

# On an Algorithm for Dynamic Reconstruction in Systems with Delay in Control

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**Abstract.** We discuss a problem of the dynamic reconstruction of unknown input controls in nonlinear vector equations. A regularizing algorithm is proposed for reconstructing these controls simultaneously with the processes. The algorithm is stable with respect to informational noises and computational errors.

**Keywords:** dynamic reconstruction, method of auxiliary models.

## 1 Introduction Problem Statement

Consider a controlled system described by the following equation

$$\dot{x}(t) = f_1(t, u_t(s), x_t(s)) + f_2(t, x_t(s))u(t) \quad (1)$$

with the initial state

$$u_{t_0}(s) = u_0(s) \in C([- \tau_m^u, 0]; R^{n_1}), \quad x_{t_0}(s) = x_0(s) \in C([- \tau_n^x, 0]; R^{n_2}). \quad (2)$$

Here  $t$  is time from a fixed interval  $T = [t_0, \vartheta]$  ( $t_0 < \vartheta < +\infty$ );  $x(t) = (x_1(t), \dots, x_{n_2}(t))$  is the phase state of the system;  $u(t) = (u_1(t), \dots, u_{n_1}(t))$  is a control; the symbols  $x_t(s)$  and  $u_t(s)$  mean the functions  $x_t(s) = x(t+s)$  for  $s \in [-\tau_n^x, 0]$  and  $u_t(s) = u(t+s)$  for  $s \in [-\tau_m^u, 0]$ , respectively. We assume that initial state (2) is Lipschitz. For simplicity, we assume also that the initial state  $x_0(s)$ ,  $u_0(s)$  is fixed and known. The control  $u = u(t) = (u_1(t), \dots, u_{n_1}(t))$  is called an admissible control if its components  $u_i(t)$ ,  $i \in [1 : n_1]$ , are Lebesgue measurable functions on the interval  $T$  and values  $u(t)$  belong to a given compact set  $P$  from Euclidean space  $R^{n_1}$  for almost all  $t \in T$ . The set of all admissible controls is denoted by  $P(\cdot)$ . Therefore,  $P(\cdot) = \{u(\cdot) \in L_2(T; R^{n_1}) : u(t) \in P \text{ for a. a. } t \in T\}$ . By the trajectory (or the solution)  $x(\cdot)$  of equation (1) with initial state (2) corresponding to some admissible control  $u(\cdot)$ , we call absolutely continuous on  $T$  function  $x = x(t)$  satisfying (1) for a.a.  $t \in T$ .

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\* This work was supported by the Russian Foundation for Basic Research (12-01-00175-a), by the Ural-Siberian Integration Project (12-C-1-1017), and by the Program for support of leading scientific schools of Russia (6512.2012.1).

**Condition 1.** The elements of matrix function

$$f_{2ij}(t, x_t(s)) = f_{2ij}(t, x(t), x(t - \tau_1^x), \dots, x(t - \tau_n^x)), \quad i \in [1 : n_2], \quad j \in [1 : n_1],$$

and vector-valued function

$$\begin{aligned} & f_{1i}(t, u_t(s), x_t(s)) = \\ & = f_{1i}(t, u(t - \tau_1^u), \dots, u(t - \tau_m^u), x(t), x(t - \tau_1^x), \dots, x(t - \tau_n^x)), \quad i \in [1 : n_2] \end{aligned}$$

satisfy the Lipschitz conditions

$$|f_{2ij}(t_1, x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)}) - f_{2ij}(t_2, x_0^{(1)}, x_1^{(2)}, \dots, x_n^{(2)})| \leq \tag{3}$$

$$\leq C_1(|t_2 - t_1| + \sum_{j=0}^n |x_j^{(1)} - x_j^{(2)}|),$$

$$|f_{1i}(t_1, u_1^{(1)}, \dots, u_m^{(1)}, x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)}) - f_{1i}(t_2, u_1^{(2)}, \dots, u_m^{(2)}, x_0^{(2)}, x_1^{(2)}, \dots, x_n^{(2)})|$$

$$\leq d_1(|t_2 - t_1| + \sum_{i=1}^m |u_i^{(1)} - u_i^{(2)}| + \sum_{j=0}^n |x_j^{(1)} - x_j^{(2)}|). \tag{4}$$

In this case, under this condition for any pair, i.e., for initial state (2) and the control  $u(\cdot) \in P(\cdot)$ , there exists a unique solution of equation (1).

