diam_d(U \cup V) \geq d(x, y) > 1/n$. Therefore, $U \cup V \in \mathcal{U}$ and $x \in U \cup V \subseteq W$. Thus, \mathcal{U} is a basis for X .

Then

$$\mathcal{H}_n \wedge \mathcal{U} = \{U \cap B : U \in \mathcal{H}_n, B \in \mathcal{U}\}$$

is a basis for each n : For it, let $x \in X$ and $x \in W$, for some open W in (X, d) . Since \mathcal{U} is a basis for X , there is $B \in \mathcal{U}$ such that $x \in B$ and $B \subseteq W$. As \mathcal{H}_n is an open cover for each n , there is $U \in \mathcal{H}_n$ such that $x \in U$. Then

$$x \in B \cap U \subseteq B \subseteq W \text{ and } U \cap B \in \mathcal{H}_n \wedge \mathcal{U}.$$

Therefore, $\mathcal{H}_n \wedge \mathcal{U}$ is a basis of (X, d) .

By given hypothesis on X , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite subsets of \mathcal{U} such that $\{St(\bigcup \mathcal{V}_n, \mathcal{H}_n \wedge \mathcal{U}) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(W_n) = 0$ for each $W_n \in \mathcal{V}_n$.

Then

$$\bigcup_{n \in \omega} \mathcal{W}_n = \{St(\bigcup \mathcal{V}_n, \mathcal{H}_n \wedge \mathcal{U}) : n \in \omega\}$$

such that each \mathcal{W}_n is finite, $\mathcal{W}_n \cap \mathcal{W}_m = \emptyset$ for $n \neq m$ and each $x \in X$, $x \in \bigcup \mathcal{W}_n$ for all but finitely many n .

Let

$$\mathcal{W}_1 = \{St(\bigcup \mathcal{V}_{m_1^1}, \mathcal{H}_{m_1^1} \wedge \mathcal{U}), \dots, St(\bigcup \mathcal{V}_{m_{i_1}^1}, \mathcal{H}_{m_{i_1}^1} \wedge \mathcal{U})\};$$

$$\mathcal{W}_2 = \{St(\bigcup \mathcal{V}_{m_1^2}, \mathcal{H}_{m_1^2} \wedge \mathcal{U}), \dots, St(\bigcup \mathcal{V}_{m_{i_2}^2}, \mathcal{H}_{m_{i_2}^2} \wedge \mathcal{U})\};$$

⋮
⋮
⋮

$$W_k = \{St(\bigcup \mathcal{V}_{m_1^k}, \mathcal{H}_{m_1^k} \wedge \mathcal{U}), \dots, St(\bigcup \mathcal{V}_{m_{i_k}^k}, \mathcal{H}_{m_{i_k}^k} \wedge \mathcal{U})\};$$

and so on.

Then $m_1 < m_2 < m_3 < \dots < m_k < \dots$ is a sequence obtained from groupability of $\{St(\bigcup \mathcal{V}_n, \mathcal{H}_n \wedge \mathcal{U}) : n \in \omega\}$ such that for each $y \in X$, for all but finitely many k there is a j with $m_k \leq j < m_{k+1}$ such that $y \in St(\bigcup \mathcal{V}_j, \mathcal{H}_j \wedge \mathcal{U})$.

Since $W_n \in \mathcal{V}_m \subseteq \mathcal{U}$, so there is k_n such that

$$U_n, V_n \in \mathcal{H}_{k_n} \text{ and } W_n = U_n \cup V_n \text{ with } diam_d(W_n) > 1/k_n.$$

For each m and for each $W_n \in \mathcal{V}_m$, select the least k_n and sets U_n and V_n from \mathcal{U}_{k_n} . Since each U_n has the property that if an open set V is a subset of an element of \mathcal{U}_n , then $V \in \mathcal{U}_n$, $U_n, V_n \in \mathcal{U}_{k_n}$. Since $\lim_{n \rightarrow \infty} diam_d(W_n) = 0$,

$$\text{for each } W_n, \text{ there is maximal } m_n \text{ such that } diam_d(W_n) < 1/m_n.$$

Then $1/k_n < diam_d(W_n) < 1/m_n$ implies that $k_n > m_n$ for each n and $\lim_{n \rightarrow \infty} m_n = \infty$. Since $\lim_{n \rightarrow \infty} diam_d(W_n) = 0$, so for each k_n , there are only finitely many W_n for which the representatives U_n, V_n are from \mathcal{U}_{k_n} and have $diam_d(U_n \cup V_n) > 1/k_n$. Let \mathcal{V}_{k_n} be the finite set of such U_n, V_n .

