Tubes of discontinuous solutions of dynamical systems and their stability

Cite as: AIP Conference Proceedings **1895**, 050011 (2017); https://doi.org/10.1063/1.5007383
Published Online: 12 October 2017

A. N. Sesekin, and N. I. Zhelonkina

Tubes of Discontinuous Solutions of Dynamical Systems and Their Stability

A. N. Sesekin[1,2,b)] and N. I. Zhelonkina[1,a)]

[1]*N. N. Krasovskii Institute of Mathematics and Mechanics, UrB of RAS, 16 S. Kovalevskaya str., 620990 Ekaterinburg, Russia*
[2]*Ural Federal University, 19 Mir str., 620990 Ekaterinburg, Russia*

[a)]Corresponding author: 312115@mail.ru
[b)]sesekin@list.ru

Abstract. The article is devoted to investigation of nonlinear dynamical systems which applies the generalized effect. The generalized effect (or impulses) is the result of the presence on the right part of the system a generalized derivative of the function of bounded variation. Such system contains incorrect operation of multiplication of discontinuous function on the generalized function from the point of view of the theory of distributions. This incorrectness is overcome by the approximation of the generalized functions in the right part of system by the sequence of smooth approximations of the generalized influences by analogy with sequential approach of the theory of the generalized functions. This sequence generates a sequence of smooth solutions. Then limit of a sequence of smooth solutions is considered. If such limit exists it is offered to be used as a solution. The solution is understood as partial pointwise limit of such sequence if a sequence of smooth solutions does not converge. Such partial limits constitute a tube of solutions. The sufficient conditions are received for stability of tubes of discontinuous solutions.

INTRODUCTION

We consider a nonlinear system of differential equations with an generalized effect at the right part of system. Such systems provide one feature. Namely, the right-hand side of the system contains an incorrect operation of multiplying a discontinuous function by a distribution. This work uses the definition of the solution based on the closure of the set of absolutely continuous trajectories in the topology of pointwise convergence [1, 2]. Here we should distinguish the following two cases. In the first case, approximating sequences of generalized actions generate sequences of smooth solutions that pointwise converge to a function of bounded variation. We propose this limit as the solution of the system in this case. In the case when the sequence of smooth solutions is not convergent, we propose as a solution all the partial limits of smooth solutions generated by smooth approximations of generalized effects. These partial limits form tubes of discontinuous solutions. We note that this definition of the solution is natural from the point of view of control theory [3]. The necessity of investigating the properties of discontinuous solutions is related to the fact that dynamic objects that allow discontinuous trajectories are encountered in mechanics, medicine, ecology, electrophysics and economy [1, 3, 4]. The interesting feature of this paper is that the Frobenius condition is not imposed on the right-hand side of the system of differential equations, which ensures a unique response of the system to each impulse actions. The paper [5] is devoted to the study of the stability of equilibrium points of a dynamical system with impulse action in the case when the dynamical system does not satisfy the Frobenius condition. The dynamical system is described by means of a differential inclusion. In the present paper, just as in [5], we do not assume the Frobenius condition, but we investigate not the asymptotic stability of the rest points of a dynamical system, but the stability of tubes of discontinuous solutions with respect to initial conditions. The property of stability of discontinuous solutions in the case when discontinuous solutions are formalized both in the monographs [6] and [7] were considered in the papers of A.M. Samoilenko, N.A. Perestyuk, D. Baynov, their students and followers. However, the stability of the tubes of discontinuous solutions was not considered in these monographs and papers. The stability of solutions in the case when there exists a unique limit of smooth solutions was considered in [8]. In [8] we used the formalization of the concept of solution, adopted in this article.

