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On Application of Fourier Method for One Class of Equations Describing Nonlinear Oscillations

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Abstract. Solutions of nonlinear partial differential equations with a small parameter are constructed as a sum of truncated Fourier series and some additional function. The coefficients of the truncated Fourier series depend on the small parameter and satisfy a nonlinear system of ordinary differential equations, which in turn is determined by nonlinear partial differential equation. A certain class of equations describing nonlinear oscillations was determined, for which the coefficients of the truncated Fourier series are bounded functions of time. This fact makes it possible to estimate the additional function and to justify the applicability of the Fourier method for the constructed class of nonlinear partial differential equations.

INTRODUCTION

We consider the initial boundary value problem for one class of nonlinear partial differential equations with two independent variables x and t and a small parameter ε. We seek a solution in the form of the truncated Fourier series and an additional function that is represented as a power series of ε. If we represent the solution of a nonlinear equation as an infinite Fourier series, we have to determine the coefficients of the series as a solution of the countable system of ordinary differential equations which, in general, can not be integrated successively. Following [1], along with the countable system which is reduced to the standard form:

$$\frac{dx_k}{dt} = \varepsilon F_k(t, x_1, x_2, \ldots), \quad k = 1, 2, \ldots,$$

we consder the "truncated" system

$$\frac{dx_k}{dt} = \varepsilon F_k(t, x_1, \ldots, x_n, 0, \ldots), \quad k = \overline{1, n},$$

that is obtained from the initial one if we put the functions to be obtained equal to zero, starting from the $(n+1)$-th function and discarding all the equations, starting with $(n+1)$-th equation. In [1] Persidskii proves a theorem which, under some assumptions as to the functions F_k, implies that the solution of the "truncated" system approximates with the preassigned accuracy the solution of the original system in a time interval if n is sufficiently large. However, the assumptions of this theorem are quite restricted.

In this work we describe a class of nonlinear equations for which we can establish the coefficients of the truncated Fourier series are bounded for all $t \geq 0$. This statement allows us to estimate an additional function v and to prove that the constructed series converges to the solution of the initial boundary value problems in time interval $0 \leq t \leq \varepsilon^{-1}$.

PROBLEM SATEMENT AND CONSTRUCTING THE SOLUTION

We consider the initial boundary value problem for the equation

$$u_{tt} + (-1)^n \frac{\partial^{2n} u}{\partial x^{2n}} = \varepsilon f(x, u, \ldots, \frac{\partial^m u}{\partial x^m}), \quad m \leq n + 1 \tag{1}$$

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with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \le x \le 1 \tag{2}$$

and the boundary conditions

$$\begin{gathered} u(0,t) = u(1,t) = 0, \quad t \ge 0, \\ \frac{\partial^{2j} u(0,t)}{\partial x^{2j}} = \frac{\partial^{2j} u(1,t)}{\partial x^{2j}} = 0, \quad j = 1, \dots, n-1 \quad (n \ge 2). \end{gathered} \tag{3}$$

Here ε is a small parameter, f is a nonlinear continuous function of its arguments, witch satisfies the condition: if for any $k \ge 1$ we set

$$u = \sum_{s=1}^{k} z_s X_s(x), \quad X_s(x) = \sin \pi s x,$$

the function f may be represented as

$$f = \sum_{s=1}^{m_k} P_s(z_1, \dots, z_k) X_s(x), \tag{4}$$

where $P_s(z_1, \dots, z_k)$ are polynomials in $z_1, \dots, z_k$, and the numbers $m_k \ge 1$. For example, if $f = uu_x$, then $m_k = 2k$.

The problem (1)–(3) arises in investigation of vibration processes in many real objects [2], for example, the problems on transverse vibrations of a string with fixed ends ($n = 1$), bending vibrations of a beam ($n = 2$) with nonlinear effects. The questions of the existence of a classical solution of nonlinear equations are considered in many works (e.g. [3], [4]).

Let the initial conditions have the form

$$u_\nu(x) = \sum_{j=1}^{J} \alpha_{\nu j} X_j(x), \quad \nu = 0, 1. \tag{5}$$

Here $\alpha_{\nu j}$ are constants. We construct the solution (1)–(3) in the form:

$$u(x,t) = \sum_{s=1}^{N} z_s(\varepsilon, t) X_s(x) + v(\varepsilon, t, x, N), \tag{6}$$

where $N \ge J$. If we substitute (6) into (1), take into account the assumptions (4), (5) and equate the expressions at the same $X_s(x)$, $s = \overline{1, N}$, we obtain the system of nonlinear differential equations for the coefficients $z_s(\varepsilon, t)$

$$\begin{gathered} z_i'' = -\omega_i^{2n} z_i + \varepsilon P_i(z_1, \dots, z_N), \quad \omega_i = \pi i, \\ z_i'(0) = \alpha_{1i}, z_i(0) = \alpha_{0i}, \quad i = \overline{1, N}. \end{gathered} \tag{7}$$

In the following this system is called the leading system.

