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Application of Admissible Functions in Studying Partial Stability of Solutions of a Nonlinear System of Differential Equations

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Abstract. A new approach related to construction of admissible functions that do not coincide with the Lyapunov functions was proposed to investigate partial stability of solutions of systems of ordinary differential equations. An example of using admissible functions for establishing partial stability of solutions for one nonlinear system of differential equations is presented.

INTRODUCTION

Lyapunov's second method is a very important approach in theory of ordinary differential equations [1]. This method is used to study asymptotic behavior of solutions of systems of ordinary differential equations, and, in investigating stability of solutions, to establish, in particular, boundedness of solutions of the systems [2]. However, construction of Lyapunov functions requires a lot of energy. In some cases construction of Lyapunov functions for nonlinear system of ordinary differential equations make it possible to prove almost periodicity of solutions of the system and evaluate its solutions, which allowed to prove the convergence of special series with recurrently computed coefficients to solution of nonlinear partial differential equations [3]. Some of the results associated with the use of Lyapunov functions for estimation of solutions of system of differential equations transferred for problems on the part of the variables [4].

A.M. Lyapunov noted a more general problem of stability of motion: it is possible to consider the problem of stability of motion not with respect to all of the variables, but only with respect to some of them. I.G. Malkin [5] pointed on the possibility of transferring some of the Lyapunovs theorems to the case of stability with respect to some of the variables. The fundamental results for partial stability problem for systems of ordinary differential equations with continuous right-hand sides were obtained by Rumyantsev [6, 7, 8, 9].

There are various approaches to study of partial stability associated with modifications of Lyapunov functions. For example, P. Hartman used such a function, but at the same time additional restrictions were imposed on the right-hand sides of initial system of differential equations [10].

In this paper we give sufficient conditions for partial globally asymptotic stability. For this purpose, the so-called admissible functions proposed in [11, 12] are used and the same admissible function was constructed for investigating partial stability of equilibrium position for a system of ordinary differential equations, considered by N.N. Krasovskii.

BASIC DEFINITIONS AND MAIN THEOREM ON GLOBALLY UNIFORMLY ASYMPTOTIC Y-STABLE

Consider a nonlinear system of ordinary differential equations of perturbed motion

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(\mathbf{0}) = 0. \tag{1}$$

Suppose that $\mathbf{x}(t) \in R^n$, $\mathbf{X} : R^n \to R^n$. The phase vector $\mathbf{x}$ is divided into two groups of variables: $\mathbf{y}$-variables, on which there is investigated the stability of the equilibrium position $\mathbf{x} = \mathbf{0}$, and the remaining $\mathbf{z}$-variables ($\mathbf{z}$-variables

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are non-controlled in studying partial stability).

$$\mathbf{x}^T = (x_1, \dots, x_n) = (y_1, \dots, y_m, z_1, \dots, z_p) = (\mathbf{y}^T, \mathbf{z}^T),$$

$$m > 0, \quad p \geq 0, \quad n = m + p, \quad y_i = x_i, \quad i = \overline{1, m}, \quad z_j = x_{m+j}, \quad j = \overline{1, p}.$$

We suppose the position of equilibrium $\mathbf{x} = \mathbf{0}$ of the system is unique. System (1) can be written in the following form

$$x_i' = X_i(x_1, \dots, x_n), \quad i = \overline{1, n}. \tag{2}$$

Usually [7, 8] partial stability is investigated under the assumption of continuity of the vector-function $\mathbf{X}=(X_1, \dots, X_n)^T$ in domain

$$t \geq 0, \quad \|\mathbf{y}\| \leq h, \quad h > 0, \quad h = \text{const}, \quad \|\mathbf{z}\| < \infty, \quad \|\mathbf{x}\| = (\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2)^{1/2} = (x_1^2 + \dots + x_n^2)^{1/2} \tag{3}$$

and under the assumption of uniqueness and $\mathbf{z}$-extendability [13] of all solutions of the system (2) in domain (3).

In addition to these, we assume that the functions $X_i(x_1, \dots, x_n)$ are continuously differentiable functions of the variables x_i, $i = \overline{1, n}$ in all space R^n.