Application of Mathematics in Technical and Natural Sciences
AIP Conf. Proc. 1895, 050011-1–050011-7; https://doi.org/10.1063/1.5007383
Published by AIP Publishing. 978-0-7354-1579-9/$30.00

STATEMENT OF THE PROBLEM

Consider the following Cauchy problem:

$$\dot{x} = f(t, x(t), v(t), \check{V}(t)) + B(t, x(t), v(t), \check{V}(t))\dot{v}(t), \qquad x(t_0) = x^0. \tag{1}$$

Here, $x(t)$ and $v(t)$ are respectively n and m are vector functions of time, $f(t, x, v, \check{V})$ is an n - vector function, and $B(t, x, v, \check{V})$ is an $n \times m$ matrix function, $\check{V}(t) = \underset{[t_0, t]}{\text{var}}\, v(\cdot)$, $v(t) \in BV_m[t_0, \vartheta]$, where $BV_m[t_0, \vartheta]$ is a Banach space of m - vector functions of bounded variation, $\underset{[t_0, t]}{\text{var}}\, v(\cdot) = \sum_{i=1}^{m} \underset{[t_0, t]}{\text{var}}\, v_i(\cdot)$ is the variation of vector function $v(t) = (v_1(t), v_2(t), ..., v_m(t))^T$,

$$\|f(t, x, v, V)\| \le \kappa(1 + \|x\|), \quad \|B(t, x, v, V)\| \le \kappa(1 + \|x\|),$$

where κ is the positive constant.

If $v(t)$ is an absolutely continuous vector function defined in $[t_0, \vartheta]$, then by the Carathéodory theorem the unique solution of Cauchy problem (1) exists and satisfies the integral equation

$$x(t) = x^0 + \int_{t_0}^{t} f(s, x(s), v(s), \check{V}(s))\, ds + \int_{t_0}^{t} B(s, x(s), v(s), \check{V}(s))\, dv(s), \tag{2}$$

where the second integral is the Riemann-Stieltjes integral.

Let the sequence $v_k(t)$ of absolutely continuous functions ($v_k(t) \in AC[t_0, \vartheta]$) converge pointwise to the function $v(t) \in BV[t_0, \vartheta]$. The corresponding sequence of solutions of the Cauchy problem (1) or (2) will be denoted by $x_k(t)$.

As in [1, 2] we say that the sequence $v_k(t)$ V-convergent to $v(t)$, if $v_k(t)$ pointwise converges to $v(t)$ and $\underset{[t_0, t]}{\text{var}}\, v_k(\cdot)$ pointwise converges to $V(t) \in BV[t_0, \vartheta]$. Denote such a convergence as follows: $v_k(t) \xrightarrow{V} v(t)$.

Note that for any two numbers a and b satisfying the inequality $t_0 \le a \le b \le \vartheta$ the following inequality holds:

$$\underset{[a, b]}{\text{var}}\, v(\cdot) \le V(b) - V(a).$$

Definition 1 *[1, 2] Every partial pointwise limit of the sequence $x_k(t)$, $k = 1, 2, \ldots$, on the interval $[t_0, \vartheta]$, generated by an arbitrary V – convergent sequence of absolutely continuous functions $v_k(t)$, $k = 1, 2, \ldots$, will be referred to as V-solution of Cauchy problem* (1).

Let $z(0) = x(\bar{t})$, $\mu(0) = v(\bar{t})$ be initial conditions for the system

$$\begin{aligned} \dot{z}(\xi) &= B(\bar{t}, z(\xi), \mu(\xi), V(\bar{t}) + \xi - \bar{t})\eta(\xi), \\ \dot{\mu}(\xi) &= \eta(\xi). \end{aligned} \tag{3}$$

Denote by $S(\bar{t}, x(\bar{t}), \Delta v(\bar{t}), V(\bar{t}), \Delta V(\bar{t}))$ (where $\bar{t} = t_i - 0$ or $\bar{t} = t_i$) the set obtained by shift on the value $-x(\bar{t})$ at the in the moment $\xi = \Delta V(\bar{t})$ the cross-section of the attainability set of (3) that

$$\mu(\Delta V(\bar{t})) = v(\hat{t}), \tag{4}$$

where $\hat{t} = \bar{t}$ if the function $V(t)$ has a left discontinuity at this point, and $\hat{t} = \bar{t} + 0$ if the function $V(t)$ at this point has a right discontinuity, the control $\eta(\xi)$ is subjected to the constraint $\|\eta(\xi)\|_1 \le 1$.

