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Modeling of the optimal programs dependence on bodies geometric constraints when moving in a viscous medium

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Abstract. In the paper a body with a variable size moving in a viscous medium is investigated. Optimal control providing movement of the body for the limited time from the initial position to the final position with minimal energy consumption is constructed. In order to minimize the risk of loss of controllability that can happen when moving along an extreme trajectory, it is suggested to dynamically change the geometry of the body. As a result of the simulation, more safe and friendly control modes were obtained.

INTRODUCTION

Here the mathematical model of solid body displacement [13] in a viscous medium is investigated. Researches are part of the theory dynamic optimization of flow [7, 8] by viscous medium and includes investigations of bodies motion in the case of multiphase viscous medium [9, 10]. Such problems of constructing optimal control with the criterion of energy saving are irregular [4]. Because an attempt to solve its by the classical variational procedures is unsuccessful because in this case the Euler–Lagrange equation does not explicitly contain the control. In this case the main problem can be reduced according to a scheme described in [4] to a minimization problem of the work of the drag forces on account of only kinematic relations. The optimal programs of the corresponding auxiliary problem can be found by means of the Euler–Lagrange variational procedure. There is an interest to investigate similar problems from the point of view of the existence of analytical relations for modeling the optimal trajectory for more complicated objects at minimal energy costs.

In this paper, mathematical modeling of the objectʼs motion is performed on the basis of the equations obtained in [13]. It should be noted that it is proposed to avoid critical values of the system parameters by adjusting the surface area of the object. Thus optimal motion program will be safer from the point of view of the possible loss of control.

MATHEMATICAL MODEL OF THE OBJECT

To construct the mathematical model shown in the Fig. 1, one can choose generalized coordinates $x$, $h$, and $\varphi$. Movement occurs in the longitudinal plane parallel to the velocity vector $\mathbf{V} = (\dot{x}; \dot{h})^T$ under the action of the control force $\mathbf{F} = (F \cos \varphi; F \sin \varphi)^T$ with the resulting angular momentum $\mathbf{U}$. The purpose is to move the object from the initial position to a given one with a minimum energy consumption.

The drag and lift forces are calculated by formulas

$$
D = (-D \cos(\varphi - \alpha); -D \sin(\varphi - \alpha))^T, \quad D_l = (-D_l \sin(\varphi - \alpha); D_l \cos(\varphi - \alpha))^T
$$

(1)

where $\alpha$ is the angle of attack.
Further, to obtain the equations of motion, it is necessary to write out the kinetic energy

$$ T = \frac{1}{2} m (\dot{x}^2 + \dot{h}^2) + \frac{1}{2} \frac{m l^2}{12} \dot{\varphi}^2 $$

and the generalized forces corresponding to the generalized coordinates

$$ Q_x = -D \cos(\varphi - \alpha) - D_l \sin(\varphi - \alpha) + F \cos(\varphi) $$
$$ Q_h = -D \sin(\varphi - \alpha) + D_l \cos(\varphi - \alpha) + F \sin(\varphi) - mg $$
$$ Q_\varphi = U $$

Then using the Lagrange equations

$$ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i $$

one can obtain movement equations

$$ m \ddot{x} = Q_x, \quad m \ddot{h} = Q_h, \quad \frac{1}{12} m l^2 \ddot{\varphi} = Q_\varphi. $$

The expression for the power of the control force and momentums is following

$$ \dot{W} = (\dot{x} \cos \varphi + \dot{y} \sin \varphi) F. $$

The system of equations (5) and (3) describes body movement.

Problem 1. It is required to find control \( F(t) \), \( 0 \leq t \leq t_k \), which moves with the minimum power expenses \( W(t_k) \) the object for the given time \( t_k \) from initial position \( x_0, h_0 \) to another one \( x_k, h_k \).

Such a problem is irregular because the Euler-Lagrange equations do not contain control elements and do not allow us to determine their optimal values in terms of phase and conjugate variables.

The drag and lift forces are calculated by the formulae

$$ D = \text{sgn}(V, D) D_1 \frac{1}{V} V, \quad D_l = \text{sgn}(V, D) s D_l \frac{1}{V} V^\perp, $$

where \( s = \text{sgn}((V, e)(V, e^\perp)) \) and \( e \) is the directing vector of the body symmetry axis.

The magnitude of the drag force acting upon the solid body is

$$ D = C_D \rho S V^2 / 2. $$

Analogously, the magnitude of the stationary lift force can be presented as

$$ D_l = C_{lD} \rho S V^2 / 2. $$
Here $S$ is the area of the body projection onto the plane perpendicular to the velocity vector of the body inertia center. According to the theory of dynamic similitude, the coefficients $C_D$ and $C_D^*$ depend on the body shape and Reynolds numbers.

