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On Global Asymptotic Stability of Solutions of a System of Ordinary Differential Equations

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Abstract. To investigate global asymptotic stability in general of an equilibrium position of an autonomous system of ordinary differential equations, considered by V.A. Pliss, a function different from the Lyapunov functions is applied. V.A. Pliss proved that for this system it is impossible to construct the Lyapunov function as a sum of a quadratic form and an integral of some nonlinear function defined by the right-hand side of the system.

INTRODUCTION

The general theory of the stability of motion is presented in monographs [1, 2, 3, 4]. The Lyapunov functions [1] are important in the study of the stability of solutions of systems of differential equations. There are different methods for constructing Lyapunov functions for same systems of differential equations [3, 5]. However, for nonlinear systems of differential equations the construction of such functions is a complex problem. It turns out that the Lyapunov functions can be used not only to study the stability of solutions, but also to prove, for example, the boundedness of solutions of systems of ordinary differential equations [6, 7].

We can use not only the second Lyapunov method to establish the stability of motion, but also functions, witch are other functions than the Lyapunov functions. For example, an admissible function was defined in paper [8]. These functions were used in paper [8] to prove the global asymptotic stability of solutions with respect to the part of the variables [9, 10]. In the present paper we will formulate the theorem about the global asymptotic stability of solutions of nonlinear systems of ordinary differential equations with respect to all variables with using the admissible functions. The admissible function will be constructed in explicit form for same particular nonlinear system of ordinary differential equations. In the monograph [5] it was proved that for this system it is impossible to construct Lyapunov function in a form that is often used to construct Lyapunov functions for nonlinear systems.

BASIC DEFINITIONS

Consider a nonlinear system of ordinary differential equations of perturbed motion

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(\mathbf{0}) = 0. \tag{1}$$

Suppose that phase vector $\mathbf{x}(t) = (x_1, \ldots, x_n)^T \in R^n$, $\mathbf{X} : R^n \to R^n$. The position of equilibrium $\mathbf{x} = \mathbf{0}$ of the system (1) is unique.

System (1) can be written in the following form

$$x_i' = X_i(x_1, \ldots, x_n), \quad i = \overline{1, n}. \tag{2}$$

Usually the stability of motion is investigated under the assumption of continuity of the vector-function $\mathbf{X}$=$(X_1, \ldots, X_n)^T$ in domain

$$t \geq 0, \quad ||\mathbf{x}|| \leq h, \quad h > 0, \quad h = \text{const}, \quad ||\mathbf{x}|| = (x_1^2 + \ldots + x_n^2)^{1/2} \tag{3}$$

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and under the assumption of uniqueness and extendability of all solutions of the system (2) in domain (3).

In addition to these, we assume that the functions $X_i(x_1, \ldots, x_n)$ are continuously differentiable functions of the variables x_i, $i = \overline{1, n}$ in all space R^n.

Let $\mathbf{x}(t) = \mathbf{x}(t; 0, \mathbf{x_0}) = (x_1(t; 0, \mathbf{x}_0), \ldots, x_n(t; 0, \mathbf{x}_0)^T$ is a solution of system (2) corresponding to initial condition for $t = 0$

$$\mathbf{x}(0) = \mathbf{x}_0 = (x_{01}, \ldots, x_{0n})^T, \quad x_{0i} = \text{const}, \quad i = \overline{1, n}.$$

Let $\mathbf{y}(t) = \mathbf{y}(t; 0, \mathbf{x}_0) = (y_1(t; 0, \mathbf{x}_0), \ldots, y_m(t; 0, \mathbf{x}_0))^T = (x_1(t; 0, \mathbf{x}_0), \ldots, x_m(t; 0, \mathbf{x}_0))^T$.

We formulate definitions for autonomous systems, following [8].

Definition 1. The set G is called a domain of *uniformly asymptotically* $\mathbf{y}$*-stability* of the unperturbed motion $\mathbf{x} = 0$, if the following properties are valid:

1) on any compact set $K_x = \{\mathbf{x} \in R^n : \|\mathbf{x}\| < r\}$, $r > 0$, $r = \text{const}$, $(K_x \subset G)$ relation $\|\mathbf{x}(t; 0, \mathbf{x}_0)\| \rightrightarrows 0$ is valid for $t \to +\infty$ and $\mathbf{x}_0 \in K_x$;

2) the unperturbed motion $\mathbf{x} = 0$ is uniformly stable, i.e. $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ such that for $\|\mathbf{x}_0\| < \delta \Longrightarrow \|\mathbf{x}(t; 0, \mathbf{x}_0)\| < \varepsilon$ for $t > 0$.