Now choose $l_1 > 1$ so large such that each W_i with $i \leq m_1$ has a representation of the form $U \cup V$ and U 's and V 's are from the sets $\mathcal{V}_{k_i}, k_i \leq l_1$. Then select j_1 so large such that for all $i > j_1$, if W_i has representatives from \mathcal{V}_{k_i} , then $k_i > l_1$.

For choosing l_2 , let m_k be least larger than j_1 , and now choose l_2 so large that if W_i with $m_k \leq i < m_{k+1}$ uses a \mathcal{V}_{k_i} , then $k_i \leq l_2$, that is, choose maximal of k_i for which $m_k \leq i < m_{k+1}$ and say l_2 , then $l_1 < k_i \leq l_2$. Now choose maximal of i for which the representation of W_i from \mathcal{V}_{k_i} where $l_1 < k_i \leq l_2$ and say j_2 , then $j_2 > j_1$ and $\forall i \geq j_2$ if W_i uses \mathcal{V}_{k_n} , then $k_n > l_2$.

Similarly we choose l_m and j_m alternately. For each m if we consider the least $m_k > l_m$, then :

1. if W_i with $m_k \leq i < m_{k+1}$ uses a \mathcal{V}_{k_i} then $l_m < k_i \leq l_{m+1}$;
2. if $i \geq j_m$ then if W_i uses \mathcal{V}_{k_n} then $k_n > l_m$.

For each $V \in \mathcal{V}_{k_n}$ with $k_n \leq l_1$, $V \in \mathcal{H}_{k_n}$ and $V \in \mathcal{U}_{k_i}$ for each $k_i \leq k_n$. Then $V \in \mathcal{U}_1$ and let \mathcal{G}_1 be collection of such $V \in \mathcal{V}_{k_n}$ with $k_n \leq l_1$. Then $\mathcal{G}_1 \subseteq \mathcal{U}_1$ is a finite subset.

Now for each $V \in \mathcal{V}_{k_n}$ with $l_p < k_n \leq l_{p+1}$ (as $p < l_p$), $V \in \mathcal{H}_{k_n}$ and $V \in \mathcal{U}_{k_i}$ for each $k_i \leq k_n$. Then $V \in \mathcal{U}_{l_p}$ and let \mathcal{G}_p be collection of such $V \in \mathcal{V}_{k_n}$ with $l_p < k_n \leq l_{p+1}$. Then $\mathcal{G}_p \subseteq \mathcal{U}_p$ is a finite subset.

Then we have that for each $y \in X$, for all but finitely many $p, y \in St(\bigcup \mathcal{G}_p, \mathcal{U}_p)$. It follows that X satisfies $U_{fin}^*(\mathcal{O}, \mathcal{O}^{gp})$ or has the star-Hurewicz property. □

Theorem 3.10 *Let (X, d) be a metric space with no isolated points. The following statements are equivalent:*

1. X satisfies $U_{fin}^*(\mathcal{O}, \mathcal{O}^{gp})$ or has the star-Hurewicz property;

2. for each basis \mathcal{B} of metric space (X, d) and for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of (X, d) , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite sets of elements of $\mathcal{B} \wedge \mathcal{U}_n$ such that $\{St(\bigcup \mathcal{V}_n, \mathcal{B} \wedge \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} \text{diam}_d(U_n) = 0$ for $U_n \in \mathcal{V}_n$;
3. for each basis \mathcal{B} of metric space (X, d) and for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of (X, d) , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite sets of elements of \mathcal{B} such that $\{St(\bigcup \mathcal{V}_n, \mathcal{B} \wedge \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} \text{diam}_d(U_n) = 0$ for $U_n \in \mathcal{V}_n$.

Proof (1) \Rightarrow (2) follows from Theorem 3.8.

(2) \Rightarrow (3) is obvious since the sets in $\mathcal{B} \wedge \mathcal{U}_n$ are subsets of \mathcal{B} .

(3) \Rightarrow (1) follows from Theorem 3.9. □

Recall that a set of reals X is *null* (or has measure zero) if for each positive ϵ there exists a cover $\{I_n\}_{n \in \omega}$ of X such that $\sum_n \text{diam}(I_n) < \epsilon$.

To generalize the notion of measure zero or null set, in 1919 [2], Borel defined a notion stronger than measure zeroness. Now this notion is known as strong measure zeroness or strongly null set.

Borel strong measure zero: Y is *Borel strong measure zero* if there is for each sequence $\langle \epsilon_n : n \in \omega \rangle$ of positive real numbers a sequence $\langle J_n : n \in \omega \rangle$ of subsets of Y such that each J_n is of diameter $< \epsilon_n$, and Y is covered by $\{J_n : n \in \omega\}$.

