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Optimization of Body Movement with Variable Structure in a Viscous Medium with Non-constant Density

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Abstract. In the class of problems in the optimal control theory for objects motion in a viscous medium, new formulations with advanced mathematical models, including additional parameters of the medium, are investigated. Such problems are degenerate and their solution requires the development of a special mathematical apparatus. In particular, the formulation of the problem of optimal energy consumption for overcoming the resistance of a viscous medium of variable density, the translational displacement of a variable form solid from one phase state to another (the displacement time is specified) is considered. An analysis of the nonlinear relationships of the physical characteristics of a viscous medium has been carried out and a mathematical model has been constructed taking into account these relationships. The features of the of optimal control problem of the movement of bodies with variable geometry in a viscous medium of variable density from the point of view of the theory of optimal control are revealed. The necessary optimality conditions are obtained. The construction of such a displacement is associated with the solution of some two–point boundary value problem for a system of Navier–Stokes equations and having a similar structure of the conjugate system.

INTRODUCTION

In previous investigations in the theory of dynamic optimization of flow, the problem of controlling the body movement in a viscous medium was considered [6, 16, 18]. In particular, optimal trajectories of displacements with large Reynolds numbers were investigated [7]. Then the problem was complicated by the presence of a multiphase medium [8, 9, 10, 11, 12]. For the motion of a rotating cylinder, the mathematical models described in [5, 13, 14]. Such problems of constructing optimal control with the criterion of energy saving are irregular [3, 4]. Because an attempt to solve its by the classical variational procedures is unsuccessful because in this case the Euler–Lagrange equation does not explicitly contain the control. In this case the main problem can be reduced according to a scheme described in [3] to a minimization problem of the work of the drag forces on account of only kinematic relations. The optimal programs of the corresponding auxiliary problem can be found by means of the Euler–Lagrange variational procedure. There is an interest to investigate similar problems from the point of view of the existence of analytical relations for modeling the optimal trajectory for more complicated objects at minimal energy costs.

Recent studies are related to the presence of additional features in control actions and parameters. Namely, the variable geometric shape of the control object makes it possible to obtain safer extremals without critical modes [15, 16].

In this paper problem statement involves the movement in a viscous medium of variable density. This is a real natural effect that occurs for example in the ocean. There is such a natural phenomenon as pycnocline [2]. A pycnocline is the layer where the density gradient $\partial\rho/\partial z$ is greatest within a body of water. An ocean current is generated by the forces such as breaking waves, temperature and salinity differences, wind, Coriolis effect, and tides caused by the gravitational pull of the Moon and the Sun. In addition, the physical properties in a pycnocline driven by density gradients also affect the flows and vertical profiles in the ocean. These changes can be connected to the transport of

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heat, salt, and nutrients through the ocean, and the pycnocline diffusion controls upwelling. Below the mixed layer, a stable density gradient (or pycnocline) separates the upper and lower water, hindering vertical transport.

MATHEMATICAL MODEL

To construct the mathematical model shown in the Fig. 1, one can choose generalized coordinates x, h, and φ. Movement occurs in the longitudinal plane parallel to the velocity vector $\mathbf{V} = (\dot{x}; \dot{h})^T$ under the action of the control force $\mathbf{F} = (F\cos\varphi; F\sin\varphi)^T$ with the resulting angular momentum $\mathbf{U}$. The purpose is to move the object from the initial position to a given one with a minimum energy consumption.

The drag and lift forces are calculated by formulas

$$\mathbf{D} = (-D\cos(\varphi-\alpha); -D\sin(\varphi-\alpha))^T, \quad \mathbf{D}_l = (-D_l\sin(\varphi-\alpha); D_l\cos(\varphi-\alpha))^T, \tag{1}$$

where α is the angle of attack.

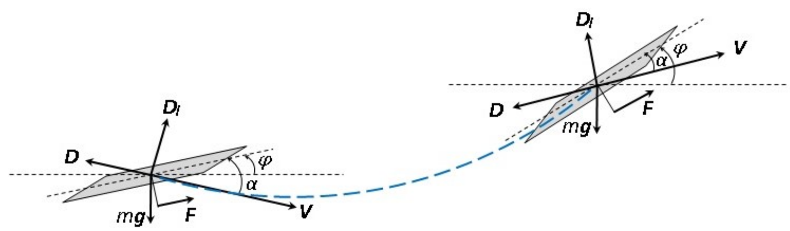

FIGURE 1. Forces and moments acting on the object

The density of sea water depends on temperature, salinity and pressure

$$\rho = f(T, S, P)\,.$$

This formula in general terms expresses the equation of state of seawater. The relationship between the density of water and its determining parameters is non-linear and a simple theoretical formula for it has not yet been obtained. Therefore, approximate equations of state are proposed.

