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Investigation of Gobal Stability of the Equilibrium Position of a System of Differential Autonomous Equations with the Help of Admissible Functions

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Abstract. To study the global stability of the zero solution, which is a single rest point for a nonlinear system of autonomous differential equations, admissible functions are used. These functions are analogous to Lyapunov functions, but do not coincide with them. Admissible functions can also be used for study the global stability of the zero solution of such systems with respect to part of the variables. An example of the application of such functions to the study of the global stability of the equilibrium state for a second-order nonlinear system is presented.

INTRODUCTION

In this paper we continue the study of the global stability of the equilibrium position for nonlinear systems of ordinary differential equations using admissible functions [1, 2, 3]. Admissible functions are some analogue of Lyapunov functions and can be used to study the global stability of the equilibrium state of a system of ordinary differential equations, including asymptotic stability with respect to part of the variables. The second Lyapunov method allows to use Lyapunov functions to investigate the stability of the equilibrium position [4]. In the general case, the construction of Lyapunov functions for nonlinear systems of ordinary differential equations is a difficult task. However, the existence and construction of such functions in an explicit form can make it possible to estimate the solutions of the systems and prove the boundedness of these solutions [5, 6]. Some of the results associated with the application of Lyapunov functions for estimation of solutions of system of differential equations transferred for problems on the part of the variables are presented in [7]. In [8] I.G. Malkin pointed on the possibility of transferring some of the Lyapunov's theorems to the case of stability with respect to part of the variables. The fundamental results for partial stability problem for systems of ordinary differential equations with continuous right-hand sides were obtained in papers [9, 10, 11, 12, 13].

In this paper we present sufficient conditions for partial global asymptotic stability of the zero solution of systems of autonomous differential equations with respect to part of the variables. For this purpose admissible functions are used and the same admissible function is constructed for investigating partial stability of equilibrium position for same system of second-order ordinary differential equations.

PRELIMINARIES

Consider a nonlinear system of ordinary differential equations of perturbed motion

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(\mathbf{0}) = \mathbf{0}. \tag{1}$$

Suppose that $\mathbf{x}(t) \in R^n$, $\mathbf{X} : R^n \to R^n$. The phase vector $\mathbf{x}$ is divided into two groups of variables: $\mathbf{y}$-variables, on which there is investigated the stability of the equilibrium position $\mathbf{x} = \mathbf{0}$, and the remaining $\mathbf{z}$-variables ($\mathbf{z}$-variables

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are non-controlled in studying partial stability).

$$\mathbf{x}^T = (x_1, \dots, x_n) = (y_1, \dots, y_m, z_1, \dots, z_p) = (\mathbf{y}^T, \mathbf{z}^T),$$

$$m > 0, \quad p \geq 0, \quad n = m + p, \quad y_i = x_i, \quad i = \overline{1, m}, \quad z_j = x_{m+j}, \quad j = \overline{1, p}.$$

We suppose the position of equilibrium $\mathbf{x} = \mathbf{0}$ of the system is unique. System (1) can be written in the following form

$$x_i' = X_i(x_1, \dots, x_n), \quad i = \overline{1, n}. \tag{2}$$

Usually [10, 11] partial stability is investigated under the assumption of continuity of the vector-function $\mathbf{X}$=$(X_1, \dots, X_n)^T$ in domain

$$t \geq 0, \quad \|\mathbf{y}\| \leq h, \quad h > 0, \quad h = \text{const}, \quad \|\mathbf{z}\| < \infty, \quad \|\mathbf{x}\| = (\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2)^{1/2} = (x_1^2 + \ldots + x_n^2)^{1/2} \tag{3}$$

and under the assumption of uniqueness and $\mathbf{z}$-extendability [14] of all solutions of the system (2) in domain (3).

In addition to these, we assume that the functions $X_i(x_1, \dots, x_n)$ are continuously differentiable functions of the variables x_i, $i = \overline{1, n}$ in all space R^n.

Let $\mathbf{x}(t) = \mathbf{x}(t; 0, \mathbf{x_0}) = (x_1(t; 0, \mathbf{x}_0), \dots, x_n(t; 0, \mathbf{x}_0))^T$ is a solution of system (2) corresponding to initial condition for $t = 0$

$$\mathbf{x}(0) = \mathbf{x}_0 = (x_{01}, \dots, x_{0n})^T, \quad x_{0i} = \text{const}, \quad i = \overline{1, n}.$$

Let $\mathbf{y}(t) = \mathbf{y}(t; 0, \mathbf{x}_0) = (y_1(t; 0, \mathbf{x}_0), \dots, y_m(t; 0, \mathbf{x}_0))^T = (x_1(t; 0, \mathbf{x}_0), \dots, x_m(t; 0, \mathbf{x}_0))^T$.