Theorem 1 *[1, 2] Every partial pointwise limit of the sequence $x_k(t)$ of solutions of* (1) *generated by a sequence $v_k(t)$, $k = 1, 2, \ldots$, of absolutely continuous functions pointwise convergent to $v(t)$ and*

$$(\lim_{k\to\infty} \underset{[t_0, t]}{\text{var}}\, v_k(\cdot) = V(t))$$

($v_k(t)$ is V-convergent to $v(t) \in BV_m[t_0, \vartheta]$) is the solution of the integral inclusion

$$x(t) \in x^0 + \int_{t_0}^{t} f(\xi, x(\xi), v(\xi), V(\xi))\, d\xi + \int_{t_0}^{t} B(\xi, x(\xi), v(\xi), V(\xi))\, dv^c(\xi)$$
$$+ \sum_{t_i \le t,\, t_i \in \Omega_-} S(t_i, x(t_i - 0), \Delta v(t_i - 0), V(t_i - 0), \Delta V(t_i - 0)) + \sum_{t_i < t,\, t_i \in \Omega_+} S(t_i, x(t_i), \Delta v(t_i + 0), V(t_i), \Delta V(t_i + 0)), \quad (5)$$

where $v^c(\xi)$ is the continuous component of the function of bounded variation $v(\xi)$, Ω_- and Ω_+ are the sets of points of the left and right discontinuities of the function $V(t)$. For every solution $x(t)$ of the integral inclusion (5) *generated by the pair $(v(t), V(t))$ there exists a sequence of absolutely continuous functions $v_k(t)$, $k = 1, 2, \dots,$ pointwise convergent to $v(t)$, $\operatorname{var}_{[t_0,t]} v_k(\cdot) \longrightarrow V(t)$; and the corresponding sequence $x_k(t)$, $k = 1, 2, \dots,$ of solutions of* (1) *is pointwise convergent to $x(t)$.*

Definition 2 *The extension of the solutions of the integral inclusion* (5) *on the interval* $[t_0, \infty)$ *will be called the solution of Eq.* (1) *on the interval* $[t_0, \infty)$.

We denote as $X(t, x_0, v(\cdot), V(\cdot))$ the tube cross section solutions of the integral inclusion (5) generated by the initial condition x_0 and a couple of functions $(v(\cdot), V(\cdot))$. Thus, each tube of discontinuous solutions is given by a pair of $v(\cdot)$ and $V(\cdot)$. The function $v(\cdot)$ defines a generalized effect on the system, and $V(\cdot)$ is a certain characteristic of the class of possible approximations of the function $v(\cdot)$. In fact, the function $V(\cdot)$ characterizes the energy possibilities of the approximating sequences. Note that if $\operatorname{var}_{[t_0,t]} v(\cdot) = V(\cdot)$, then, according to [1, 2], the tube of discontinuous solutions becomes a single trajectory.

Definition 3 *We will say that a tube of solutions of the integral inclusion* (1) *$X(t, x_0, v(\cdot), V(\cdot))$ is stable if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ that if $|\overline{x} - x_0| < \delta$ then*

$$\rho(X(t, x_0, v(\cdot), V(\cdot)), X(t, \overline{x}, v(\cdot), V(\cdot))) < \varepsilon$$

where $\rho(A, B)$ is Hausdorff distance between sets A and B.

