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On the Stability of Discontinuous Solutions of Bilinear Systems With Impulse Action, Constant and Linear Delays

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Abstract. This paper is devoted to the study of the stability properties of solutions bilinear system of differential equations with generalized effects in the system matrix, constant and linear delays in phase coordinates. Sufficient stability conditions are obtained.

INTRODUCTION

This paper is devoted to the investigation of the stability property of solutions of a bilinear system of differential equations with a matrix of a system containing generalized actions (impulse components) and two types of delay in phase coordinates constant and linear. Systems of this type are found (see, for example, [1]) when describing models of managing advertising costs when selling new products. Let $S(t)$ be the number of people who become potential buyers at time t, $P(t)$ is the number of sales of a product at time t. According to [1], $S(t)$ and $P(t)$ values satisfy the equations

$$\dot{S}(t) = k_1 \cdot u(t) \cdot \left(1 - \frac{S(t)}{M}\right) - b_1 \cdot S(t) + d \cdot P(t),$$

$$\dot{P}(t) = k_2 \cdot S(t) + k_3 \cdot S(t - \tau) + k_4 \cdot S(\mu \cdot t) - b_2 \cdot P(t).$$

Here $u(t)$ is the current investment in advertising, b_1, b_2 is the rate at which the potential number of consumers and sales are reduced, respectively ($b_i = const$), M is the maximum possible number of people in the target audience ($M = const$), k_1 - the rate at which the number of potential buyers who learned about the product from advertisements grows (k_1 = const), d - the rate at which people who purchased goods return again, k_2 - the rate at which potential buyers who fall into the category of "innovators" purchase goods, k_3 - the rate at which n bathe product people from the category of "early adopters".

Previously, similar questions were considered in [2] for systems without delay, in papers [3] and [4] for systems with constant or linear delay. In contrast to the mentioned works, this paper considers a system containing both fixed and linear delay.

An important feature of the system under consideration is that the right side of the system contains an operation that is incorrect from the point of view of the theory of generalized functions and that the multiplication of a discontinuous function by a generalized one. As a consequence, the concept of a solution is formalized on the basis of the closure of the set of smooth solutions in the space of functions of bounded variation. Details on the development of this approach for different classes of systems of differential equations can be found in [5, 6, 7, 8], to solve this problem uses an approach based on the closure of a set of smooth solutions in the space of functions of bounded variation. This approach is natural in terms of control theory [9], where the impulse controls often represent idealized processes with large change parameters for short periods of time. Another formalization of the concept of a discontinuous solution is considered in the papers [10]. And also we will note two publications [11, 12] in which the question of solution stability is considered for a linear system with constant and linear delays.

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FORMULATION OF THE PROBLEM

We will consider a bilinear homogeneous system of differential equations with linear and constant delays.

$$\dot{x}(t) = (A(t) + \sum_{j=1}^{m} D_j(t)\dot{v}_i(t))x(t) + B(t)x(t-\tau) + C(t)x(\mu t) \ t \geqslant t_0 \geq 0. \tag{1}$$

Here $A(t)$, $C(t)$, $D_j(t)$ $(j \in \overline{1,m})$ are continuous bounded matrix functions of dimension $n \times n$, $B(t)$ is continuous bounded $m \times n$ matrix functions, matrices $D_i(t)\,(j \in \overline{1,m})$ mutually commutative, $v_i(t)$ s the components of the vector function of bounded variation $v(t) = (v_1(t), v_2(t), \ldots, v_m(t))^T$, $\tau > 0$ is a constant delay, μt is a linear delay $(0 < \mu < 1)$, $\varphi(t)$ is the initial function of bounded variation, which is defined on the interval $[\min\{t_0 - \tau, \mu t_0\},\ t_0]$.

A feature of system (5) is that its right-hand side contains the incorrect operation of multiplying a discontinuous function by a distribution. This is explained as follows. If the function $v(t)$ is discontinuous at some time moment t, then the system is subjected to a impulse at that moment. As a result, the function $x(t)$ at this moment becomes discontinuous and in $\sum_{j=1}^{m} D_j(t)\dot{v}_i(t)\,x(t)$ an incorrect operation of multiplication of a distribution by a discontinuous occurs. This leads to the need to formalize the concept of a solution.

As in [2, p. 386], [3, p. 230], [7, p. 226] we shall mean by an approximable solution of equation (5) on an arbitrary finite interval$[t_0, \vartheta]$ the function of bounded variation$x(t)$, which is the pointwise limit of the sequence of absolutely continuous functions $x_k(t)$ generated by the sequence of absolutely continuous functions $v_k(t)$, which converge pointwise to the vector function of bounded variation $v(t)$, if the limit does not depend on the choice of the sequence.