The main factor affecting the density is temperature, so the Boussinesq approximation is sometimes used for oceanological calculations

$$\rho = 1.028(1-\beta T)\,,$$

where β is a tabular coefficient.

Currently used the International equation of sea water state adopted in 1980. In this equation, the density ρ seawater with a pressure in the unit standard atmosphere ($P = 0$) is calculated from the temperature T and salinity S according to the following equation

$$\begin{aligned}\rho(T,S,0) = \rho_w + (8.24493\cdot10^{-1} - 4.0899\cdot10^{-3}T + \\ +7.6438\cdot10^{-5}T^2 - 8.2467\cdot10^{-7}T^{-3} + 5.3875\cdot10^{-9}T^4)S - \\ -(5.72466\cdot10^{-3} - 1.0227\cdot10^{-4}T + \\ +1.6546\cdot10^{-6}T^2S^{3/2} + 4.8314\cdot10^{-4}S^2)\end{aligned} \tag{2}$$

where ρ_w is the density of the average ocean water, adopted by the standard, is defined as

$$\begin{aligned}\rho_w = 999.842594 + 6.793952\cdot10^{-2}T - 9.095290\cdot10^{-3}T^2 + \\ 1.001685\cdot10^{-4}T^3 + 1.120083\cdot10^{-6}T^4 + 6.536332\cdot10^{-9}T^5\end{aligned} \tag{3}$$

A plot of the density of depth is shown in Fig. 2. The International equation of seawater state at atmospheric pressure is valid in salinity ranges from 0 to 42% and temperatures from −2 to +40^oC.

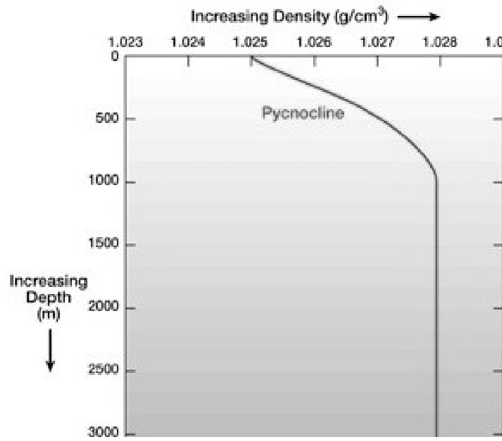

FIGURE 2. Forces and moments acting on the object

Further, to obtain the equations of motion, it is necessary to write out the kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{h}^2) + \frac{1}{2}\frac{ml^2}{12}\dot{\varphi}^2 \tag{4}$$

and the generalized forces corresponding to the generalized coordinates

$$\begin{aligned} Q_x &= -D\cos(\varphi-\alpha) - D_l\sin(\varphi-\alpha) + F\cos(\varphi) \\ Q_h &= -D\sin(\varphi-\alpha) + D_l\cos(\varphi-\alpha) + F\sin(\varphi) - mg \\ Q_\varphi &= U \end{aligned} \tag{5}$$

Then using the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{6}$$

one can obtain movement equations

$$m\ddot{x} = Q_x, \quad m\ddot{h} = Q_h, \quad \frac{1}{12}ml^2\ddot{\varphi} = Q_\varphi. \tag{7}$$

The expression for the power of the control force and momentums is following

$$\dot{W} = (\dot{x}\cos\varphi + \dot{y}\sin\varphi)F. \tag{8}$$

The system of equations (7) and (5) describes body movement.

Now the optimization problem can be formulated.

Problem 1. It is required to find control $F^0(t)$, $\quad 0 \leqslant t \leqslant t_k$, which moves with the minimum power expenses $W(t_k)$ the object for the given time t_k from initial position x_0, h_0 to another one x_k, h_k.

Such a problem is irregular because the Euler-Lagrange equations do not contain control elements and do not allow us to determine their optimal values in terms of phase and conjugate variables.

The drag and lift forces are calculated by the formulae

$$\mathbf{D} = \operatorname{sgn}(\mathbf{V}, \mathbf{D})D\frac{1}{V}\mathbf{V}, \quad \mathbf{D}_l = \operatorname{sgn}(\mathbf{V}, \mathbf{D})sD_l\frac{1}{V}\mathbf{V}^\perp, \tag{9}$$

where $s = \operatorname{sgn}((\mathbf{V}, \mathbf{e})(\mathbf{V}, \mathbf{e}^\perp))$ and $\mathbf{e}$ is the directing vector of the body symmetry axis.