Definition 4 *We will say that a tube $X(t, x_0, v(\cdot), V(\cdot))$ of solutions of the integral inclusion* (1) *is asymptotically stable if it is stable and if we have the equality*

$$\lim_{t \longrightarrow \infty} \rho(X(t, x_0, v(\cdot), V(\cdot)), X(t, \overline{x}, v(\cdot), V(\cdot))) = 0.$$

Consider a bilinear system that is a the special case of system (1)

$$\dot{x}(t) = (A(t) + \sum_{i=1}^{m-1} D_i(t)\dot{v}_i(t))x(t) + f(t)\dot{v}_m(t) + g(t) \quad (6)$$

where $A(t)$, $D_i(t)$ $(i \in \overline{1, m-1})$ are continuous matrices of dimension $n \times n$, $f(t)$ and $g(t)$ are continuous vector functions of dimension n, respectively, $v(t) = (v_1(t), v_2(t), ...v_m(t))^T$, $\operatorname{var}_{[t_0,t]} v(t) = \sum_{i=1}^{m} \operatorname{var}_{[t_0,t]} v_i(\cdot)$.

Let $V(t) = \lim_{j \longrightarrow \infty} \operatorname{var}_{[t_0,t]} v^j(\cdot)$ and $v^j(\cdot)$ be the sequence of absolutely continuous vector functions which converge pointwise to $v(t) \in BV$.

According to Theorem 1, all V-solutions of Eq. (6) satisfy the following integral inclusion

$$x(t) \in x^0 + \int_{t_0}^{t} A(\xi)x(\xi)\, d\xi + \int_{t_0}^{t} g(\xi)\, d\xi + \sum_{i=1}^{m-1} \int_{t_0}^{t} D_i(\xi)x(\xi)\, dv_i^c(\xi) + \int_{t_0}^{t} f(\xi)\, dv_m^c(\xi)$$
$$+ \sum_{t_i \le t,\, t_i \in \Omega_-} S(t_i, x(t_i - 0), \Delta v(t_i - 0), V(t_i - 0), \Delta V(t_i - 0)) + \sum_{t_i < t,\, t_i \in \Omega_+} S(t_i, x(t_i), \Delta v(t_i + 0), V(t_i), \Delta V(t_i + 0)),$$

where $v^c(t)$ is the continuous component of the function of bounded variation $v(t)$.

The set $S(\bar{t}, x(\bar{t}), \Delta v(\bar{t}), V(\bar{t}), \Delta V(\bar{t}))$ (where $\bar{t} = t_i - 0$ if $t_i \in \Omega_-$ and $\bar{t} = t_i$ if $t_i \in \Omega_+$) is defined as the shift on the value $-x(\bar{t})$ at the moment $\xi = \Delta V(\bar{t})$ of the cross section ($\mu(\Delta V(\bar{t})) = v(t_i)$ if $\bar{t}_i \in \Omega_-$ and $\mu(\Delta V\bar{t}) = v(t_i + 0)$ if $\bar{t}_i \in \Omega_+$) of the attainability set of systems

$$\begin{aligned} \dot{z}(\xi) &= \sum_{i=1}^{m-1} D_i(\bar{t}) z(\xi)\eta_i(\xi) + f(\bar{t})\eta_i(\xi) \\ \dot{\mu}(\xi) &= \eta(\xi) \end{aligned} \tag{7}$$

at the moment $\xi = \Delta V(\bar{t})$, where the control $\eta(\xi)$ satisfies the constraint $\|\eta(\xi)\| \le 1$, where $\|\eta(\xi)\| = \sum_{i=1}^{m} |\eta_i(\xi)|$.

Theorem 2 *Suppose that the fundamental matrix $Y(t, s)$ of system $\dot{x} = A(t)x$ satisfy the condition*

$$\|Y(t, s)\| \le c e^{-\alpha(t-s)}, \tag{8}$$

where α and c are some constants, $\alpha \ge 0,\ c \ge 1,\ t \ge s \ge t_0$. Furthermore, the following inequalities are valid

$$\|\tilde{D}_i(t)\| \le K, \forall t \in [t_0, \infty), i \in \overline{1, m-1}, \tag{9}$$

where K is some positive constants. Then if $x(t)$ and $\overline{x}(t)$ are solutions of the integral inclusion generated by the initial condition x_0 and $\overline{x}_0$, and the same system of functions $\eta_K(\xi)$, generating the jump of the trajectory $x(t)$ and $\overline{x}(t)$, then the next estimation will be true:

$$\|\overline{x}(t) - x(t)\| \le \|\overline{x}_0 - x_0\| \times e^{-(\alpha(t-t_0) - cKe(t-t_0+var_{[t_0,t]} V(\cdot)))}. \tag{10}$$