However, Borel was unable to construct a nontrivial (that is, an uncountable) example of a strongly null set. He, therefore, conjectured that there exists no such examples.

In 1928, Sierpinski observed that every Luzin set is strongly null, thus the Continuum Hypothesis implies that Borel's Conjecture is false.

Sierpinski asked whether the property of being strongly null is preserved under taking homeomorphic (or even continuous) images.

In 1941, the answer given by Rothberger is negative under the Continuum Hypothesis. This led Rothberger to introduce the following topological version of strong measure zero (which is preserved under taking continuous images).

A space X is said to have the Rothberger property if it satisfies the selection principles $S_1(\mathcal{O}, \mathcal{O})$.

In 1988 [1], Miller and Fremlin proved that a space Y has the Rothberger property ($S_1(\mathcal{O}, \mathcal{O})$) if and only if it has Borel strong measure zero with respect to each metric on Y which generates the topology of Y .

In 2004 [6], Babinkostova et al. defined the following property: a metric space (X, d) is Hurewicz measure zero if for each sequence $\langle \epsilon_n : n \in \omega \rangle$ of positive real numbers there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that:

1. for each n , \mathcal{V}_n is a finite set of open subsets in X ;
2. for each n , each member of \mathcal{V}_n has d -diameter less than ϵ_n ;
3. $\bigcup_{n \in \omega} \mathcal{V}_n$ is a groupable cover of X .

Theorem 3.11 [6] *Let (X, d) be a zero-dimensional separable metric space with no isolated points. The following statements are equivalent:*

1. X has the Hurewicz property;

2. X has the Hurewicz measure zero property.

Now we give similar description for star selection principle related to the star-Hurewicz property.

First we define star version of Hurewicz measure zero property as follows :

Definition 3.12 A metric space (X, d) is star-Hurewicz measure zero if for each sequence $\langle \epsilon_n : n \in \omega \rangle$ of positive real numbers there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that:

1. for each n , \mathcal{V}_n is a finite set of open subsets of X ;
2. for each n , each member of \mathcal{V}_n has d -diameter less than ϵ_n ;
3. $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X , where $\mathcal{U}_n = \{U \subset X : U \text{ is open set with } diam_d(U) < \epsilon_n\}$ for each n .

Theorem 3.13 Let (X, d) be a zero-dimensional separable metric space with no isolated points. The following statements are equivalent:

1. X has the star-Hurewicz property;
2. X is star-Hurewicz measure zero with respect to every metric which gives X the same topology as d does.

Proof For (1) \Rightarrow (2), let X have the star-Hurewicz property and $\langle \epsilon_n : n \in \omega \rangle$ be a sequence of positive real numbers. For each n , define

$$\mathcal{U}_n = \{U \subset X : U \text{ is open set with } diam_d(U) < \epsilon_n\}.$$

Then \mathcal{U}_n is a large open cover of X for each n . Apply star-Hurewicz property to $\langle \mathcal{U}_n : n \in \omega \rangle$, there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n , \mathcal{V}_n is a finite subset of \mathcal{U}_n and for each $y \in X$, $y \in St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n . Thus, $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and X is star-Hurewicz measure zero with respect to every metric on X which gives X the same topology as d does.

For (2) \Rightarrow (1), let d be an arbitrary metric on X which gives X the same topology as the original one. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of X . Since X is a zero-dimensional metric space, replace \mathcal{U}_n by

$$\{U \subseteq X : U \text{ clopen, } diam_d(U) < 1/n \text{ and } \exists V \in \mathcal{U}_n \text{ such that } U \subseteq V\} \text{ for each } n.$$

Also X is a separable metric space, replace last cover by a countable subcover $\{U_m : m \in \omega\}$. Since the cover is countable and sets are clopen, we can make it disjoint clopen cover refining \mathcal{U}_n for each n . Also for each n , each member of this new cover has $diam_d \leq 1/n$. Also by taking intersections of each new cover with the next new cover we obtain new cover. Now name this new cover \mathcal{U}_n^* for each n . Therefore, $\langle \mathcal{U}_n^* : n \in \omega \rangle$ is a sequence of open covers such that for each n :

1. \mathcal{U}_n^* is clopen disjoint cover of X refining \mathcal{U}_n ;
2. for each $V \in \mathcal{U}_n^*$, $diam_d(V) \leq 1/n$;
3. \mathcal{U}_{n+1}^* refines \mathcal{U}_n^* .