To determine the angle of attack, one can use the formula
\[
\alpha = -s \arccos |\mathbf{e}, \mathbf{V}/V|.
\] (10)

The problem reduction is proved by that object movement occurs in a potential gravity field. And the changeable part of work of control force is used on change of kinetic energy. Therefore the varied part of work will be equivalent to power expenses for overcoming of hydrodynamic forces of resistance and will be equal to scalar product $(\mathbf{F}^T \mathbf{V})$
\[
N = -DV = -C_D \rho S_0 \sin \alpha \frac{V^3}{2}.
\] (11)

Power of hydrodynamic forces is equal to
\[
N = -\frac{1}{2} C_D \rho S_0 (\dot{V}^3 \cos \alpha + 3 V^2 \sin \alpha).
\] (12)

Now it is possible to consider dynamics of the object, having assigned function of control to derivatives of the generalized coordinates. It should be noted that the variable geometry can be accounted by introducing the function $S_0 = f(t)$. Thus the initial problem is to an equivalent following problem.

**Problem 2.** It is required to find functions $\mathbf{V}(t) = (V_x(t), V_y(t))^T$ and $\omega(t)$, minimizing terminal functional $N(t_k)$ at dynamical relations (12) and restrictions
\[
x(t_k) = x_k, \quad h(t_k) = h_k, \quad \varphi(t_k) = \varphi_k, \\
V^2 = \dot{x}^2 + \dot{h}^2.
\] (13)

According to classical Euler–Lagrange procedure it is necessary to write out Hamiltonian
\[
H = \lambda_0 \dot{N} + \lambda_1 \dot{x} + \lambda_2 \dot{h} + \lambda_3 \dot{\varphi}
\]
and conjugated system with boundary conditions
\[
-\lambda_0 = \frac{\partial H}{\partial N} = 0, \quad \lambda_0(t_k) = \frac{\partial \Phi}{\partial N(t_k)}
\]
\[
-\lambda_1 = \frac{\partial H}{\partial x}, \quad \lambda_1(t_k) = \frac{\partial \Phi}{\partial x(t_k)}
\]
\[
-\lambda_2 = \frac{\partial H}{\partial h}, \quad \lambda_2(t_k) = \frac{\partial \Phi}{\partial h(t_k)}
\]
\[
-\lambda_3 = \frac{\partial H}{\partial \varphi}, \quad \lambda_3(t_k) = \frac{\partial \Phi}{\partial \varphi(t_k)}
\] (14)

Here $\Phi = N(t_k) + v_1(x(t_k) - x_k) + v_2(h(t_k) - h_k) + v_3(\varphi(t_k) - \varphi_k)$ is functional describing boundary conditions.

Euler–Lagrange equations
\[
\frac{\partial H}{\partial x} = \lambda_1 + \frac{\partial N}{\partial x} = 0 \\
\frac{\partial H}{\partial h} = \lambda_2 + \frac{\partial N}{\partial h} = 0 \\
\frac{\partial H}{\partial \varphi} = \lambda_3 + \frac{\partial N}{\partial \varphi} = 0
\] (15)

allow to calculate Lagrange multipliers and at having substituted them in the conjugated system (14) to write out the equations of optimal movement
\[
\dot{x} = V_x, \quad \frac{d}{dt} \left( \frac{\partial N}{\partial V_x} \right) = \frac{\partial N}{\partial x} \\
\dot{h} = V_h, \quad \frac{d}{dt} \left( \frac{\partial N}{\partial V_h} \right) = \frac{\partial N}{\partial h} \\
\dot{\varphi} = \omega, \quad \frac{d}{dt} \left( \frac{\partial N}{\partial \omega} \right) = \frac{\partial N}{\partial \varphi}
\] (16)

This is the solution of the original problem.
MODEL EXAMPLE

Now consider the following model example. Let our object be in some initial state, see left part of Fig. 2. And it is required to transfer object to another final state (right part of the figure). Namely, to change its position and direction with magnitude of the velocity vector $V$.

A strong increase in engine thrust $F$ with $S = const$ leads to a critical angle of attack $\alpha$ and to decrease in the velocity value $V$ to zero after moment $t_1$ (see Fig. 3). One can avoid this negative effect by lessening object surface area. This is shown by the blue line in the figure.

Thus, changing the surface area value can avoid critical modes and reduce the risk of loss of object control.
ACKNOWLEDGMENTS

The investigation was supported by the Russian Foundation for Basic Research, project no. 16-01-00505-a.

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