Let  $u(\cdot)$  be an admissible control realizing during the given time interval  $T$ ;  $x(\cdot)$  be the real motion generated by this control. We assume that the phase states  $x(\tau_i)$  of the system are inaccurately measured at frequent enough time moments  $\tau_i \in T$  in the process. Measurement results  $\xi^h(\tau_i) \in R^{n_2}$  satisfy the inequalities

$$|\xi^h(\tau_i) - x(\tau_i)| \leq h. \tag{5}$$

Here, the quantity  $h \in (0, 1)$  specifies the measurement error.

In the present paper, we construct an algorithm that reconstructs the control  $u(\cdot)$  on the basis of the current information  $\xi^h(\cdot)$  in real time. Since the exact reconstruction is impossible due to the error of measurements  $\xi^h(\cdot)$  we require that the algorithm should generate some approximation. Namely, it is required to construct an algorithm allowing us, on the basis of the inaccurate measurements  $\xi^h(\cdot)$ , and in real time, to form the admissible control  $v^h(\cdot)$  such that the mean-square deviation of  $v^h(\cdot)$  from  $u(\cdot)$ ; i.e.,

$$|v^h(\cdot) - u(\cdot)|_{L_2(T)}^2 = \int_{t_0}^{\vartheta} |v^h(t) - u(t)|^2 dt, \tag{6}$$

is arbitrarily small for the sufficiently small measurement error  $h$ . Since the measurements are inaccurate it is in general impossible to identify  $u(t)$  precisely, therefore the problem is to approximate the input by some function  $v^h(t)$ .

Here and below, the symbol  $|\cdot|$  stands for both the Euclidean norm and the corresponding matrix norm and for the modulo of a number. In what follows, we set  $\tau_m^u = \tau_n^x = \tau$  for simplicity, and by  $\xi^h(\cdot)$  we denote the function  $\xi^h(t)$ ,  $t \in [t_0 - \tau, \vartheta]$  such that  $\xi^h(t) = x_0(t - t_0)$  for  $t \in [t_0 - \tau, t_0]$ ,  $\xi^h(t) = \xi^h(\tau_i)$  for  $t \in [\tau_i, \tau_{i+1})$ ,  $i \in [0 : d - 1]$ , where  $\tau_i = \tau_{h,i}$ ,  $d = d_h$ ,  $\xi^h(\tau_i)$  satisfies (5).

The suggested solution outline is the following ([1–6]). An auxiliary control system (model  $M$ ) described by equation of the form

$$\dot{w}(t) = F(t, \xi_t^h(s), v_t^h(s)), \quad w_{t_0}(s) = w_0(s), \quad t \in T \tag{7}$$

is associated with the real dynamical system (1). Here the vector  $w \in R^{n_2}$  characterizes state of the model, the form of function  $F$  is corrected below, vector  $v^h$  is control action. After that, the problem of reconstruction of input  $u(\cdot)$  is replaced by the problem of positional control of the model. This process is realized on the time interval  $T$  in such a way that control  $v^h(\cdot)$  “approximates” appropriately  $u(\cdot)$ . First, one takes a uniform net  $\Delta = \{\tau_i\}_{i=0}^m$ ,  $\tau_{i+1} = \tau_i + \delta$ ,  $\delta > 0$ ,  $i \in [0 : m]$ ,  $\tau_0 = 0$ ,  $\tau_m = T$  with the step  $\delta$ . Then, on the interval  $t \in [\tau_i, \tau_{i+1})$  the model is acted upon the controls

$$v_i^h = V_h(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) \tag{8}$$

calculated at the moment  $\tau_i$  by use of some rule, which hereinafter we shall identify with mapping  $V_h$ . Thus, the controls in the model are realized by the method of feedback control. Its value on the interval  $[\tau_i, \tau_{i+1}]$  depends on the measurement results  $\xi^h(\cdot)$  corresponding to the phase state  $x(\cdot)$  of the system (1) and state  $w$  of the model (7). The described process forms the piece-wise function

$$v^h(t) = v_i^h, \quad t \in [\tau_i, \tau_{i+1})$$

in real time synchro with the motion of real system (1). Thus, to solve the problem above, we should specify a model and a control law for this model.