Function $v(\varepsilon, t, x, N)$ satisfies an equation of type (1)

$$v_{tt} + (-1)^n \frac{\partial^{2n} v}{\partial x^{2n}} = \varepsilon [f(x, v, \dots, \frac{\partial^m v}{\partial x^m}) + f_1(x, v, \dots, \frac{\partial^m v}{\partial x^m}) + \sum_{s=N+1}^{m_1} P_s(z_1, \dots, z_N) X_s(x)], \tag{8}$$

but with the zero initial and boundary conditions

$$\begin{gathered} v(x,0) = v_t(x,0) = 0, \\ v(0,t) = v(1,t) = 0, \quad t \ge 0, \\ \frac{\partial^{2j} v(0,t)}{\partial x^{2j}} = \frac{\partial^{2j} v(1,t)}{\partial x^{2j}} = 0, \quad j = \overline{1, n-1}. \end{gathered} \tag{9}$$

Here function f_1 is determined by nonlinear function f and a number N in (6).

We construct the solution of the initial-boundary value problem (8)–(9) in the form of a power series of ε

$$v(x,t) = \sum_{i=1}^{\infty} \varepsilon^i v_i(x,t,\varepsilon). \tag{10}$$

Here the functions v_i depend on ε, since the functions z_j, $j = \overline{1,N}$, witch are a solution of the leading system (7) also depend on ε.

If we substitute (10) into (8), we obtain the linear non-uniform partial equations for the functions $v_i(x,t,\varepsilon)$

$$\frac{\partial^2 v_i}{\partial t^2} + (-1)^n \frac{\partial^{2n} v_i}{\partial x^{2n}} = F_i(t,x,v_1,\ldots,v_{i-1}) \tag{11}$$

with the zero initial and boundary conditions

$$\begin{gathered} v_i(x,0) = \frac{\partial v_i(x,0)}{\partial t} = 0, \\ v_i(0,t) = v_i(1,t) = 0, \quad t \geq 0, \\ \frac{\partial^{2j} v_i(0,t)}{\partial x^{2j}} = \frac{\partial^{2j} v_i(1,t)}{\partial x^{2j}} = 0, \quad j = \overline{1,n-1}, i \geq 1. \end{gathered} \tag{12}$$

For $i = 1$ we have

$$F_1 = \sum_{j=N+1}^{m_1} P_j(z_1,\ldots,z_N) X_j.$$

Since function f satisfies (4), we may construct the solution of the problem (11), (12) in the form of the sums

$$v_i = \sum_{j=1}^{m_i} q_{ij}(t,\varepsilon) X_j, \quad i \geq 1,$$

where $m_i = [(r_f - 1)i + 1]N + N_0$, r_f is the degree of the polynomials $P_s(z_1,\ldots,z_N)$, the number $N_0 \geq 0$ is determined by the function f, and the coefficients q_{ij} are determined successively as solutions of linear ordinary equations.

Thus, we can rewrite the solution of the original problem (1)–(3) in the form of a series

$$u(x,t) = \sum_{i=1}^{N} z_i(\varepsilon,t) X_i + \sum_{i=1}^{\infty} \varepsilon^i \sum_{j=1}^{m_i} q_{ij}(t,\varepsilon) X_j. \tag{13}$$

We can represent the solution of the equation (1) also in the form of a series by powers of specially chosen functions, using the method of special series [5, 6, 7, 8, 9]. The method of special series, in contrast to Fourier method for nonlinear partial differential equations, makes it possible to find the coefficients of these series recurrently. Special series are also used to obtain solutions in bounded domains for solving the initial-boundary value problems [10, 11, 12, 13].