Proof. According to [1] $x(t)$ and $\overline{x}(t)$ will satisfy the integral equation

$$\begin{aligned} x(t) = x^0 &+ \int_{t_0}^{t} A(\xi)x(\xi)\, d\xi + \sum_{i=1}^{m-1} \int_{t_0}^{t} D_i(\xi)x(\xi)\, dv_i^c(\xi) + \int_{t_0}^{t} f(\xi)\, dv_m^c(\xi) \\ &+ \sum_{t_i \le t,\, t_i \in \Omega_-} \widetilde{S}(t_i, x(t_i - 0), \eta_{t_i - 0}(\cdot), V(t_i - 0)) + \sum_{t_i < t,\, t_i \in \Omega_+} \widetilde{S}(t_i, x(t_i), \eta_{t_i}(\cdot), V(t_i)), \end{aligned} \tag{11}$$

where $\widetilde{S}(\bar{t}, x(\bar{t}), \eta_{\bar{t}}(\cdot), V(\bar{t})) = z(\Delta V(\bar{t})) - x(\bar{t})$, $z(\xi)$ is the solution of the equation

$$\begin{aligned} \dot{z}(\xi) &= \sum_{i=1}^{m-1} D_i(\bar{t}) z(\xi)\eta_i(\xi) + f(\bar{t})\eta_m(\xi) \\ z(0) &= x(\bar{t}). \end{aligned} \tag{12}$$

According to the Cauchy formula the solution $x(t)$ and $\overline{x}(t)$ will satisfy the integral equation

$$\begin{aligned} x(t) = Y(t, t_0)x_0 &+ \sum_{i=1}^{m-1} \int_{t_0}^{t} Y(t, \xi) D_i(\xi)x(\xi)\, dv_i^c(\xi) + \int_{t_0}^{t} f(\xi)\, dv_m^c(\xi) \\ &+ \sum_{t_i \le t,\, t_i \in \Omega_-} Y(t, t_i)\widetilde{S}(t_i, x(t_i - 0), \eta_{t_i - 0}(\cdot), V(t_i - 0), \Delta V(t_i - 0)) \\ &+ \sum_{t_i < t,\, t_i \in \Omega_+} Y(t, t_i)\widetilde{S}(t_i, x(t_i), \eta_{t_i}(\cdot), V(t_i), \Delta V(t_i)) + \int_{t_0}^{t} Y(t, t_i) g(\xi)\, dv_m^c(\xi). \end{aligned} \tag{13}$$

From (13) we have

$$\overline{x}(t)-x(t)=Y(t,t_0)(\overline{x}_0-x_0)+\sum_{i=1}^{m-1}\int_{t_0}^{t}Y(t,\xi)D_i(\xi)(\overline{x}(\xi)-x(\xi))\,dv_i^c(\xi)$$
$$+\sum_{t_i\le t,\,t_i\in\Omega_-}Y(t,t_i)(\widetilde{S}(t_i,\overline{x}(t_{i-0}),\eta_{t_{i-0}}(\cdot),V(t_{i-0}),\Delta V(t_{i-0})-\widetilde{S}(t_i,x(t_i-0),\eta_{t_i-0}(\cdot),V(t_i-0),\Delta V(t_i-0)))$$
$$+\sum_{t_i<t,\,t_i\in\Omega_+}Y(t,t_i)(\widetilde{S}(t_i,\overline{x}(t_i),\eta_{t_i}(\cdot),V(t_i),\Delta V(t_i-0)-\widetilde{S}(t_i,x(t_i),\eta_{t_i}(\cdot),V(t_i),\Delta V(t_i))). \tag{14}$$

