



Conference Paper

New Approach to Study Numeric Triangles

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Abstract

In this paper we deal with numerical triangles defined by generating functions in the power of k . We present new approach to study such triangles that allow us to get methods for obtaining generating functions of the diagonals of the triangles. Methods for obtaining generating functions of the central coefficients and the diagonal $T_{2n,n}$ of the triangle $T_{n,k}$ are discussed. Some further ideas for application are given.

1. Introduction

Triangles play an important role in combinatorics, number theory, mathematical analysis, and other math areas. By a triangle we mean here a numeric triangle that consists of numbers. The triangles were studied by the following mathematicians: B. Pascal, L. Euler, J. Bernoulli, J. Riordan, J. Knuth, V.I. Arnold, L. Shapiro, R. Sprugnoli, O.V. Kuzmin, P. Luschny, P. Barry, etc.

There are a lot of such triangles, for instance, the Bernoulli-Euler triangle, the Catalan triangle, the Motzkin triangle. That is reflected in many papers and books [1-3]. But the most famous triangle is the Pascal triangle (Figure 1).

Pascal's triangle is the triangle of the binomial coefficients $\binom{n}{k}$.

Pascal's triangle is well studied. For example,

- the sum of the entries in the n -th row of Pascal's triangle is the n -th power of 2;
- the triangle is symmetrical. The numbers on the left side have the identical matching numbers on the right side;
- Pascal's triangle determines the coefficients which arise in binomial expansions $(x + y)^n$;
- in a row p where p is a prime number, all the terms in that row except the first and the last are multiples of p ;

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0								1							
1							1	1							
2						1	2	1							
3					1	3	3	1							
4				1	4	6	4	1							
5			1	5	10	10	5	1							
6		1	6	15	20	15	6	1							
7	1	7	21	35	35	21	7	1							
8	1	8	28	56	70	56	28	8	1						
...														

Figure 1: Pascal's triangle.

- each term in the triangle is equal to the number of ways to get to it from the top, moving either to the right-down or to the left-down;
- Pascal's triangle has properties related to the Fibonacci numbers and the Catalan numbers and many other interesting properties.

Pascal's triangle may be constructed in the following way: in row 0 (the topmost row), there is a unique nonzero entry 1. Each entry of each subsequent row is constructed by adding the number above and to the left with the number above and to the right, treating blank entries as 0. Also it could be constructed by the formula of binomial coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

or by the Riordan array of the form

$$\left(\frac{1}{1-x}, \frac{x}{1-x} \right).$$

In general, triangles may be constructed by the following ways:

- by an explicit formula;
- by a recurrence relation;
- by a generating function or a Riordan array.

In common, the topmost row in the triangles starts from $T_{0,0}$ (Figure 2).

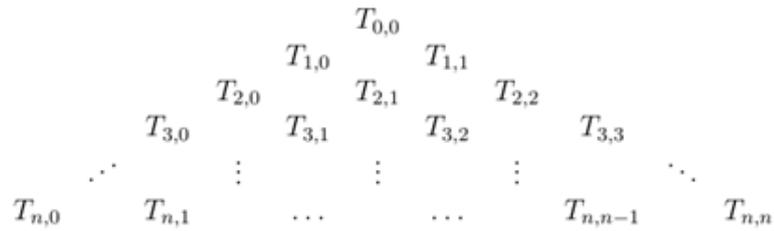


Figure 2: Common view of the triangles.

The scope of the paper is a development of approach to study numeric triangles and its application for finding generating functions of diagonals of the triangles. Some works of P. Barry [4] were carried out to obtain some relations for a generating function of the central diagonal, but in general, a study of generating functions of diagonals has not been presented.

2. Generating functions and Riordan arrays

Generating functions are a powerful tool to study such triangles. By the generating function we mean the following definition [3]:

Definition 1. Let a_0, a_1, a_2, \dots is an infinite integer sequence. Then a power series of the form

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a generating function for the integer sequence a_n .

As we mentioned above triangles may be constructed by Riordan arrays. The Riordan array defined by a pair of generating functions $(F(x), G(x))$ was introduced by L. Shapiro in 1991 [5].

The Riordan array is a triangle defined by the following way:

$$R_{n,k} = [x^n] F(x) G^k(x),$$

where $[x^n]$ is the coefficient of x^n in the generating function. Depend on variants of the generating functions there will be constructed different triangles. For instance, for Pascal's triangle we have

$$R_{n,k} = [x^n] \frac{1}{1-x} \left(\frac{x}{1-x} \right)^k.$$

There exist many operations on such triangles defined by the Riordan arrays: diagonalization, linearization, composition, inversion, etc. (see the paper of D. Merlini, R. Sprugnoli, M. C. Verri [6]). However, the implementation of these operations is represented by examples and in general form the solution is not given.

3. The approach for constructing triangles based on powers of generating functions

Studying triangles defined by the Riordan arrays, we noticed that triangles are constructed exactly by powers of the generating function $G^k(x)$. So we suggest to construct triangles by coefficients of powers of the generating function $G^k(x)$ without the generating function $F(x)$. This notation coincides with the concept of Riordan array $(1, G(x))$ or $(\frac{G(x)}{x}, G(x))$, which is given by Shapiro [5]. But there is no need to study 2 generating functions.

Let $G(x)$ be a power series without a constant term, i.e.,

$$G(x) = \sum_{n>0} g_n x^n,$$

where $g_0 = 0$ and $g_1 \neq 0$.