According to [7, p. 226], under the assumptions made, on any finite interval $[t_0, \vartheta]\,(\vartheta > t_0)$ there exists an approximable solution of system (5), which satisfies the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} A(\xi)x(\xi)\,d\xi + \sum_{j=1}^{m}\int_{t_0}^{t} D_j(\xi)x(\xi)\,dv_j^c(\xi) +$$
$$+ \int_{t_0}^{t} B(\xi)x(\xi-\tau)d\xi + \int_{t_0}^{t} C(\xi)x(\mu\xi)\,d\xi +$$
$$+ \sum_{t_i \leq t,\, t_i \in W_-} S\,(t_i, x(t_i - 0), \Delta v(t_i - 0)) + \sum_{t_i < t,\, t_i \in W_+} S\,(t_i, x(t_i), \Delta v(t_i + 0)), \tag{2}$$

where

$$S\,(t, x, \Delta v) = z(1) - z(0), \tag{3}$$

$$\dot{z}(\xi) = \sum_{j=1}^{m} D_j(t)z(\xi)\,\Delta v_i(t), \qquad z(0) = x, \tag{4}$$

W_- and W_+ respectively, the points of the left and right discontinuities of the vector function $v(t)$, $v_i^c(t)$ are the continuous component of the function of bounded variation $v_i(t)$, $\Delta v(t-0) = v(t) - v(t-0)$ and $\Delta v(t+0) = v(t+0) - v(t)$.

We will be understood as an approximable solution of equation (5) on the infinite interval, the corresponding continuation of the solution of integral equation (2) to the interval $[t_0, \infty)$.

SUFFICIENT CONDITIONS OF STABILITY

We introduce the following notation

$$h(t) = \sup_{s \in [\min\{t-\tau, \mu t\}, t]} \|x(s)\|. \tag{5}$$

Here $\|x\|$ is the norm of vector x in l_1, namely $\|x\| = \sum_{i=1}^{n} |x_i|$. Further, we will use the concept of variation of a vector function, which, depending on the choice of the norm, is determined ambiguously. Here we will determine the variation of the vector function using the norm in l_1.

Theorem 1. Let the fundamental matrix of system

$$\dot{x}(t) = A(t)x(t)$$

satisfy the estimate

$$\|Y(t, s)\| \leqslant ce^{-\alpha(t-s)}, \quad c \geqslant 1, \tag{6}$$

where α and c is some positive constants. In addition, we assume that the following estimates are fair.

$$\|D_j(t)\| \leqslant K, \ \|B(t)\| \leqslant K, \ \|C(t)\| \leqslant K \ \forall t \in [t_0, \infty), \ \ j \in \overline{1, m}. \tag{7}$$

Here K is a positive constant. Then if the inequality

$$0 \leqslant \alpha(t - t_0) - Kc \sum_{j=1}^{m} \operatorname*{var}_{[t_0, t]} v_j(\cdot) \leqslant \beta(t - t_0). \tag{8}$$

holds for all$t \in [t_0, \infty)$, then the zero solution of system (5) is asymptotically stable and the we have the inequality

$$\|x(t)\| \leq e^{-(\alpha(t-t_0)-Kc((1+\tau)(t-t_0)-\operatorname*{var}_{[t_0,t]} v(\cdot))} c\, h(t_0). \tag{9}$$

Proof. According to [7, p. 226], the approximate solution of systems (5) is described by an integral equation (2). Let $Y(t, s)$ is the fundamental matrix of the system $\dot{x}(t) = A(t)x(t)$. Then using the Cauchy formula [**?**] for (2) we obtain the representation of the solution in the form

$$x(t) = Y(t, t_0)\varphi(t_0) + \sum_{i=1}^{m} \int_{t_0}^{t} Y(t, \xi)D_i(\xi)x(\xi)\, dv_i^c(\xi) + \int_{t_0}^{t} Y(t, \xi)B(\xi)x(\xi - \tau)\, d\xi + \int_{t_0}^{t} Y(t, \xi)C(\xi)x(\mu\xi)\, d\xi$$

$$+ \sum_{t_i \leq t,\, t_i \in W_-} Y(t, t_i)S(t_i, x(t_i - 0), \Delta v(t_i - 0)) + \sum_{t_i < t,\, t_i \in W_+} Y(t, t_i)S(t_i, x(t_i), \Delta v(t_i + 0)). \tag{10}$$

According to (3) and (4)

$$S(t, x, \Delta v) = z(1) - x = \int_0^1 \sum_{j=1}^{m} D_j(t)z(\xi)\Delta v_j\, d\xi.$$

Adding and subtracting under the integral to $z(\xi)$ x and calculating the norms the left and right sides of this equality in view of the previously made assumptions we obtain

$$\|z(1) - x\| \leq K\|\Delta v\|\, \|x\| + \int_0^1 K\|\Delta v\|\, \|z(\xi) - x\|\, d\xi.$$

Applying to the last inequality the Gronwall lemma [13], we obtain

$$\|S(t, x, \Delta v)\| = \|z(1) - x\| \leq K\|\Delta v\|\, \|x\|(e^{K\|\Delta v\|} - 1). \tag{11}$$