For the difference $\widetilde{S}(t,\overline{x}(t),\eta_t(\cdot),V(t),\Delta V(t))-\widetilde{S}(t,x(t),\eta_t,V(t),\Delta V(t))$, according to (3) the representation is true

$$\widetilde{S}(t,\overline{x}(t),\eta_t(\cdot),V(t),\Delta V(t))-\widetilde{S}(t,x(t),\eta_t(\cdot),V(t),\Delta V(t))$$
$$=\overline{z}(\Delta V(t))-\overline{x}(t)-(z(\Delta V(t))-x(t))=\int_0^{\Delta V(t)}\sum_{i=1}^{m-1}D_i(t)(\overline{z}(\xi)-z(\xi))\eta_t(\xi)d\xi. \tag{15}$$

Adding and subtracting $\sum_{i=1}^{m-1}D_i(t)\int_0^{\Delta V(t)}(\overline{x}(t)-x(t))\eta_t(\cdot)d\xi$ on the right side of the last equality and then calculating the norms of the left and right parts of the resulting expression, taking into account the assumptions made earlier, we obtain

$$\|\overline{z}(\Delta V(t)-\overline{x}(t)-(z(\Delta V(t))-x(t)))\|\le K\|\overline{x}(t)-x(t)\|\cdot\int_0^{\Delta V(t)}\|\eta_t(\xi\|d\xi+\int_0^{\Delta V(t)}K\|\overline{z}(\xi)-z(\xi)\|\|\eta_t(\xi)\|d\xi.$$

Applying Gronwall's lemma [9] to the last inequality, we obtain

$$\|\overline{z}(1)-\overline{x}(t)-(z(1)-x(t))\|\le K\int_0^{\Delta V(t)}\|\eta_t(\xi)\|d\xi\ \|\overline{x}(t)-x(t)\|e^{K\int_0^{\Delta V(t)}\|\eta_t(\xi)\|d\xi}. \tag{16}$$

It was shown in [10] that the inequality $ae^a\le e^{\beta a}-1$ holds for all $a\ge 0\ \ \beta\ge e$. Then the inequality (16) can be written in the form

$$\|\overline{z}(1)-\overline{x}(t)-(z(1)-x(t))\|\le\|\overline{x}(t)-x(t)\|\ (e^{eK\int_0^{\Delta V(t)}\|\eta_t(\xi)\|d\xi}-1),$$

and the last inequality due to the fact that $\|\eta_t(\xi)\|\le 1$, is followed by the inequality

$$\|\overline{z}(1)-\overline{x}(t)-(z(1)-x(t))\|\le\|\overline{x}(t)-x(t)\|\ (e^{eK\Delta V(t)}-1). \tag{17}$$

Calculating the norms of the left and right sides in (17) and taking into account (8), (9), (15), (16), (17) and using the notation

$$y(t)=\overline{x}(t)-x(t), \tag{18}$$

we have

$$\|y(t)\|\le c[e^{-\alpha(t-t_0)}\|\overline{x}_0-x_0\|+K\int_0^t e^{-\alpha(t-\xi)}\|y(\xi)\|\,d\underset{[t_0,\xi]}{\mathrm{var}}\,v^c(\cdot)]$$
$$+c\sum_{t_i<t,\,t_i\in\Omega_-}e^{-\alpha(t-t_0)}(e^{Ke|\Delta V(t_i-0)|}-1)\,\|y(t_i-0)\|+\sum_{t_i<t,\,t_i\in\Omega_+}e^{-\alpha(t-t_0)}(e^{Ke|\Delta V(t_i)|}-1)\,\|y(t_i)\|.$$