Definition 2. The unperturbed motion $\mathbf{x} = 0$ is called *globally uniformly asymptotically stable*, if the set G coincides with the whole space.

Consider the line segment, which does not include any point of direction field for the system (2). Equation of this segment we write in parametric form

$$\overline{\mathbf{x}} = \mathbf{x}^0 + u(\mathbf{x}^1 - \mathbf{x}^0), \quad u \in [0, 1]. \tag{4}$$

Definition 3. The surface in R^n which consists of positive semi-trajectories of system (2) for $t \geq 0$ started from segment $[\mathbf{x}^0, \mathbf{x}^1]$ we will call *integral surface* (**IS**).

On this surface **IS** we introduce Poincares coordinates (u, t), where the first coordinate u determines a trajectory corresponding to parameter u in (4) and the second coordinate t determines its time length from the segment (4). Then equation of integral surface **IS** can be written as $\mathbf{x}$=$\mathbf{x}(t, u)$. Consider a family of curves on **IS** defined by equation $t = t(c, u)$, where c is an arbitrary parameter. Let

$$\frac{\partial x_i}{\partial u} = p_i(t, u), \quad i = \overline{1, n},$$

or in vector form

$$\frac{\partial x}{\partial u} = \mathbf{p}(t, u), \qquad \mathbf{p} = (p_1, \ldots, p_n)^T.$$

THEOREM ON GLOBALLY UNIFORMLY ASYMPTOTICALLY STABLE

Let $V(\mathbf{x}, \mathbf{p}, t) : R^n \times R^n \times [0, \infty) \to R$ is continuously differentiable with respect to all the variables positive functions. The family of curves on **IS** can be described by the equation

$$\mathbf{x} = \mathbf{x}(t(c, u), u).$$

By $\mathbf{g}$ let denote the vector $\mathbf{g} = (g_1, \ldots, g_n)^T$, whose coordinates g_i, $i = \overline{1, n}$ satisfy the equality

$$\|g\|^2 = \sum_{i>j} (X_i p_j - X_j p_i)^2. \tag{5}$$

The following lemma is valid.

Lemma 1. *Let on* **IS** *function* $V(\mathbf{x}, \mathbf{p}, t)$ *satisfies the following inequality:*

$$\|\mathbf{X}\| V(\mathbf{x}, \mathbf{p}, t) \geq \nu \|\mathbf{g}\|, \tag{6}$$

where $\nu > 0$, $\quad\|\mathbf{X}\| = (X_1^2 + \ldots + X_n^2)^{1/2}$ *and coordinates of vector* $\mathbf{g}$ *satisfy equality* (5).

Then there is a family of curves on **IS** *for which the equality is valid*

$$ds = V(\mathbf{x}, \mathbf{p}, t)du,$$

where $ds^2 = \sum_{i=1}^{n} dx_i{}^2$.

It was shown that these curves satisfy equation

$$\frac{dt}{du} = \frac{-(\mathbf{X} \cdot \mathbf{p}) \pm \sqrt{V^2\|\mathbf{X}\|^2 - \|\mathbf{g}\|^2}}{\|\mathbf{X}\|^2}. \tag{7}$$

Definition 4. The curves, which equations satisfy the equation (7), are called *φ-curves*.

In the particular case, when

$$V(\mathbf{x}, \mathbf{p}, t) = \|\mathbf{p}\| = \left(\sum_{i=1}^{n} p_i^2\right)^{1/2}$$

from equation (7) we have

$$\frac{dt}{du} = \frac{-(\mathbf{X} \cdot \mathbf{p}) \pm |(\mathbf{X} \cdot \mathbf{p})|}{\|\mathbf{X}\|^2}.$$

that allow to build φ-curves on which

$$\frac{dt}{du} = 0, \qquad or \qquad t = \text{const}.$$

Lemma 2. *The following equality is valid*

$$\|\mathbf{X}\|^2\|\mathbf{p}\|^2 - (\mathbf{X} \cdot \mathbf{p})^2 = \|\mathbf{g}\|^2.$$

Definition 5. Let function $V(\mathbf{x}, \mathbf{p}, t)$ is continuously differentiable, positive on set G_1 in variables $\mathbf{x}, \mathbf{p}, t$. We will call the function *admissible function (A-functin)*, if the following conditions are valid:

1. $V(\mathbf{x}, \mathbf{p}, t)$ satisfies inequality (6);
2. $V(\mathbf{x}, \mathbf{p}, t) = 0$.