Now define a metric d^* on X by $d^*(x, y) = 1/n + 1$ where n is the least such that there exist $U \in \mathcal{U}_n^*$ with $x \in U$ and $y \notin U$. It can be easily seen that d^* generates the same topology on X as d does. Then X has star-Hurewicz measure zero with respect to d^* . By setting $\epsilon_n = 1/n + 1$ for each n , there are finite sets \mathcal{V}_n such that $diam_d^*(U)$ is less than $\epsilon_n (= 1/n + 1)$ whenever $U \in \mathcal{V}_n$, and $\{St(\bigcup \mathcal{V}_n, \mathcal{H}_n) : n \in \omega\}$ is a groupable cover of X , where

$$\mathcal{H}_n = \{U \subset X : U \text{ is an open set in } (X, d^*) \text{ with } diam_d^*(U) < \epsilon_n\}.$$

Let $\langle \mathcal{W}_n : n \in \omega \rangle$ be a sequence of finite subsets of $\{St(\bigcup \mathcal{V}_n, \mathcal{H}_n) : n \in \omega\}$ such that $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$ whenever $m \neq n$, and $\bigcup_{n \in \omega} \mathcal{W}_n = \{St(\bigcup \mathcal{V}_n, \mathcal{H}_n) : n \in \omega\}$, and for each $y \in X$, for all but finitely many $n, y \in \bigcup \mathcal{W}_n$.

Now

$$\mathcal{W}_1 = \{St(\bigcup \mathcal{V}^{k_1}, \mathcal{H}_{k_1}), St(\bigcup \mathcal{V}^{k_2}, \mathcal{H}_{k_2}), \dots, St(\bigcup \mathcal{V}^{k_n}, \mathcal{H}_{k_n})\}$$

and say $i_1 = k_n$. Since \mathcal{W}_n 's are finite and pairwise disjoint, so the set $\{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i \leq i_1\}$ is finite and exhausted in a finite number of \mathcal{W}_k 's and choose j_1 such that for each $k \geq j_1$,

$$St(\bigcup \mathcal{V}_k, \mathcal{H}_k) \notin \{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i \leq i_1\}.$$

Now

$$\mathcal{W}_{j_1} = \{St(\bigcup \mathcal{V}^{t_1}, \mathcal{H}_{t_1}), St(\bigcup \mathcal{V}^{t_2}, \mathcal{H}_{t_2}), \dots, St(\bigcup \mathcal{V}^{t_{j_1}}, \mathcal{H}_{t_{j_1}})\}$$

and say $i_2 = t_{j_1}$. Since \mathcal{W}_n 's are finite and pairwise disjoint, the set $\{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i_1 < i \leq i_2\}$ is finite and exhausted in a finite number of \mathcal{W}_k 's and choose j_2 such that for each $k \geq j_2$,

$$St(\bigcup \mathcal{V}_k, \mathcal{H}_k) \notin \{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i_1 < i \leq i_2\}.$$

Alternatively, we choose sequences $1 < i_1 < i_2 < \dots < i_m < \dots$ and $j_0 = 1 < j_1 < j_2 < \dots < j_m < \dots$ such that :

1. Each element of \mathcal{W}_1 belongs to $\{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i \leq i_1\}$;
2. For each $i \geq j_k$, if $U \in \mathcal{W}_{j_k}$, then $U \notin \{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i \leq i_k\}$;
3. Each element of \mathcal{W}_{j_k} belongs to $\{St(\bigcup \mathcal{V}_i, \mathcal{H}_i) : i_k < i \leq i_{k+1}\}$.

Then for each element $V \in \mathcal{V}_i$ such that $St(\bigcup \mathcal{V}_i, \mathcal{H}_i)$ is an element of \mathcal{W}_{j_k} , V has d^* -diameter less than $\epsilon_{i_k} = 1/i_k + 1 \leq 1/k + 1$ since $i_k \geq k$. As V is open set in (X, d^*) and $diam_d^*(V) < 1/k + 1$, then $V \subseteq B_d^*(x, 1/k + 1)$, where $B_d^*(x, 1/k + 1)$ is an open ball centered at $x \in V$ and of radius $1/k + 1$ in (X, d^*) .
Now

$$B_d^*(x, 1/k + 1) = \{y \in X : d^*(x, y) < 1/k + 1\}.$$

Therefore, for each $y \in B_d^*(x, 1/k + 1)$, $d^*(x, y) < 1/k + 1$, there is $U \in \mathcal{U}_n^*$ such that $x \in U$ and $y \notin U$ for some $n > k$. Therefore, for all $k \leq n$, there is no set $U \in \mathcal{U}_k^*$ such that $x \in U$ and $y \notin U$. Thus, for all $k \leq n$, there is a set $U \in \mathcal{U}_k^*$ such that $x, y \in U$ for all $x, y \in B_d^*(x, 1/k + 1)$, that is, $B_d^*(x, 1/k + 1) \subseteq U \in \mathcal{U}_k^*$ for all $k \leq n$.