Calculating the norms of the left and right parts in (10) and taking attention to the inequality (6), (7) and (11) we will to have

$$\|x(t)\| \leq c\left[e^{-\alpha(t-t_0)}\|\varphi(t_0)\| + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}\|x(\xi)\|\, d\sum_{j=1}^{m} \operatorname*{var}_{[t_0,\xi]} v_j^c(\cdot) + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}\|x(\xi-\tau)\|\, d\xi + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}\|x(\mu\xi)\|\, d\xi\right] +$$

$$+c\sum_{t_i \leq t, t_i \in W_-} e^{-\alpha(t-t_i)}K\|\Delta v(t_i-0)\|(e^{K\|\Delta v(t_i-0)\|}-1)\|x(t_i-0)\| + c\sum_{t_i<t, t_i \in W_+} e^{-\alpha(t-t_i)}K\|\Delta v(t_i+0)\|(e^{K\|\Delta v(t_i+0)\|}-1)\|x(t_i)\|. \tag{12}$$

Given the assessment of

$$cKm\|\Delta v(t)\|(e^{K\|\Delta v(t)\|} - 1) < e^{cK\|\Delta v(t)\|} - 1,$$

in equity which is not difficult to verify taking advantage of decomposition of the exponent in a row, the inequality (12) can be written as

$$\|x(t)\| \le c\left[e^{-\alpha(t-t_0)}\|\varphi(t_0)\| + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}\|x(\xi)\|\, d\sum_{j=1}^{m}\operatorname*{var}_{[t_0,\,\xi]} v_j^c(\cdot) + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}\|x(\xi-\tau)\|\, d\xi + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}\|x(\mu\xi)\|\, d\xi\right]$$

$$+ \sum_{t_i\le t, t_i\in W_-} e^{-\alpha(t-t_i)}(e^{K\|\Delta v(t_i-0)\|}-1)\|x(t_i-0)\| + \sum_{t_i< t, t_i\in W_+} e^{-\alpha(t-t_i)}(e^{K\|\Delta v(t_i+0)\|}-1)\|x(t_i)\|. \tag{13}$$

Given the notation (**??**) from (13), it is not difficult to get following inequality

$$h(t) \le c\left[e^{-\alpha(t-t_0)}h(t_0) + K\int_{t_0}^{t} e^{-\alpha(t-\xi)}h(\xi)\, d\sum_{j=1}^{m}\operatorname*{var}_{[t_0,\,\xi]} v_j^c(\cdot) + 2K\int_{t_0}^{t} e^{-\alpha(t-\xi)}h(\xi)\, d\xi\right]$$

$$+ \sum_{t_i\le t, t_i\in W_-} e^{-\alpha(t-t_i)}(e^{K\|\Delta v(t_i-0)\|}-1)h(t_i-0) + \sum_{t_i< t, t_i\in W_+} e^{-\alpha(t-t_i)}(e^{K\|\Delta v(t_i+0)\|}-1)h(t_i). \tag{14}$$

Multiply (14) by $e^{\alpha(t-t_0)}$. We introduce the notation

$$y(t) = e^{\alpha(t-t_0)}h(t). \tag{15}$$

As a result, we obtain

$$y(t) \le ch(t_0) + cK\int_{t_0}^{t} y(\xi)\, d(2\xi + \sum_{j=1}^{m}\operatorname*{var}_{[t_0,\,\xi]} v_j^c(\cdot)) +$$

$$+ \sum_{t_i\le t, t_i\in W_-} (e^{cK\|\Delta v(t_i-0)\|}-1)y(t_i-0) + \sum_{t_i< t, t_i\in W_+} (e^{cK\|\Delta v(t_i+0)|}-1)y(t_i).$$

According to Lemma 5.4.3 from [5], every solution the last inequality satisfies the estimate

$$y(t) \le e^{Kc(2(t-t_0)+\operatorname*{var}_{[t_0,\,t]} v(\cdot))} ch(t_0), \tag{16}$$

where

$$\operatorname*{var}_{[t_0,\,t]} v(\cdot) = \sum_{j=1}^{m}\operatorname*{var}_{[t_0,\,t]} v_i^c(\cdot) + \sum_{t_i\le t, t_i\in W_-}\|\Delta v(t_i-0)\| + \sum_{t_i< t, t_i\in W_+}\|\Delta v(t_i+0)\|.$$

Multiplying the inequality (16) by $e^{-\alpha(t-t_0)}$ and considering (15), we have the estimate

$$h(t) \le e^{-(\alpha(t-t_0)-Kc(2(t-t_0)-\operatorname*{var}_{[t_0,\,t]} v(\cdot))} ch(t_0), \tag{17}$$

from which, in particular, the estimate (9) follows. □

CONCLUSIONS

In this article, we considered the stability conditions for the solution of a bilinear system with a generalized action in the system matrix and terms containing the phase vector at $t-\tau$ and μt moments of times, which means that there is a constant and linear delay in the right-hand part of the bilinear system. We obtained sufficient conditions for the stability of the zero solution for such a system.

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