Let $\mathbf{x}(t) = \mathbf{x}(t; 0, \mathbf{x_0}) = (x_1(t; 0, \mathbf{x}_0), \dots, x_n(t; 0, \mathbf{x}_0))^T$ is a solution of system (2) corresponding to initial condition for $t = 0$

$$\mathbf{x}(0) = \mathbf{x}_0 = (x_{01}, \dots, x_{0n})^T, \quad x_{0i} = \text{const}, \quad i = \overline{1, n}.$$

Let $\mathbf{y}(t) = \mathbf{y}(t; 0, \mathbf{x}_0) = (y_1(t; 0, \mathbf{x}_0), \dots, y_m(t; 0, \mathbf{x}_0))^T = (x_1(t; 0, \mathbf{x}_0), \dots, x_m(t; 0, \mathbf{x}_0))^T$.

We formulate definitions for autonomous systems, following [6, 7, 8, 13].

Definition 1. A set G is called a domain of *uniformly asymptotic* $\mathbf{y}$*-stability* of unperturbed motion $\mathbf{x} = 0$, if the following properties are valid:

1) on any compact set $K_x = \{\mathbf{x} \in R^n : \|\mathbf{x}\| < R\}$, $(K_x \subset G)$ relation $\|\mathbf{y}(t; 0, \mathbf{x}_0)\| \rightrightarrows 0$ is valid for $t \to +\infty$ and $\mathbf{x}_0 \in K_x$;

2) unperturbed motion $\mathbf{x} = 0$ is uniformly $\mathbf{y}$-stable, i.e. $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ such that for $\|\mathbf{x}_0\| < \delta \Longrightarrow \|\mathbf{y}(t; 0, \mathbf{x}_0)\| < \varepsilon$ for $t > 0$.

Definition 2. An unperturbed motion $\mathbf{x} = 0$ is called *globally uniformly asymptotic* $\mathbf{y}$*-stable*, if the set G coincides with the whole space.

Assertion. *Suppose that the system* (2) *has a domain G such that $\exists \mathbf{y_0} \in \partial G$ and $\|\mathbf{y}(t, \mathbf{y_0})\| \nrightarrow 0$ for $t \to +\infty$.*
Then there is a line segment

$$\mathbf{x} = \mathbf{y}^1 + u(\mathbf{y_0} - \mathbf{y}^1), \quad u \in [0, 1], \quad \mathbf{y}^1 \in G, \tag{4}$$

which does not include any point of direction field for (2) *in the space R^m.*

Proof. Let a segment (4) will be a part of a line passing through point $\mathbf{y_0}$ and it is perpendicular to the direction field of system (2). In this case, point $\mathbf{y}^1$ will be the begin of this segment (for $u = 0$) located in domain G, and point $\mathbf{y_0}$ is the end of this segment (for $u = 1$), located on the boundary of domain G. This is obtained from the corollary of Picard's theorem [10] which for our case can be formulated as follows:

if the initial conditions differ by small values from the initial conditions $\mathbf{y_0}$ by $\mathbf{v}$, then the corresponding solution will differ by the small from the solution passing through the point $\mathbf{y_0}$, and the family of solutions corresponding to the initial conditions $\mathbf{v}$ satisfies Lipschitz condition in the variable $\mathbf{v}$, and the time interval for existence of these solutions will not depend on $\mathbf{v}$.

This assertion allows us to introduce the definition of an integral surface.

Definition 3. A surface in R^n which consists of positive semi-trajectories of system (2) for $t \geq 0$ started from segment $[\mathbf{y}^0, \mathbf{y}^1]$ we will call an *integral surface* (**IS**).

On this **IS** we introduce Poincares coordinates (u, t), where the first coordinate u determines a trajectory corresponding to parameter u in (4) and the second coordinate t determines its time length from segment (4). Then equation of integral surface **IS** can be written as $\mathbf{x}$=$\mathbf{x}(t, u)$. Consider a family of curves on **IS** defined by equation $t = t(c, u)$, where c is an arbitrary parameter. Let

$$\frac{\partial x_i}{\partial u} = p_i(t, u), \quad i = \overline{1, n},$$

or in vector form

$$\frac{\partial x}{\partial u} = \mathbf{p}(t, u), \qquad \mathbf{p} = (p_1, \ldots, p_n)^T.$$

MAIN THEOREM ON GLOBALLY UNIFORMLY ASYMPTOTIC Y-STABLE

Let $V(\mathbf{x}, \mathbf{p}, t) : R^n \times R^n \times [0, \infty) \to R$ is continuously differentiable with respect to all the variables positive functions. The family of curves on **IS** can be described by the equation

$$\mathbf{x} = \mathbf{x}(t(c, u), u).$$

By $\mathbf{g}$ let denote the vector $\mathbf{g} = (g_1, \ldots, g_n)^T$, with coordinates g_i, $i = \overline{1, n}$ satisfy the equality

$$||g||^2 = \sum_{i>j} (X_i p_j - X_j p_i)^2. \tag{5}$$

The following lemmas are valid.