Definition 6. For A-function $V(\mathbf{x}, \mathbf{p}, t)$ the expression

$$\partial_t V = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i + \sum_{i=1}^{n} \frac{\partial V}{\partial p_i} \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} p_j + \frac{\partial V}{\partial t}$$

we call *a partial derivative of function* $V(\mathbf{x}, \mathbf{p}, t)$ *with respect to t on integral surface*.

The following theorem holds.

Theorem. *Let system* (1) *has the following properties:*

1°. *Set G includes sphere* $\|\mathbf{x}\| \leq \rho$.

2°. *There is an A-function* $V(\mathbf{x}, \mathbf{p}, t)$ *in set* $G_1 = \{\mathbf{x} : \|\mathbf{x}\| \geq r, r < \rho\}$ *on any* **IS** *with non-positive partial derivative of function* $V(\mathbf{x}, \mathbf{p}, t)$ *with respect to t on integral surface.*

Then unperturbed motion $\mathbf{x} = 0$ *is globally uniformly asymptotically stable.*

Proof of the theorem is by contradiction.

Example of system of ordinary differential equations, considered by V.A. Pliss

Consider system of ordinary differentia equation, considered by V.A. Pliss [5]

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - f(x_1) \equiv X_1 \\ \frac{dx_2}{dt} &= x_3 - x_1 \equiv X_2 \\ \frac{dx_3}{dt} &= -ax_1 - bf(x_1) \equiv X_3, \end{aligned} \tag{8}$$

where a, b are constants and $f(x_1) \in C^1[0, \infty)$.

V.A. Pliss assumed the following conditions:

$$a > 0, \qquad 0 \leq b < 1, \qquad f'(x_1) > \frac{a}{1-b}. \tag{9}$$

V.A. Pliss proved that for system (8) it is impossible to construct the Lyapunov's function as a sum of a quadratic form and an integral of some nonlinear function defined by the right-hand side of the system.

We prove that equilibrium position of system (8) is uniformly global asymptotically stable. We apply the theorem for the proof.

Let us verify the condition 1° of the theorem.

Indeed, the roots of the characteristic equation of the first approximation system for system (8) are found from equation

$$\lambda^3 + f'(0)\lambda^2 + \lambda + a + bf'(0) = 0. \tag{10}$$

If conditions (9) are satisfied, the roots of this equation have negative real parts. On the basis of Lyapunov's theorem on the stability of the solution with respect to the first approximation, we verify the asymptotic stability the equilibrium position of system (8) in the small. The condition 1° of the theorem is satisfied.

To verify condition 2° of the theorem, we take an admissible function in the form

$$V(\mathbf{x}, \mathbf{p}) = \sqrt{w}, \tag{11}$$

where w is quadratic form

$$w = g_1^2 + (1-b)g_2^2 + kg_3^2 + 2bg_1g_3 + 2ag_2g_3, \tag{12}$$

in whitch

$$g_1 = X_3p_2 - X_2p_3, \quad g_2 = X_3p_1 - X_1p_3, \quad g_3 = X_2p_1 - X_1p_2, \quad k = (2a^2 + b(1-b))/(1-b).$$

The function (11) is positive. This fact can easily be verified using the Sylvester criterion for the quadratic form (12).