## 2 Algorithm for Solving the Problem

As a model, we take the following system of linear ordinary differential equation

$$\dot{w}(t) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h + 2(\xi^h(\tau_i) - w(\tau_i)), \tag{9}$$

$$w \in R^{n_2}, \quad t \in [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{h,i}, \quad v_{t_0}^h(s) = u_0(s),$$

with the initial state  $w(t_0) = \xi^h(t_0)$ . The solution of this equation  $w(\cdot) = w(\cdot; t_0, w_{t_0}(s), v^h(\cdot))$  is understood in the sense of Caratheodory. So, the right-hand side of equation of the model (7) has the form

$$F(t, \xi_t^h(s), v_t^h(s)) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h + 2(\xi^h(\tau_i) - w(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}).$$

Introduce the following notation:  $\Delta^{(j)} = [t_j, t_{j+1}]$ ,  $t_j = t_0 + \tau_1^x j$ ; the symbol  $l$  stands for the integer part of the number  $\tau/\tau_1^x$ ;  $j_* = \max\{j : t_j < \vartheta\}$ ,

$$g_j(h) = h^{(1/3)^j}, \quad j \in [1 : j_*].$$

Fix a partition of the interval  $T$  with a step  $\delta = \delta(h)$  depending on the measurement error  $h$ , i.e.,

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{d_h}, \quad \tau_i = \tau_{h,i}, \quad \tau_{h,0} = t_0, \quad \tau_{h,d_h} = \vartheta, \quad (10)$$

(for simplicity, we assume that  $\tau_i - \tau_{i-1} = \delta = \delta(h)$ ). Without loss of generality, we can suppose that the partition  $\Delta_h$  is chosen in such a way that  $t_j \in \Delta_h$ . Define the law of forming the control  $v_i^h$  in the model (for  $\tau_i \in [t_j, t_{j+1}) \cap T$ ) by the relations

$$\begin{aligned} V_h(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) &= V_j(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) \\ &= \arg \min\{2(l_i, f_2(\tau_i, \xi_{\tau_i}^h(s))v) + \alpha_j |v|^2 : v \in P\}. \end{aligned} \quad (11)$$

Here  $\alpha_j$  is a parameter,  $j \in [0 : j_*]$ ,  $l_i = w(\tau_i) - \xi^h(\tau_i)$ .

**Condition 2.** Let  $n_2 \geq n_1$ , and let there exists a number  $c_* > 0$  such that the matrix  $f_2(t, x_i(s))$  has a minor of order  $n_1$  with the property: the  $n_1 \times n_1$ -dimensional matrix  $\bar{f}_2(t) = \bar{f}_2(t, x_i(s))$  corresponding to this minor satisfies the inequality

$$|\bar{f}_2(t)u| \geq c_*|u|$$

for each  $t \in T$  and all  $u \in R^{n_1}$ .

We choose the parameter  $\alpha_j$  which plays the role of the regularizer, as follows:

$$\alpha_0 = Ch^{2/3}, \quad \alpha_j = Cg_j^{2/3}(h), \quad j \geq 1, \quad C = \text{const} > 0. \quad (12)$$

Let us describe the algorithm for solving the problem above.

Before the initial moment the value  $h$  and the partition  $\Delta = \Delta_h$  with diameter  $\delta = \delta(h)$  are fixed. The work of the algorithm starting at time  $t = 0$  is decomposed into  $m_h - 1$  steps. At the  $i$ -th step carried out during the time interval  $\delta_i = [\tau_i, \tau_{i+1})$ ,  $\tau_i = \tau_{h,i}$ , the following actions take place. First, at time moment  $\tau_i$  vector  $v_i^h$  is calculated by formula (11). Then the control  $v^h(t) = v_i^h$  is fed onto the input of the model (9). After that, we transform the state  $w_{\tau_i}(s)$  of the model into  $w_{\tau_{i+1}}(s)$ . The procedure stops at time  $\vartheta$ .

The following theorem is true.