We formulate definitions for autonomous systems, following [9, 10, 11, 14].

Definition 1. A set G is called a domain of *uniformly asymptotic* $\mathbf{y}$-*stability* of unperturbed motion $\mathbf{x} = \mathbf{0}$, if the following properties are valid:

1) on any compact set $K_x = \{\mathbf{x} \in R^n : \|\mathbf{x}\| < R\}$, $(K_x \subset G)$ relation $\|\mathbf{y}(t; 0, \mathbf{x}_0)\| \rightrightarrows 0$ is valid for $t \to +\infty$ and $\mathbf{x}_0 \in K_x$;

2) unperturbed motion $\mathbf{x} = 0$ is uniformly $\mathbf{y}$-stable, i.e. $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ such that for $\|\mathbf{x}_0\| < \delta \Longrightarrow \|\mathbf{y}(t; 0, \mathbf{x}_0)\| < \varepsilon$ for $t > 0$.

Definition 2. An unperturbed motion $\mathbf{x} = \mathbf{0}$ is called *global uniformly asymptotic* $\mathbf{y}$-*stable*, if the set G coincides with the whole space.

Assertion. *Suppose that the system* (2) *has a domain G such that $\exists \mathbf{y_0} \in \partial G$ and $\|\mathbf{y}(t, \mathbf{y_0})\| \nrightarrow 0$ for $t \to +\infty$.*
Then there is a line segment

$$\mathbf{x} = \mathbf{y}^1 + u(\mathbf{y_0} - \mathbf{y}^1), \quad u \in [0, 1], \quad \mathbf{y}^1 \in G, \tag{4}$$

which does not include any point of direction field for (2) *in the space R^m.*

Proof. Let a segment (4) will be a part of a line passing through point $\mathbf{y_0}$ and it is perpendicular to the direction field of system (2). In this case, point $\mathbf{y}^1$ will be the begin of this segment (for $u = 0$) located in domain G, and point $\mathbf{y_0}$ is the end of this segment (for $u = 1$), located on the boundary of domain G. This is obtained from the corollary of Picard's theorem [13] which for our case can be formulated as follows: if the initial conditions differ by small values from the initial conditions $\mathbf{y_0}$ by $\mathbf{v}$, then the corresponding solution will differ by the small from the solution passing through the point $\mathbf{y_0}$, and the family of solutions corresponding to the initial conditions $\mathbf{v}$ satisfies Lipschitz condition in the variable $\mathbf{v}$, and the time interval for existence of these solutions will not depend on $\mathbf{v}$.

This assertion allows us to introduce the definition of an integral surface.

Definition 3. A surface in R^n which consists of positive semi-trajectories of system (2) for $t \geq 0$ started from segment $[\mathbf{y}^0, \mathbf{y}^1]$ we will call an *integral surface* (**IS**).

On this **IS** we introduce Poincare's coordinates (u, t), where the first coordinate u denotes a trajectory corresponding to parameter u in (4) and the second coordinate t determines its time length starting from segment (4). Then equation of integral surface **IS** can be written as $\mathbf{x}$=$\mathbf{x}(t, u)$. Consider a family of curves on **IS** defined by equation $t = t(c, u)$, where c is an arbitrary parameter. Let

$$\frac{\partial x_i}{\partial u} = p_i(t, u), \quad i = \overline{1, n},$$

or in vector form

$$\frac{\partial x}{\partial u} = \mathbf{p}(t, u), \qquad \mathbf{p} = (p_1, \ldots, p_n)^T.$$

THEOREM OF PARTIAL ASYMPTOTIC STABILITY IN THE LARGE FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Let $V_{(1)}(\mathbf{x}, \mathbf{p}, t) : R^n \times R^n \times [0, \infty) \to R$ is continuously differentiable with respect to all the variables positive functions. The family of curves on **IS** can be described by equation

$$\mathbf{x} = \mathbf{x}(t(c, u), u).$$

By $\mathbf{g}$ let denote vector $\mathbf{g} = (g_1, \ldots, g_n)^T$, with coordinates g_i, $i = \overline{1, n}$ satisfy equality

$$||\mathbf{g}||^2 = \sum_{i>j} (X_i p_j - X_j p_i)^2. \tag{5}$$

The following lemmas are valid.

Lemma 1. *Let on* **IS** *function* $V_{(1)}(\mathbf{x}, \mathbf{p}, t)$ *satisfies the following inequality:*

$$||\mathbf{X}||V_{(1)}(\mathbf{x}, \mathbf{p}, t) \geq \nu||\mathbf{g}||, \tag{6}$$

where $\nu > 0$, $||\mathbf{X}|| = (X_1^2 + \ldots + X_n^2)^{1/2}$ *and coordinates of vector* $\mathbf{g}$ *satisfy equality* (5).