To verify that the function (11) is A-function, it is enough to check that the following condition is valid:

$$||\mathbf{X}||^2 \geq \nu||\mathbf{x}||^2. \tag{13}$$

We have $f(x_1) = f'(c_1)x_1$, where $0 < c_1 < x_1$. Then

$$||\mathbf{X}||^2 = (x_2 - f'(c_1)x_1)^2 + (x_3 - x_1)^2 + (a + bf'(c_1))^2x_3^2. \tag{14}$$

For the characteristic roots of the quadratic form (14) the following equation is obtained:

$$(1-\lambda)[\lambda^2 - \lambda(3 + (f'(c_1))^2) + 1] = 0,$$

which has positive roots

$$\lambda_1 = 1, \quad \lambda_{2,3} = \left([3 + (f'(c_1))^2] \pm \sqrt{[3 + (f'(c_1))^2]^2 - 4}\right)/2,$$

from which the minimal root should be chosen to prove the inequality (13). The inequality (13) is valid by

$$\nu = \min\left\{1, \frac{3}{2}\cdot\left(1 - \frac{\sqrt{3(1-b)^4 + 6(1-b)^2a^2 + a^4}}{3(1-b)^2 + a^2}\right)\right\}.$$

Let us find the partial derivative

$$\partial_t V = \frac{\partial_t w}{2\sqrt{w}} \tag{15}$$

and

$$\partial_t w = g_1\partial_t g_1 + (1-b)g_2\partial_t g_2 + kg_3\partial_t g_3 + b\partial_t g_1 g_3 + bg_1\partial_t g_3 + a\partial_t g_2 g_3 + ag_2\partial_t g_3. \tag{16}$$

In the expression (16) we find $\partial_t g_1$, $\partial_t g_2$, $\partial_t g_3$. We have

$$\partial_t g_1 = \left(\frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3}\right)g_1 - \frac{\partial X_2}{\partial x_1}g_2 - \frac{\partial X_3}{\partial x_1}g_3 = g_2 + (a + bf'(x_1))g_3, \tag{17}$$

$$\partial_t g_2 = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_3}{\partial x_3}\right)g_2 - \frac{\partial X_1}{\partial x_2}g_1 - \frac{\partial X_3}{\partial x_3}g_3 = -f'(x_1)g_2 - g_1, \tag{18}$$

$$\partial_t g_3 = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right)g_3 - \frac{\partial X_1}{\partial x_3}g_1 - \frac{\partial X_2}{\partial x_3}g_2 = -f'(x_1)g_3 - g_2. \tag{19}$$

Substituting the expressions (17)—(19) in the formula (16), we obtain a quadratic form

$$\partial_t w = -\left[\left((1-b)f'(x_1)+a\right)g_2^2 + \left((k_0-b^2)f'(x_1)-ba\right)g_3^2 + (2af'(x_1)+k_0-b)g_2g_3\right]. \tag{20}$$

Using the Sylvester criterion, we verify that the quadratic form (20) is nonpositive. Consequently, the partial derivative (15) is nonpositive and all the conditions of the theorem are satisfied and unperturbed motion $\mathbf{x} = 0$ is globally uniformly asymptotically stable.

CONCLUSION

To study global asymptotic stability of systems of ordinary equations, admissible functions, different from the Lyapunovs functions were proposed. For the system studied by V.A. Pliss, the global asymptotic stability of the equilibrium position with the help of an admissible function was proved.

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REFERENCES

[1] A. M. Lyapunov, *The General Problem of the Stability of Motion* (Taylor and Francis, London.Washington, 1992).
[2] I. G. Malkin, *On the stability of motion in the sense of Lyapunov* (American Mathematical Society, 1951).
[3] N. N. Krasovskii, *Some problems of the theory of stability of motions* (Fizmatgiz, Moscow, 1959), pp. 1–211.
[4] P. Hartman, *Ordinary differential equations* (Wiley, 1973), pp. 1–612.
[5] V. A. Pliss, *Same Problems of Global Stability Theory* (LGU, Leningrad, 1958), pp. 1–183.
[6] T. Yoshizawa, *Stability Theory by Liapounov's Second Method* (Math. Soc. Japan, Tokyo, 1966).
[7] M. Y. Filimonov, Russian Journal of Numerical Analysis and Mathematical Modelling **11**, 27–39 (1996).
[8] M. Y. Filimonov, AIP Conference Proceedings **1789**, p. 040023 (2016), http://aip.scitation.org/doi/pdf/10.1063/1.4968476 .
[9] V. V. Rumyantsev and A. S. Oziraner, *Stability and Stabilization of Motion in Part of Variables* (Nauka, Moscow, 1987), pp. 1–256.
[10] V. I. Vorotnikov, *Partial Stability and Control* (Birkhauser, Boston, 1998), pp. 1–429.