**Theorem 1.** *Let  $\delta = \delta(h) \leq h$ . Then the inequalities*

$$\nu^{(j)} \equiv |v^h(\cdot) - u(\cdot)|_{L_2(\Delta^{(j-1)}; R^{n_1})}^2 \leq c_j g_j(h), \quad j \in [1 : j_*],$$

are valid. Here,  $v^h(t) = u(t)$  for  $t \in [t_0 - \tau, t_0]$ ,  $v^h(t) = u_0(-\tau)$  for  $t \in [t_0 - \tau - \tau_1^u, t_0 - \tau)$ .

The proof of the theorem is based on auxiliary statements, which are used in forthcoming considerations. Introduce two systems

$$\begin{aligned} \dot{p}(t) &= f_1(t) + f_2(t)u_1(t), \quad t \in T, \\ \dot{q}(t) &= F_1(t) + F_2(t)u_2(t), \end{aligned}$$

where  $p(t), q(t) \in R^n, f_1(\cdot), F_1(\cdot) \in L_2(T; R^n), f_2(\cdot) \in L_2(T; R^{n \times r}), F_2(\cdot) \in L_2(T; R^{n \times r}), u_1(\cdot), u_2(\cdot) \in L_2(T; R^r), |u_l(\cdot)|_{L_\infty(T; R^r)} \leq K, l = 1, 2.$

Introduce the notation:  $\Delta_*^{(j)} = [t_j^*, t_{j+1}^*] \cap T, t_j^* = t_0 + \tau_* j, j \in [0 : j_0], \Delta^{(-1)} = [t_0 - \tau_*, t_0], \tau_* = \text{const} \in (0, \vartheta - t_0), j_0 = \max\{j : t_j^* \leq \vartheta\}.$  Let  $r \leq n$  and let there exists a number  $c > 0$  such that the matrix  $f_2(t)$  has a minor of order  $r$  such that the  $r \times r$ -matrix  $\bar{f}_2(t)$  corresponding to this minor satisfies the following inequality:  $|\bar{f}_2(t)u| \geq c|u|$  for each  $t \in T$  and all  $u \in R^r.$

It is easy to verify the following lemmas.

**Lemma 1.** *Let the function  $t \rightarrow (\bar{f}_2(t))^{-1}u_1(t)$  be a function of bounded variation on  $T$  and let the conditions*

$$\begin{aligned} |f_1(\cdot) - F_1(\cdot)|_{L_2(\Delta_*^{(j)}; R^n)}^2 &\leq a_1^{(j)}, \quad |f_2(\cdot) - F_2(\cdot)|_{L_2(\Delta_*^{(j)}; R^{n \times r})}^2 \leq a_2^{(j)}, \\ |p(t) - q(t)|^2 + \tilde{\alpha}_j \int_{t_j^*}^t \{|u_2(\nu)|^2 - |u_1(\nu)|^2\} d\nu &\leq a_3^{(j)} \quad t \in [t_j^*, t_{j+1}^*], \\ |p(t_j^*) - q(t_j^*)|^2 &\leq a_4^{(j)}, \quad \tilde{\alpha}_j = \text{const} \in (0, +\infty) \end{aligned}$$

be true. Then the inequality

$$|u_1(\cdot) - u_2(\cdot)|_{L_2(\Delta_*^{(j)}; R^r)}^2 \leq K_j \left\{ \sum_{l=1}^4 (a_l^{(j)})^{1/2} + \tilde{\alpha}_j^{-1/2} \right\} + a_3^{(j)} / \tilde{\alpha}_j$$

is valid.

**Lemma 2.** *The bunches of solutions of systems (1) and (9) are bounded in the space  $W^{1,\infty}(T; R^{n_2}) = \{x(\cdot) \in L_2(T; R^{n_2}); \dot{x}(\cdot) \in L_2(T; R^{n_2})\}.$*

We use the relation

$$\varepsilon_j(t) = |x(t) - w(t)|^2 + \alpha_j \int_{t_j}^t \{|v^h(\nu)|^2 - |u(\nu)|^2\} d\nu, \quad j \in [0 : j_*], \quad t \in T.$$

**Lemma 3.** *The following inequalities*

$$\varepsilon_j(t) \leq b_j, \quad t \in \Delta^{(j)} \cap T, \quad j \in [0 : j_*],$$

are valid, where

$$b_j = |x(t_j) - w(t_j)|^2 + c_j^{(1)}(h + \delta) + c_j^{(2)} \sum_{k=j-l}^j \nu^{(k)},$$

$c_j^{(1)}, c_j^{(2)}$  are some constants, which can be explicitly written.