The magnitude of the drag force acting upon the solid body is

$$D = C_D\rho S V^2/2. \tag{10}$$

Analogously, the magnitude of the stationary lift force can be presented as

$$D_l = C_D^{\perp} \rho S V^2/2. \tag{11}$$

Here S is the area of the body projection onto the plane perpendicular to the velocity vector of the body inertia center.

According to the theory of dynamic similitude, the coefficients C_D and $C_D^{\perp}$ depend on the body shape and Reynolds numbers [1].

To determine the angle of attack, one can use the formula

$$\alpha = -s \arccos |(\mathbf{e}, \mathbf{V}/V)|. \tag{12}$$

The problem reduction is proved by that object movement occurs in a potential gravity field. And the changeable part of work of control force is used on change of kinetic energy. Therefore the varied part of work will be equivalent to power expenses for overcoming of hydrodynamic forces of resistance and will be equal to scalar product $(\mathbf{D}^T \mathbf{V})$

$$N = -D\,V = -C_D\,\rho\,S_0\,\sin\alpha\,\frac{V^3}{2}\,. \tag{13}$$

Power of hydrodynamic forces is equal to

$$\dot{N} = -\frac{1}{2}\,C_D\,\rho\,S_0\,(\dot{\alpha}\,V^3\cos\alpha + 3\,\dot{V}\,V^2\,\sin\alpha)\,. \tag{14}$$

Now it is possible to consider dynamics of the object, having assigned function of control to derivatives of the generalized coordinates. It should be noted that the variable geometry can be accounted by introducing the function $S_0 = f(t)$. Thus the initial problem is to an equivalent following problem.

Problem 2. It is required to find functions $\mathbf{V}(t) = (V_x(t), V_h(t))^T$ and $\omega(t)$, minimizing terminal functional $N(t_k)$ at dynamical relations (14) and restrictions

$$\begin{aligned} &x(t_k) = x_k\,, \quad h(t_k) = h_k\,, \quad \varphi(t_k) = \varphi_k\,, \\ &V^2 = \dot{x}^2 + \dot{h}^2\,. \end{aligned} \tag{15}$$

NECESSARY OPTIMALITY CONDITIONS

According to classical Euler–Lagrange procedure it is necessary to write out Hamiltonian

$$H = \lambda_0 \dot{N} + \lambda_1 \dot{x} + \lambda_2 \dot{h} + \lambda_3 \dot{\varphi}$$

and conjugated system with boundary conditions

$$\begin{aligned} -\dot{\lambda}_0 &= \frac{\partial H}{\partial N} = 0\,, & \lambda_0(t_k) &= \frac{\partial \Phi}{\partial N(t_k)} \\ -\dot{\lambda}_1 &= \frac{\partial H}{\partial x}\,, & \lambda_1(t_k) &= \frac{\partial \Phi}{\partial x(t_k)} \\ -\dot{\lambda}_2 &= \frac{\partial H}{\partial h}\,, & \lambda_2(t_k) &= \frac{\partial \Phi}{\partial h(t_k)} \\ -\dot{\lambda}_3 &= \frac{\partial H}{\partial \varphi}\,, & \lambda_3(t_k) &= \frac{\partial \Phi}{\partial \varphi(t_k)} \end{aligned} \tag{16}$$

Here $\Phi = N(t_k) + \nu_1(x(t_k) - x_k) + \nu_2(h(t_k) - h_k) + \nu_3(\varphi(t_k) - \varphi_k)$ is functional describing boundary conditions.

Euler–Lagrange equations

$$\begin{aligned} \frac{\partial H}{\partial \dot{x}} &= \lambda_1 + \frac{\partial \dot{N}}{\partial \dot{x}} = 0 \\ \frac{\partial H}{\partial \dot{h}} &= \lambda_2 + \frac{\partial \dot{N}}{\partial \dot{h}} = 0 \\ \frac{\partial H}{\partial \dot{\varphi}} &= \lambda_3 + \frac{\partial \dot{N}}{\partial \dot{\varphi}} = 0 \end{aligned} \tag{17}$$

allow to calculate Lagrange multipliers and at having substituted them in the conjugated system (16) to write out the equations of optimal movement

$$\begin{aligned} \dot{x} &= V_x\,, \quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial V_x}\right) = \frac{\partial \dot{N}}{\partial x} \\ \dot{h} &= V_h\,, \quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial V_h}\right) = \frac{\partial \dot{N}}{\partial h} \\ \dot{\varphi} &= \omega\,, \quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial \omega}\right) = \frac{\partial \dot{N}}{\partial \varphi} \end{aligned} \tag{18}$$

This is the solution of the original problem.

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