Then a triangle $T_{n,k}$ is defined as follows:

$$(G(x))^k = \sum_{n \geq k} T_{n,k} x^n.$$

Here we assume that $G^0(x) = T_{0,0} = 1$.

Hence, the generating function $G(x)$ in the power of k gives the following triangle $T_{n,k}$ (Figure 3):

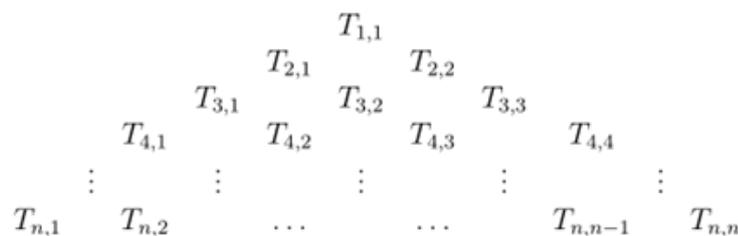


Figure 3: View of the triangles defined by powers.

That is, a numbering of terms of the triangle goes with (1,1) not (0,0).

The following notation will be used throughout this paper. In paper [7, 8] the notion of the composita of the given generating function $G(x) = \sum_{n>0} g_n x^n$ was introduced.

Definition 2. The composita is the function of two variables defined by

$$G^\Delta(n, k) = \sum_{\pi_k \in C_n} g_{\lambda_1} g_{\lambda_2} \dots g_{\lambda_k},$$

where n, k, λ_i are integers that are greater than 0, C_n is the set of all compositions of n , and π_k is the composition into k parts exactly ($\sum_{i=1}^k \lambda_i = n$).

The composita is a two variable function of coefficients of the generating function $G(x)$ in the power of k . So the generating function for the composita is defined as follows:

$$G^k(x) = \sum_{n \geq k} G^\Delta(n, k) x^n = \sum_{n \geq k} T_{n,k} x^n.$$

According above notation we get the following relations for Pascal's triangle. Terms of Pascal's triangle are defined by

$$T_{n,k} = G^\Delta(n, k) = \binom{n-1}{k-1},$$

where n and k start with 1.

Then Pascal's triangle is defined by the following expression:

$$G^k(x) = (xH(x))^k = \left(\frac{x}{1-x}\right)^k = \sum_{n \geq k} \binom{n-1}{k-1} x^n,$$

where $H(x) = \frac{1}{1-x} = \sum_{n \geq 0} h_n x^n$.

4. The approach for constructing triangles based on powers of generating functions

Considering triangle as coefficients of generating function in the power of k allow us to obtain very interesting results related to obtaining generating functions for diagonals of the triangle.

Firstly we show an application to get a generating function of central diagonal of the triangle [9].

Lemma 1. Suppose

$$\bullet H(x) = \sum_{n \geq 0} h_n x^n \text{ is a generating function such that } h_0 \neq 0,$$

- $G(x) = xH(x)$ with $G^k(x) = \sum_{n \geq k} G^\Delta(n, k)x^n = \sum_{n \geq k} T_{n,k}x^n$,
- $A(x) = \sum_{n > 0} a_n x^n$ is the generating function, which is obtained from the functional equation $A(x) = xH(A(x))$.

Then the generating function $F(x)$ of the central coefficients of the triangle $T_{n,k}$ is equal to the first derivative of the function $A(x)$:

$$F(x) = A'(x) = \sum_{n > 0} G^\Delta(2n - 1, n)x^{n-1}.$$

As application, using Lemma 1 it easy to obtain the well-known generating function of the central coefficients of Pascal's triangle.

Since for Pascal's triangle there hold $H(x) = \frac{1}{-1-x} = \sum_{n \geq 0} h_n x^n$, then solving the functional equation

$$A(x) = \frac{x}{1 - A(x)},$$

we get

$$A(x) = \frac{1 - \sqrt{1 - 4x}}{2}.$$

After differentiating $A(x)$ respect to x , we arrive at the desired result:

$$F(x) = A'(x) = \frac{1}{\sqrt{1 - 4x}}.$$

Also the inverse problem was solved. In [9] there was given method for obtaining a unique triangle, when we know only the generating function of the central coefficients of the triangle.

Recently in [10], we had shown how to find a generating function of the diagonal $T_{2n,n}$.

Lemma 2. Suppose

- $H(x) = \sum_{n \geq 0} h_n x^n$ is a generating function such that $h_0 \neq 0$,
- $G(x) = xH(x)$ with $G^k(x) = \sum_{n \geq k} G^\Delta(n, k)x^n = \sum_{n \geq k} T_{n,k}x^n$,
- $A(x) = \sum_{n > 0} a_n x^n$ is the generating function, which is obtained from the functional equation $A(x) = xH(A(x))$.

Then the generating function $F(x) = \sum_{n \geq 0} f_n x^n$ of the diagonal $T_{2n,n}$ of the triangle $T_{n,k}$ is defined by the following expression:

$$F(x) = \frac{x A'(x)}{A(x)}.$$

Also it should be noted that by using the proposed approach P. Barry [11] found formulas for the central coefficients of Riordan matrices and Cyril Banderier and other [12] found explicit formulas for enumeration of lattice paths.

5. Conclusions and further directions of study

New approach to study triangles defined by the powers of generating functions is presented. That allow us to get methods for obtaining generating functions of diagonals of such triangles. Further developing the ideas presented in the paper will allow us to obtain:

- methods for finding generating functions of the diagonals of form $T_{2n+k,n}$ in general;
- methods for solving iterative equations of the form $A(A(x)) = F(x)$;
- methods for solving functional equations $A(x) = F(xA^m(x))$, where m is integer.

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