*Proof.* Fix  $\tau_i \in \Delta^{(j)}$ . Then for  $t \in \Delta^{(j)} \cap \delta_i = [\tau_i, \tau_{i+1}]$ , we obtain

$$\varepsilon_j(t) \leq \varepsilon_j(\tau_i) + \sum_{j=1}^4 \Lambda_{ji}(t), \tag{13}$$

where

$$\Lambda_{1i}(t) = 2(s_i, \int_{\tau_i}^t \{f_1(\nu, u_\nu(s), x_\nu(s)) - f_1(\tau_i, v_\nu^h(s), \xi_{\tau_i}^h(s))\} d\nu), \quad s_i = x(\tau_i) - w(\tau_i),$$

$$\begin{aligned} \Lambda_{2i}(t) = & 2(s_i, \int_{\tau_i}^t \{f_2(\nu, x_\nu(s))u(\nu) - \\ & - f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h\} d\nu) + \alpha_j \int_{\tau_i}^t \{|v^h(\nu)|^2 - |u(\nu)|^2\} d\tau, \end{aligned}$$

$$\Lambda_{3i}(t) = -2(t - \tau_i)(s_i, \xi^h(\tau_i) - w(\tau_i)), \quad \Lambda_{4i}(t) = (t - \tau_i) \int_{\tau_i}^t |\dot{w}(\tau) - \dot{x}(\tau)|^2 d\tau.$$

By virtue of lemma 2, we have

$$\Lambda_{4i}(t) \leq K_*^{(j)}(t - \tau_i)^2, \quad t \in \delta_i. \tag{14}$$

Note that  $v^h(\tau_i + s) = v^h(t + s)$  for  $s \geq t_0 - \tau_i$ ,  $t \in [\tau_i, \tau_{i+1}]$  and in addition

$$|\xi^h(\tau_i + s) - x(t + s)| \leq K_*(h + t - \tau_i) \quad \text{for } \tau_i + s \geq t_0 - \tau. \tag{15}$$

Taking into account lemma 2, as well as the Lipschitz property of the functions  $u_0(s)$  and  $x_0(s)$ , inequalities (4) and the relation

$$|\xi^h(\tau_i + s) - x(t + s)| \leq K_*(h + t - \tau_i) \quad \text{for } \tau_i + s \geq t_0 - \tau, \tag{16}$$

we obtain for  $t \in \delta_i$  the estimate

$$\begin{aligned} & \int_{\tau_i}^t |f_1(\nu, u_\nu(s), x_\nu(s)) - f_1(\tau_i, v_\nu^h(s), \xi_{\tau_i}^h(s))| d\nu \leq \\ & \leq K_1^{(j)}(t - \tau_i)(h + t - \tau_i) + K_2^{(j)}(t - \tau_i)^{1/2} \sum_{k=1}^m \left( \int_{\tau_i - \tau_k^u}^{t - \tau_k^u} |u(\nu) - v^h(\nu)|^2 d\nu \right)^{1/2}. \end{aligned}$$

Here,  $\tau_0^x = 0$ . In this case, the inequality

$$\Lambda_{1i}(t) \leq 2(t - \tau_i)|x(\tau_i) - w(\tau_i)|^2 + K_3^{(j)}\{(t - \tau_i)(h + t - \tau_i)\}^2 +$$

$$+ \sum_{k=1}^m \int_{\tau_i - \tau_k^u}^{t - \tau_k^u} |u(\nu) - v^h(\nu)|^2 d\nu \tag{17}$$

holds for  $t \in \delta_i$ . In view of (5), we have

$$A_{3i}(t) \leq -2(t - \tau_i)|x(\tau_i) - w(\tau_i)|^2 + K_4^{(j)}h(t - \tau_i), \quad t \in \delta_i. \tag{18}$$

Moreover, from (5), (3), and (16), we derive

$$|f_2(\nu, x_\nu(s))u(\nu) - f_2(\tau_i, \xi_{\tau_i}^h(s))u(\nu)| \leq K_0(h + \nu - \tau_i)$$

for  $\nu \in [\tau_i, \tau_{i+1}]$ . In this case,

$$A_{2i}(t) \leq K_5^{(j)}(t - \tau_i)(h + t - \tau_i) + \int_{\tau_i}^t \{2(l_i, f_2(\tau_i, \xi_{\tau_i}^h(s))\{v_i^h - u(\nu)\} + \alpha_j\{|v_i^h|^2 - |u(\nu)|^2\}) d\nu.$$