Taking into account the estimate $c(e^{a}-1) < e^{ca}-1$ $(c \geq 1)$, which validity is not difficult to check by using the expansion of the exponential in to series, the last inequality can be written in the form

$$\|y(t)\| \leq c[e^{-\alpha(t-t_0)}\|\overline{x}_0 - x_0\| + K\int_0^t e^{-\alpha(t-\xi)}\|y(\xi)\|d\operatorname*{var}_{[t_0,\xi]} v^c(\cdot)] \\ + c\sum_{t_i<t,\, t_i\in\Omega_-} e^{-\alpha(t-t_0)}(e^{cKe|\Delta V(t_i-0)|}-1) + \sum_{t_i<t,\, t_i\in\Omega_+} e^{-\alpha(t-t_0)}(e^{cKe|\Delta V(t_i)|}-1).$$

Multiplying the last inequality by $e^{\alpha(t-t_0)}$ and introducing the notation

$$q(t) = e^{\alpha(t-t_0)}\|y(t)\|, \tag{19}$$

we get

$$q(t) \leq c\|\overline{x}_0 - x_0\| + cK\int_{t_0}^t q(\xi)d\operatorname*{var}_{[t_0,\xi]} v^c(\cdot) + \sum_{t_i<t,\, t_i\in\Omega_-}(e^{cKe|\Delta V(t_i-0)|}-1)q(t_i-0) + \sum_{t_i<t,\, t_i\in\Omega_+}(e^{cKe|\Delta V(t_i)|}-1)q(t_i).$$

We multiply the integral in the last inequality by e and replace $\operatorname*{var}_{[t_0,\xi]} v^c(\cdot)$ by $\operatorname*{var}_{[t_0,\xi]} V(\cdot)$. The inequality will only intensify. As a result, we have

$$q(t) \leq c\|\overline{x}_0 - x_0\| + cKe\int_{t_0}^t q(\xi)d\operatorname*{var}_{[t_0,\xi]} v^c(\cdot) \\ + \sum_{t_i<t,\, t_i\in\Omega_-}(e^{cKe\|\Delta V(t_i-0)\|}-1)q(t_i-0) + \sum_{t_i<t,\, t_i\in\Omega_+}(e^{cKe\|\Delta V(t_i)\|}-1)q(t_i). \tag{20}$$

According to the Lemma [5.4.3] in [1], every solution of the inequality (20) satisfies the estimate

$$q(t) \leq e^{cKe(t-t_0+\operatorname*{var}_{[t_0,t]} V(\cdot))}\|\overline{x}_0 - x_0\|.$$

Multiplying this inequality by $e^{-\alpha(t-t_0)}$ and taking into account the notation (19), we obtain the estimate (10).

Theorem 3 *Suppose, that there is a number β such that the inequality*

$$\alpha(t-t_0) - ck(t-t_0+\operatorname*{var}_{[t_0,t]} V(\cdot)) > \beta \tag{21}$$

is satisfied for every $t \in [t_0,\infty]$. Then the tube of discontinuous solutions $X(t, x_0, v(\cdot), V(\cdot))$ will be stable.

If, however, the condition

$$\lim_{t\longrightarrow\infty}(\alpha(t-t_0) - ck(t-t_0+\operatorname*{var}_{[t_0,t]} V(\cdot)) > \beta) = +\infty, \tag{22}$$

is satisfied, then the tube of discontinuous solutions will be asymptotically stable.

Proof. Let $\rho(X,Y) = \|\overline{x} - \widetilde{y}\|$, where $\overline{x}\epsilon X, \widetilde{y}\epsilon Y$, X and Y is the bounded closed set. Suppose that there is a one-to-one correspondence $\overline{\overline{x}}_n \longleftrightarrow \overline{\overline{y}}_n$ between the elements of the set X and Y. It is known that for any pair $\overline{\overline{x}}_n, \overline{\overline{y}}_n$ the following inequality holds $\|\overline{\overline{x}}_n - \overline{\overline{y}}_n\| \leq \gamma$, where γ is some positive constant.