Then there is a family of curves on **IS** *for which the following equality is valid:*

$$ds = V_{(1)}(\mathbf{x}, \mathbf{p}, t)du,$$

where $ds^2 = \sum_{i=1}^{n} dx_i{}^2$.

It was shown that these curves satisfy equation

$$\frac{dt}{du} = \frac{-(\mathbf{X} \cdot \mathbf{p}) \pm \sqrt{V^2||\mathbf{X}||^2 - ||\mathbf{g}||^2}}{||\mathbf{X}||^2}. \tag{7}$$

Definition 4. The curves, which equations satisfy equation (7), are called *φ-curves*.

In the particular case, when

$$V_{(1)}(\mathbf{x}, \mathbf{p}, t) = ||\mathbf{p}|| = \left(\sum_{i=1}^{n} p_i^2\right)^{1/2}, \tag{8}$$

from equation (7) we have

$$\frac{dt}{du} = \frac{-(\mathbf{X} \cdot \mathbf{p}) \pm |(\mathbf{X} \cdot \mathbf{p})|}{||\mathbf{X}||^2},$$

that allow to build φ-curves on which

$$\frac{dt}{du} = 0, \qquad or \qquad t = \text{const}.$$

Lemma 2. *The following equality is valid*

$$\|\mathbf{X}\|^2\|\mathbf{p}\|^2 - (\mathbf{X}\cdot\mathbf{p})^2 = \|\mathbf{g}\|^2. \tag{9}$$

Proof. The following equalities are valid

$$\|\mathbf{X}\|^2\|\mathbf{p}\|^2 - (\mathbf{X}\cdot\mathbf{p})^2 = \sum_{i=1}^{n} X_i^2 \sum_{j=1}^{n} p_j^2 - \Big(\sum_{i=1}^{n} X_i p_i\Big)^2$$

$$= \sum_{i>j} X_i^2 p_j^2 + \sum_{i=1}^{n} X_i^2 p_i^2 + \sum_{i>j} X_j^2 p_i^2 - 2\sum_{i>j} X_i p_j X_j p_i - \sum_{i=1}^{n} X_i^2 p_i^2$$

$$= \sum_{i>j} (X_i p_j - X_j p_i)^2 = \|\mathbf{g}\|^2.$$

Lemma 2 is proved.

Definition 5. Let function $V_{(1)}(\mathbf{x},\mathbf{p},t)$ is continuously differentiable, positive on set G_1 in variables $\mathbf{x},\mathbf{p},t$. We will call the function an *admissible function (A-function)*, if the following conditions are valid:
1. $V_{(1)}(\mathbf{x},\mathbf{p},t)$ satisfies inequality (6);
2. $V_{(1)}(\mathbf{x},\mathbf{0},t) = 0$.

Definition 6. For A-function $V_{(1)}(\mathbf{x},\mathbf{p},t)$ the expression

$$\partial_t V_{(1)} = \sum_{i=1}^{n} \frac{\partial V_{(1)}}{\partial x_i} X_i + \sum_{i=1}^{n} \frac{\partial V_{(1)}}{\partial p_i} \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} p_j + \frac{\partial V_{(1)}}{\partial t}$$

we will call *a partial derivative of function* $V_{(1)}(\mathbf{x},\mathbf{p},t)$ *with respect to t on integral surface for system* (1) .
The following main theorem [1, 2, 3] holds.

Theorem 1.. *Let system* (1) *has the following properties:*
1°. *Set G includes sphere* $\|\mathbf{x}\| \le \rho$.
2°. *There is an A-function* $V_{(1)}(\mathbf{x},\mathbf{p},t)$ *in set* $G_1 = \{\mathbf{x} : \|\mathbf{x}\| \ge r, r < \rho\}$ *on any* **IS** *with non-positive partial derivative of function* $V_{(1)}(\mathbf{x},\mathbf{p},t)$ *with respect to t on integral surface, probably exclude a set* $M \subset G$, *which consists of the whole trajectories belonging G.*
Then unperturbed motion $\mathbf{x} = \mathbf{0}$ *is global uniformly asymptotic* **y**-*stable.*
Proof of the theorem is carried out by contradiction.