The rule for forming the control  $v_i^h$  (11) and the last inequality imply

$$A_{2i}(t) \leq K_5^{(j)}(t - \tau_i)(h + t - \tau_i). \tag{19}$$

Finally, taking into account (13)–(19), we conclude that for  $t \in \Delta^{(j)} \cap \delta_i$

$$\varepsilon_j(t) \leq \varepsilon_j(\tau_i) + K_6^{(j)}\delta(h + \delta) + K_3^{(j)} \sum_{k=1}^m \int_{\tau_i - \tau_k^u}^{t - \tau_k^u} |u(\nu) - v^h(\nu)|^2 d\nu,$$

i.e., for  $t \in \Delta^{(j)} = [t_j, t_{j+1}]$ ,

$$\varepsilon_j(t) \leq \varepsilon_j(t_j) + K_7^{(j)}(h + \delta) + K_8^{(j)} \int_{t_j - \tau}^{t_{j+1} - \tau_1^u} |u(\nu) - v^h(\nu)|^2 d\nu.$$

Note that  $\tau = l\tau_1^u + \gamma$ ,  $\gamma \geq 0$ . Therefore,  $t_{j+1} - \tau_1^u = t_j$ ,  $t_{j-l-1} \leq t_j - \tau \leq t_{j-l}$ . In this case, for  $t \in \Delta^{(j)}$  we have

$$\varepsilon_j(t) \leq \varepsilon_j(t_j) + K_7^{(j)}(h + \delta) + K_9^{(j)} \sum_{k=j-l}^j \nu^{(k)}.$$

Here, constants  $K_k^{(j)}$ ,  $k \in [0 : 9]$  are written explicitly. Thus, one can assume that  $c_j^{(1)} = K_7^{(j)}$  and  $c_j^{(2)} = K_9^{(j)}$ . The lemma is proved.

**Lemma 4.** *Let  $\delta \leq h$  and values  $\alpha_j$  be given by (12). Then the inequalities*

$$\nu^{(j)} \leq c_j g_j(h), \tag{20}$$

$$b_j \leq c_j^{(0)} g_j(h) \tag{21}$$

are valid.

*Proof.* For simplicity, set  $t_{j_*+1} = \vartheta$ . By virtue of lemma 3, we have for  $t \in \Delta^{(j)}$

$$|x(t) - w(t)| \leq \left( \varepsilon_j(t) + \alpha_j \int_{t_j}^t \{|v^h(\nu)|^2 + |u(\nu)|^2\} d\nu \right)^{1/2} \leq \left( b_j + \alpha_j \rho_A \right)^{1/2}, \quad (22)$$

where  $\rho_A = 2\tau_* d^2(P)$  and  $d(P) = \sup\{|u| : u \in P\}$ . Taking into account the inclusion  $t_j \in \Delta_h$ , we conclude that for any  $j \in [0 : j_*]$ , one can specify the number  $i = i_j(h)$  such that  $t_j = \tau_{i_j}(h)$ . Introduce the notation  $\varrho_j \equiv |f_1(\cdot) - F_1(\cdot)|_{L_2(\Delta^{(j)}; R^{n_2})}^2$ . In this case, by virtue of lemma 2, as well as of (4) and (16), we obtain

$$\varrho_j \leq d_j^{(1)} \sum_{i=i_j(h)}^{i=i_{j+1}(h)-1} \int_{\tau_i}^{\tau_{i+1}} \{\delta^2 + h^2 + \gamma^h(\nu) + \gamma_i^h(\nu) + |\xi^h(\tau_i) - w(\tau_i)|^2\} d\nu,$$

where

$$\gamma^h(\nu) = \sum_{k=1}^m |u(\nu - \tau_k^u) - v^h(\nu - \tau_k^u)|^2, \quad \gamma_i^h(\nu) = \sum_{k=0}^n |x(\nu - \tau_k^x) - \xi^h(\tau_i - \tau_k^x)|^2.$$