Thus, by definition of d^* , for each element $V \in \mathcal{V}_i$ such that $St(\bigcup \mathcal{V}_i, \mathcal{H}_i)$ is an element of \mathcal{W}_{j_k} , V is a subset of an element of \mathcal{U}_k^* , each of which in turn is a subset of an element of \mathcal{U}_k . For each k and for each element $V \in \mathcal{V}_i$ such that $St(\bigcup \mathcal{V}_i, \mathcal{H}_i)$ is an element of \mathcal{W}_{j_k} , choose a $U \in \mathcal{U}_k$ with $V \subseteq U$ and let \mathcal{T}_k be the finite set of such chosen U 's. Then \mathcal{T}_n is a finite subset of \mathcal{U}_n for each n and $\bigcup \mathcal{V}_i \subseteq \bigcup \mathcal{T}_n$ for each $St(\bigcup \mathcal{V}_i, \mathcal{H}_i) \in \mathcal{W}_{j_n}$.

Now it is enough to show that if for each n , $V \in \mathcal{H}_n$, then there is $U \in \mathcal{U}_n$ such that $V \subseteq U$.

Let $V \in \mathcal{H}_n$, V is open set in (X, d^*) and $diam_d^*(V) < 1/n + 1$, then $V \subseteq B_d^*(x, 1/n + 1)$, where $B_d^*(x, 1/n + 1)$ is an open ball centered at $x \in V$ and of radius $1/n + 1$ in (X, d^*) . Now

$$B_d^*(x, 1/n + 1) = \{y \in X : d^*(x, y) < 1/n + 1\}.$$

Therefore, for each $y \in B_d^*(x, 1/n + 1)$, $d^*(x, y) < 1/n + 1$, there is $U \in \mathcal{U}_k^*$ such that $x \in U$ and $y \notin U$ for some $k > n$. Therefore, for all $k \leq n$, there is no set $U \in \mathcal{U}_k^*$ such that $x \in U$ and $y \notin U$. Thus, for all $k \leq n$, there is a set $U \in \mathcal{U}_k^*$ such that $x, y \in U$ for all $x, y \in B_d^*(x, 1/n + 1)$, that is, $B_d^*(x, 1/n + 1) \subseteq U \in \mathcal{U}_k^*$ for all $k \leq n$.

By definition of \mathcal{U}_n^* , for each element $V \in \mathcal{U}_n^*$ is a subset of an element of \mathcal{U}_n . For each $V \in \mathcal{V}_n \subseteq \mathcal{H}_n$, there is $U \in \mathcal{U}_n$ such that $V \subseteq U$. Therefore, for each n , $St(A, \mathcal{H}_n) \subseteq St(B, \mathcal{U}_n)$, where $A \subseteq B$ and $\bigcup \mathcal{W}_n \subseteq St(\bigcup \mathcal{T}_n, \mathcal{U}_n)$.

Then for each $x \in X$, $x \in \{St(\bigcup \mathcal{T}_k, \mathcal{U}_k) : k \in \omega\}$ for all but finitely many k . □

Theorem 3.14 *Let (X, d) be a zero-dimensional separable metric space with no isolated points. The following statements are equivalent:*

1. X has the Hurewicz property;
2. X has the star-Hurewicz property;
3. for each basis \mathcal{B} of metric space (X, d) and for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of (X, d) , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite sets of elements of $\mathcal{B} \wedge \mathcal{U}_n$ such that $\{St(\bigcup \mathcal{V}_n, \mathcal{B} \wedge \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$ for $U_n \in \mathcal{V}_n$;
4. for each basis \mathcal{B} of metric space (X, d) and for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of (X, d) , there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of finite sets of elements of \mathcal{B} such that $\{St(\bigcup \mathcal{V}_n, \mathcal{B} \wedge \mathcal{U}_n) : n \in \omega\}$ is a groupable cover of X and $\lim_{n \rightarrow \infty} diam_d(U_n) = 0$ for $U_n \in \mathcal{V}_n$.
5. X has the Hurewicz basis covering property;
6. X is Hurewicz measure zero with respect to every metric on X which gives X the same topology as d does;
7. X is star-Hurewicz measure zero with respect to every metric on X which gives X the same topology as d does.

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