AN EXAMPLE OF A-FUNCTION

We give an example of an admissible function depending only on **p**. Consider the quadratic form

$$V_{(1)}(\mathbf{p}) = \mathbf{A}\sqrt{\omega}, \tag{10}$$

where $A > 0$ and $\omega = \sum\limits_{i,j=1}^{n} a_{ij} p_i p_j$ is a positive definite quadratic form with constant coefficients a_{ij}. Taking into account (9) we have

$$V_{(1)}^2 \|\mathbf{X}\|^2 \ge A^2 \lambda_{min} \|\mathbf{p}\|^2 \|\mathbf{X}\|^2 \ge \nu \|\mathbf{p}\|^2,$$

where A can choose so that $\nu = A^2\lambda_{min} = 1$. To prove this inequality we used the well-known property of a definitely positive quadratic form

$$\lambda_{min} \sum_{i=1}^{n} {p_i}^2 \le \sum_{i,j=1}^{n} a_{ij} p_i p_j \le \lambda_{max} \sum_{i=1}^{n} {p_i}^2.$$

Here $\lambda_{min} > 0$ and $\lambda_{max} > 0$ are the smallest and the largest roots of characteristic equation, respectively, for a definitely positive quadratic form ω, and norm $\|\mathbf{g}\|$ satisfies equality (9). Therefore, the function $V_{(1)} = A\sqrt{\omega}$ is admissible function.

Remark 1. *The function (8) is a special case of a quadratic form (10). Therefore, the function (8) is also A-function.*

AN EXAMPLE OF THE APPLICATION OF ADMISSIBLE FUNCTIONS TO THE STUDY OF PARTIAL GLOBAL STABILITY

Consider the following system of ordinary differential equation:

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1 + cf(x_2 - x_1) \equiv X_1(x_1, x_2), \\ \frac{dx_2}{dt} &= bx_2 + cf(x_2 - x_1) \equiv X_2(x_1, x_2), \end{aligned} \tag{11}$$

where a, b, c are constant and $f(z)$ is continuously differentiable function with respect to z, $f(z) \in C^1(-\infty, +\infty)$ and $f(0) = 0$.

Let consider two different cases:

1) $a = b < 0$;

2) $a < 0$, $b > 0$.

Case 1. The following theorem is valid.

Theorem 2.. *Let in system* (11) $a = b < 0$ *and the following conditions are valid:*

1) the unperturbed motion $\mathbf{x} = (x_1, x_2)^T = \mathbf{0}$ *is uniformly stable;*

2) $f(0) = 0$, $|f'(z)| \le q < -a$, $q = \text{const}$.

Then unperturbed motion $\mathbf{x} = \mathbf{0}$ *is global uniformly asymptotic stable.*

Proof. We assume the existence of domain G and that the condition 1° of Theorem 1 is valid. The position of equilibrium of the system (11) can be only $x_1 = x_2 = 0$.

Consider for system (11) the following function

$$V_{(11)}(\mathbf{x}, \mathbf{p}, t) = \|\mathbf{p}\| = \left(\sum_{i=1}^{2} p_i^2\right)^{1/2}. \tag{12}$$

Function (12) is a special case of function (8). Taking into account Remark 1, we obtain that this function is A-function.

We find the partial derivative $V_{(11)}(\mathbf{x}, \mathbf{p}, t)$ with respect to t on integral surface in accordance with Definition 6.

$$\partial_t V_{(11)} = \frac{1}{2\sqrt{p_1^2 + p_2^2}}\left[p_1\left(\frac{\partial X_1}{\partial x_1}p_1 + \frac{\partial X_1}{\partial x_2}p_2\right) + p_2\left(\frac{\partial X_2}{\partial x_1}p_1 + \frac{\partial X_2}{\partial x_2}p_2\right)\right]$$

$$= \frac{1}{2\sqrt{p_1^2 + p_2^2}}\left[(a - f')p_1^2 + f'p_1p_2 - f'p_1p_2 + (b + f')p_2^2\right] \le \frac{q_1(p_1^2 + p_2^2)}{2\sqrt{p_1^2 + p_2^2}} \le q_1\|\mathbf{p}\| \le 0,$$

where $q_1 < 0$.

Thus, condition 2° of Theorem 1 is satisfied. Consequently, by Theorem 1, the equilibrium position of system (11) $x_1 = x_2 = 0$ will be global uniformly asymptotic stable.

Case 2. The following theorem is valid.

Theorem 3. *Let in system* (11) $a < 0$, $b > 0$, $a + b < 0$ *and the following conditions are valid:*

1) the unperturbed motion $\mathbf{x} = (x_1, x_2)^T = \mathbf{0}$ *is uniformly* x_1*-stable;*

2) $f(0) = 0$, $\dfrac{b}{c(a-b)} > 1$, $a < f'(z) < \min\left[-b, -\dfrac{ab}{c(a-b)}\right]$.

Then unperturbed motion $\mathbf{x} = \mathbf{0}$ *is global uniformly asymptotic* x_1*-stable.*

CONCLUSION

To study partial global asymptotic stability of systems of ordinary equations, admissible functions, different from the Lyapunov's functions were proposed. An example of a system of differential equations and the corresponding admissible function is given, which is used to prove the global asymptotic stability of the equilibrium position of the system of ordinary equations.

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