Note that

$$\int_{t_j}^{t_{j+1}} \gamma^h(\nu) d\nu \leq d_j^{(2)} \int_{t_{j-l-1}}^{t_j} |u(\nu) - v^h(\nu)|^2 d\nu = d_j^{(2)} \sum_{k=j-l}^j \nu^{(k)}, \quad (23)$$

$$\int_{t_j}^{t_{j+1}} \gamma_i^h(\nu) d\nu \leq d_j^{(3)} (h^2 + \delta^2). \quad (24)$$

In addition,

$$\nu^{(k)} = 0 \quad k \in [-l : 0]. \quad (25)$$

Therefore, combining inequalities (22)–(24), we obtain the estimates

$$\varrho_j \leq d_j^{(5)} \{h^2 + \delta^2 + \sum_{k=j-l}^j \nu^{(k)} + b_j + \alpha_j\}, \quad j \in [0 : j_*]. \quad (26)$$

One can easily see that the following estimates also hold:

$$|f_2(\cdot) - F_2(\cdot)|_{L_2(\Delta^{(j)}; R^{n_2 \times n_1})}^2 \leq d_j^{(5)} (h^2 + \delta^2), \quad j \in [0 : j_*]. \quad (27)$$

Here  $d_j^{(1)} - d_j^{(5)}$  are some constants, which can be explicitly written. By lemma 3, (22), and (25), for  $\delta \leq h$ , we have the inequalities

$$\varepsilon_0(t) \leq b_0 \leq c_0^* h, \quad t \in \Delta^{(0)}, \quad (28)$$



$$|x(t_1) - w(t_1)|^2 \leq \rho_A \alpha_0 + c_0^* h \leq c_* h^{2/3}. \tag{29}$$

Taking into account (25)–(28), for  $h \in (0, 1)$ , we obtain

$$\varrho_0 \leq d_0^{(1)} \{h^2 + \delta^2 + b_0 + h^{2/3}\} \leq d_0^* h^{2/3}, \quad |f_2(\cdot) - F_2(\cdot)|_{L_2(\Delta^{(0)}; R^{n_2 \times n_1})}^2 \leq c_j^{(*)} h^2.$$

By virtue of condition 1, one can use lemma 1. Set  $p = x, q = w, u_1 = u, u_2 = v^h, f_1(t) = f_1(t, u_t(s), x_t(s)), f_2(t) = f_2(t, x_t(s)), F_1(t) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + 2(\xi^h(\tau_i) - w(\tau_i)), F_2(t) = f_2(\tau_i, \xi_{\tau_i}^h(s)) \quad t \in [\tau_i, \tau_{i+1})$ . Then, assuming  $a_1^{(0)} = d_0^* h^{2/3}, a_2^{(0)} = c_j^{(*)} h^2, a_3^{(0)} = c_0^* h, a_4^{(0)} = c_* h^{2/3}, \tilde{\alpha}_0 = \alpha_0 = ch^{2/3}$ , we have

$$\nu^{(1)} = |u(\cdot) - v^h(\cdot)|_{L_2(\Delta^{(0)}; R^{n_1})}^2 \leq \tilde{c}_1 h^{1/3} = c_1 g_1(h). \tag{30}$$

It means that inequality (20) holds for  $j = 1$ . Further, by using (29) and (30), we deduce that

$$b_1 = |x(t_1) - w(t_1)|^2 + c_1^{(1)}(h + \delta) + c_1^{(2)} \sum_{k=1-l}^1 \nu^{(k)} \leq \tilde{c}_1^{(0)} h^{1/3} = c_1^{(0)} g_1(h).$$

Inequality (21) for  $j = 1$  is also verified. It follows from (22) that

$$|x(t_j) - w(t_j)|^2 \leq b_{j-1} + \rho_A \alpha_{j-1}, \quad j \in [1 : j_* - 1]. \tag{31}$$

Consequently, in view of relations (31), as well as of the rule for definition  $b_j$ , we have the inequality

$$b_j \leq b_{j-1} + d_j \left( h + \alpha_{j-1} + \sum_{k=j-l}^j \nu^{(k)} \right), \quad d_j = \text{const} \in (0, +\infty). \tag{32}$$