Let $\overline{x} = \overline{x_n}$. It is obvious that

$$\|\overline{x}_n - \widetilde{y}\| \leq \|\overline{x}_n - \overline{y}_n\| \leq \gamma.$$

Then, by Theorem 2

$$\rho(X(t, x_0, v(\cdot), V(\cdot)); X(t, \overline{x}_0, v(\cdot), V(\cdot))) \leq \|\overline{x}_0 - x_0\|\; e^{-(\alpha(t-t_0)-cKe(t-t_0+var_{[t_0,t]}V(\cdot)))}. \tag{23}$$

If condition (21) is fulfilled, then

$$\rho(X(t, x_0, v(\cdot), V(\cdot)); X(t, \overline{x}_0, v(\cdot), V(\cdot))) \leq \|\overline{x}_0 - x_0\| e^{-\gamma},$$

This provides the stability of the tube $X(t, x_0, v(\cdot), V(\cdot))$.

If the condition (22) holds, then according to (23) we have

$$\lim_{t \longrightarrow \infty} \rho(X(t, x_0, v(\cdot), V(\cdot)); X(t, \overline{x}_0, v(\cdot), V(\cdot))) = 0,$$

this ensures the asymptotic stability of the tube $X(t, x_0, v(\cdot), V(\cdot))$.

Remark. *If the function $V(t)$ is piecewise constant with a finite number of discontinuity points (finite number of impulses acting on the system (1)), then under condition $\alpha = 0$ we will have a stability of the tube $X(t, x_0, v(\cdot), V(\cdot))$, and under condition $\alpha > 0$ we will have an asymptotic stability of the tube.*

CONCLUSIONS

In this paper we defined the notions of stability and asymptotic stability of tubes of discontinuous solutions of a bilinear system. Tubes of discontinuous solutions arise as a result of a possible response of the system to impulse action. Using the technique of integral inequalities, we obtained the sufficient conditions for stability and asymptotic stability of the tubes of discontinuous solutions. Sufficient conditions for stability and asymptotic stability can be used in designing impulse controls for systems when the response of the system to the impulse effect can be ambiguous due to its dependence on the method of approximating the impulse action.

ACKNOWLEDGEMENTS

The research was supported by Russian Science Foundation (RSF)(project No.16-11-10146).

REFERENCES

[1] S. T. Zavalishchin and A. N. Sesekin, *Dynamic Impulse Systems: Theory and Applications* (Kluwer Academic Publishers, Dordrecht, 1997).

[2] A. N. Sesekin, "Dynamic systems with nonlinear impulse structure," in *Proceedings of the Steklov Institute of Mathematics*, (MAIK, "Nauka/Interperiodika, Moscow, 2000), pp. 159–173.

[3] N. N. Krasovskii, *Motion Control Theory* (Nauka, Moscow, 1968). [in Russian]

[4] B. M. Miller and E. Ya. Rubinovich (2013) Discontinuous solutions in the optimal control problems and their representation by singular space-time transformations, *Automation and Remote Control* **74**, 1969–2006.

[5] F. L. Pereira and G. N. Silva (2005) Lyapunov stability of measure driven impulsive systems, *Differential Equations* **40**(8), 1122–1130.

[6] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations* (World Scientific Publishing Co. Inc. River Edge, NJ, 1995).

[7] V. Lakshmikantham, D. Bainov, and P. Simeonov, *Theory of Impulsive Differential Equations* (World Scientific, Singapore, 1989).

[8] A. N. Sesekin and N. I. Zhelonkina, "Stability of nonlinear dynamical systems containing the product of discontinuous functions and distributions," in *AMEE'16*, AIP Conference Proceedings 1789, (American Institute of Physics, Melville, NY, 2016), paper 040010, 8p.

[9] R. Bellman, *Stability Theory of Differential Equations* (Dover Books on Mathematics, Dover Publications, 2008).

[10] A. N. Sesekin and N. I. Zhelonkina, "On the stability of linear systems with generalized action and delay," in *IFAC-PapersOnLine, Proceedings of the 18th IFAC World Congress Milano, Italy, 2011*, pp. 13404–13407, ISSN 1474-6670.