Lemma 1. *Let on* **IS** *function* $V(\mathbf{x}, \mathbf{p}, t)$ *satisfies the following inequality:*

$$||\mathbf{X}|| V(\mathbf{x}, \mathbf{p}, t) \geq \nu ||\mathbf{g}||, \tag{6}$$

where $\nu > 0$, $||\mathbf{X}|| = (X_1^2 + \ldots + X_n^2)^{1/2}$ *and coordinates of vector* $\mathbf{g}$ *satisfy equality* (5).

Then there is a family of curves on **IS** *for which the equality is valid*

$$ds = V(\mathbf{x}, \mathbf{p}, t) du,$$

where $ds^2 = \sum_{i=1}^{n} dx_i{}^2$.

It was shown that these curves satisfy equation

$$\frac{dt}{du} = \frac{-(\mathbf{X} \cdot \mathbf{p}) \pm \sqrt{V^2 ||\mathbf{X}||^2 - ||\mathbf{g}||^2}}{||\mathbf{X}||^2}. \tag{7}$$

Definition 4. The curves, which equations satisfy equation (7), are called *φ-curves*.

In the particular case, when

$$V(\mathbf{x}, \mathbf{p}, t) = ||\mathbf{p}|| = \left(\sum_{i=1}^{n} p_i^2 \right)^{1/2},$$

from equation (7) we have

$$\frac{dt}{du} = \frac{-(\mathbf{X} \cdot \mathbf{p}) \pm |(\mathbf{X} \cdot \mathbf{p})|}{||\mathbf{X}||^2},$$

that allow to build φ-curves on which

$$\frac{dt}{du} = 0, \qquad or \qquad t = \text{const}.$$

Lemma 2. *The following equality is valid*

$$||\mathbf{X}||^2 ||\mathbf{p}||^2 - (\mathbf{X} \cdot \mathbf{p})^2 = ||\mathbf{g}||^2.$$

Definition 5. Let function $V(\mathbf{x}, \mathbf{p}, t)$ is continuously differentiable, positive on set G_1 in variables $\mathbf{x}, \mathbf{p}, t$. We will call the function an *admissible function (A-function)*, if the following conditions are valid:

1. $V(\mathbf{x}, \mathbf{p}, t)$ satisfies inequality (6);
2. $V(\mathbf{x}, \mathbf{p}, t) = 0$.

Definition 6. For A-function $V(\mathbf{x}, \mathbf{p}, t)$ the expression

$$\partial_t V = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i + \sum_{i=1}^{n} \frac{\partial V}{\partial p_i} \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} p_j + \frac{\partial V}{\partial t}$$

we call *a partial derivative of function $V(\mathbf{x}, \mathbf{p}, t)$ with respect to t on integral surface*.

The following main theorem [11, 12] holds.

Theorem. *Let system* (1) *has the following properties:*

1°. *Set G includes sphere $\|\mathbf{x}\| \leq \rho$.*

2°. *There is an A-function $V(\mathbf{x}, \mathbf{p}, t)$ in set $G_1 = \{\mathbf{x} : \|\mathbf{x}\| \geq r, r < \rho\}$ on any* **IS** *with non-positive partial derivative of function $V(\mathbf{x}, \mathbf{p}, t)$ with respect to t on integral surface, probably exclude a set $M \subset G$, which consists of the whole trajectories belonging G.*

Then unperturbed motion $\mathbf{x} = 0$ is globally uniformly asymptotic $\mathbf{y}$*-stable.*

Proof of the theorem is carried out by contradiction.

AN EXAMPLE OF APPLYING A-FUNCTION

Consider system

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + R_1(x_1, x_2) \equiv X_1(x_1, x_2), \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + R_2(x_1, x_2) \equiv X_2(x_1, x_2), \end{aligned} \tag{8}$$

where $a_{ij} = \text{const}$ and the order of functions R_1, R_2 exceeds 1 at the point $x_1 = x_2 = 0$.