Setting  $a_1^{(j)} = d_j^{(4)} \{h^2 + \delta^2 + \sum_{k=j-l}^j \nu^{(k)} + a_3^{(j)} + \alpha_j\}, a_3^{(j)} = b_j, a_2^{(j)} = d_j^{(5)}(h^2 + \delta^2), a_4^{(j)} = b_{j-1} + \rho_A \alpha_{j-1}, j \in [1 : j_*]$  for  $j \geq 1$  in lemma 1 and taking into account inequalities (32), we obtain

$$\nu^{(j+1)} \leq c^{(j)} \{h^{1/2} + \left( \sum_{k=j-l}^j \nu^{(k)} \right)^{1/2} + b_{j-1}^{1/2} + \alpha_{j-1}^{1/2} + \alpha_j^{1/2}\} + b_j \alpha_j^{-1}, \quad j \in [1 : j_*].$$

Here, we used lemma 3 and inequalities (27), (28), and (31) for choosing values  $a_i^{(j)}$ . Now, to proof inequalities (20) and (21), one can use the proof by induction. The lemma is proved.

### 3 Example

The algorithm was tested by a model example. The following system

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t-1) + u(t) \\ \dot{x}_2(t) &= x_2(t-1) + x_1(t) + u(t-1), \quad t \in T = [0, 2], \end{aligned} \tag{33}$$

with initial conditions  $x_0(s) = y_0(s) = 1$ ,  $u(s) = 0$  for  $s \in [-1, 0]$  and control  $u(t) = t$  was considered. The solution  $x(t) = \{x_1(t), x_2(t)\}$  of system (33) was calculated analytically. During the experiment, we assumed that  $\xi^h(\tau_i) = x_1(\tau_i) + h$ . As a model, we took the system (9), which has the form

$$\begin{aligned} \dot{w}^{(0)}(t) &= 2\xi_1^h(\tau_i - 1) + v_i^h + 2(\xi_1^h(\tau_i) - w^{(0)}(\tau_i)) \quad \text{for } t \in [\tau_i, \tau_{i+1}) \\ \dot{w}^{(1)}(t) &= \xi_2^h(\tau_i - 1) + \xi_1^h(\tau_i) + v^h(\tau_i - 1) + 2(\xi_2^h(\tau_i) - w^{(1)}(\tau_i)), \end{aligned} \tag{34}$$

with the initial condition  $w^{(0)}(s) = w^{(1)}(s) = 1$ , for  $s \in [-1, 0]$ . Here  $v^h(\tau_i) = v_i^h$  for  $t \in [\tau_i, \tau_{i+1})$ ,  $i \geq 0$ ,  $v^h(s) = 0$  for  $s \in [-1, 0)$ . The controls  $v_i^h$  in model (34) were calculated by the following formula (see (11))

$$v_i^h = \arg \min\{2l_i v + \alpha_j |v|^2 : |v| \leq K\},$$

where  $l_i = w^{(0)}(\tau_i) - \xi_1^h(\tau_i)$ .

In figures 1 and 2 the results of calculations are presented for the case when  $\delta = 10^{-4}$ ,  $\alpha_0 = Ch^{2/3}$ ,  $\alpha_1 = Ch^{2/9}$ ,  $C = 0.2$ ,  $K = 10$ . Fig. 1 corresponds to the case when  $h = 0.001$ , fig. 2 —  $h = 0.02$ . In these figures the solid (dashed) lines represent the model control  $v^h(\cdot)$  (the real control  $u(\cdot)$ ). The equations were solved by the Euler method with step  $\delta$ .

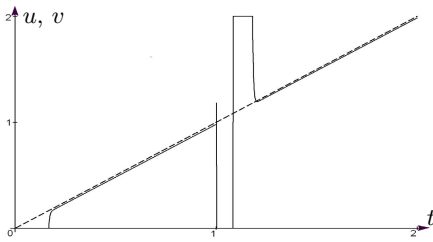


Fig. 1.

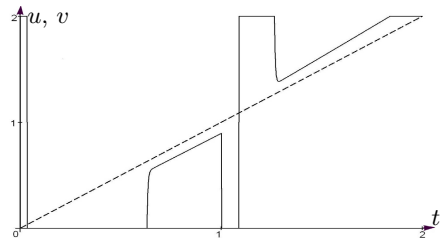


Fig. 2.

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