N.N. Krasovskii [14] investigated the position of equilibrium x_1=x2=0 of system (8) for asymptotic stability in the large. It was assumed that the system

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2, \end{aligned} \tag{9}$$

which is a linear part of the system (8) with $R_1 = R_2 = 0$ has characteristic roots with negative real parts and the right-hand sides of the system satisfy (8) the following conditions:

$$l(x_1^2 + x_2^2) \leq X_1^2 + X_2^2 \tag{10}$$

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} < -\beta, \tag{11}$$

which should be valid for all x_1, x_2 and positive constants l, β.

Consider the case when the characteristic roots of the linear part of the system (8) λ_1 and λ_2 have real roots of different signs. Let $\lambda_1 < 0$ and $\lambda_2 > 0$. Taking into account the fact that $\lambda_1 < 0$ we will investigate globally uniformly asymptotic x_1-stability equilibrium position of the system (8). Suppose that the right-hand side of the system (8) is such that there exists a domain of uniformly asymptotic x_1-stability of unperturbed motion $x_1 = x_2 = 0$. We assume the existence of domain G and that the condition 1° of the main theorem is valid.

Note that in [15], in particular, it was shown that for the system (9) with the characteristic roots $\lambda_1 < 0$ and $\lambda_2 > 0$ for globally uniformly asymptotic x_1-stability equilibrium position it is necessary and sufficient that the coefficient $a_{12} = 0$.

Consider function

$$V = \sqrt{g^2}, \tag{12}$$

where $g = X_2 p_1 - X_1 p_2$.

We verify that function (12) is an admissible function. Taking into account (10), we obtain

$$V^2 \|\mathbf{x}\|^2 \geq g^2 l(x_1^2 + x_2^2) \geq \nu g^2, \qquad \nu > 0.$$

Therefore the first condition of definition 5 is hold. Since $V(bf x,\ bf 0, t) = 0$, we find the partial derivative $V(\mathbf{x}, \mathbf{p}, t)$ with respect to t on integral surface in accordance with definition 6.

$$\begin{aligned}
\partial_t V = \frac{g}{\sqrt{g^2}}\partial_t g = \frac{g}{\sqrt{g^2}}\Bigg[p_1\left(\frac{\partial X_2}{\partial x_1}X_1 + \frac{\partial X_2}{\partial x_2}X_2\right) + X_2\left(\frac{\partial X_1}{\partial x_1}p_1 + \frac{\partial X_1}{\partial x_2}p_2\right) \\
-p_2\left(\frac{\partial X_1}{\partial x_1}X_1 + \frac{\partial X_1}{\partial x_2}X_2\right) - X_1\left(\frac{\partial X_2}{\partial x_1}p_1 + \frac{\partial X_2}{\partial x_2}p_2\right)\Bigg] \\
= \frac{g}{\sqrt{g^2}}\left[\frac{\partial X_2}{\partial x_2}(X_2 p_1 - X_1 p_2) + \frac{\partial X_1}{\partial x_1}(X_2 p_1 - X_1 p_2)\right] = \frac{g^2}{\sqrt{g^2}}\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right) \\
= \sqrt{g^2}\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right) = V\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right) \leq 0.
\end{aligned}$$

The last inequality is obtained under the condition (11).

Thus, condition 2° of the main theorem is satisfied. Consequently, by the main theorem, the equilibrium position of system (8) $x_1 = x_2 = 0$ will be globally uniformly asymptotic x_1-stable.

Remark. *If the roots of the characteristic equation $\lambda_1 < 0$ and $\lambda_2 < 0$, then we can obtain globally stability of the system* (8)*, if we take* $\mathbf{y} = (x_1, x_2)^T$.

Indeed, condition 1° of the main theorem is satisfied on the basis of Lyapunov's theorem [14]. We choose function (12) as an admissible function V and apply the main theorem, since both conditions of this theorem are satisfied.

CONCLUSION

To study partial globally uniformly asymptotic stability of systems of ordinary differential equations, there are proposed admissible functions, different from the Lyapunovs functions. For the system studied by N.N. Krasovskii, the globally asymptotic x_1-stability of the equilibrium position is proved with the help of an admissible function.

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