PROPERTIES OF P-SYSTEMS

In [14] it is shown that if the following relations hold for the right-hand sides $P_i(z_1,\ldots,z_N)$

$$\frac{\partial P_i}{\partial z_k} = \frac{\partial P_k}{\partial z_i}, \quad k,i = \overline{1,N} \tag{14}$$

then a positive definite Lyapunov function for the system (7) exists in the form

$$V_N = \frac{1}{2} \sum_{i=1}^{N} (\dot{z}_i^2 + \omega_i^{2n} z_i^2) - \varepsilon Q_N(z_1,\ldots,z_N), \tag{15}$$

which the time-derivative along the trajectories of the system (7) is zero

$$\left(\frac{dV_N}{dt}\right)_{(7)} = 0. \tag{16}$$

Here function $Q_N(z_1, \ldots, z_N)$ has continuous partial derivatives $\partial Q_N/\partial z_i$, $i = \overline{1, N}$, $Q_N(0, \ldots, 0) = 0$ and may be represented as

$$Q_N = \sum_{i=1}^{N} \int_0^{z_i} P_i^{[1,\ldots,i-1]}(z_1, \ldots, z_N) dz_i,$$

where by $P_i^{[1,\ldots,i-1]}(z_1, \ldots, z_N)$ we denote the right-hand sides of (7) with z_j=0, j=$\overline{1, i{-}1}$.

Definition. If the equalities (14) are valid for the leading system (7) corresponding to function $f(x, u, \ldots, \frac{\partial^m u}{\partial x^m})$, then such system is called *P-system*.

Example of P-system. We consider the nonlinear wave equation, which describes nonlinear vibrations of a string with fixed ends

$$u_{tt}=u_{xx}(1 + \varepsilon u_x), \tag{17}$$

$$u(x, 0) = \sin \pi x, \qquad u_t(x, 0) = 0, \tag{18}$$

$$u(0, t) = u(1, t) = 0, \quad t \geq 0. \tag{19}$$

The corresponding leading system (7) for equation (17) has the form

$$z_i''=-\omega_i^2 z_i+0.5\varepsilon\pi^3 P_i(z_1, \ldots, z_N), \ \ \omega_i=\pi i, \tag{20}$$

with the initial conditions

$$z_i'(0)=0, \ \ i=\overline{1, N}, \quad z_1(0)=1, \ \ z_k(0)=0, \ \ k=\overline{2, N},$$

where

$$P_i = -\Big(\sum_{s+l=i} sl^2 z_s z_l + i \sum_{l-s=i} sl z_s z_l\Big).$$

The equalities (14) are valid for the leading system (20) corresponding to function $f = u_{xx}u_x$ and the leading system (20) is P-system.

For this P-system the positive definite Lyapunov function exists for any N in the form

$$V_N = \frac{1}{2} \sum_{i=1}^{N} (\dot{z}_i^2 + \omega_i^{2n} z_i^2) - \varepsilon Q_N(z_1, \ldots, z_N),$$

which the time-derivative along the trajectories of the system (20) is zero. Here the function Q_N is represented in the form

$$Q_N=\frac{\pi^3}{2} \sum_{i=2}^{N} \sum_{k\leq i/2} k(i{-}k)\Big\{i{-}\frac{i}{2}\Big[\mathrm{sign}\Big(\frac{k}{i}{-}\frac{1}{2}\Big){+}1\Big]\Big\} z_k z_i z_{i-k}.$$

We can also construct a solution of the problem (17)–(19) in the form of a special series [13, 15, 16]

$$u(x, t) = \sum_{i,j=0}^{\infty} g_{ij}(t) P^i(x) Q^j(x)$$

with special functions $P(x)$, $Q(x)$. In order to exactly satisfy the boundary conditions, it was sufficient to require that these functions satisfy the differential system of equations

$$P' = \sum_{m+2n\leq l-1} a_{m,2n+1} P^m Q^{2n+1},$$

$$Q' = \sum_{m+2n\le l} b_{m,2n} P^m Q^{2n},$$

where $a_{m,2n+1}, b_{m,2n}$ =const. In particular, a pair of functions $P(x) = \cos \pi x$ and $Q(x) = \sin \pi x$ satisfy these conditions. For these functions we have the differential system of equations

$$\begin{aligned} P' &= -\pi Q(P^2 + Q^2), \\ Q' &= \pi P(P^2 + Q^2). \end{aligned} \tag{21}$$

Thus, the solution of the problem (17)–(19) is representable in the form of the series

$$u(x,t) = \sum_{i,j=0}^{\infty} g_{ij}(t) \sin^i \pi \cos^j \pi x. \tag{22}$$

It is necessary to use differential relations (21) for the recurrently calculation of the coefficients of series (22).

JUSTIFICATION OF APPLICABILITY OF FOURIER METHOD FOR SOME CLASS OF EQUATIONS

We describe a some class of nonlinear partial differential equations (1), for which the applicability of the Fourier method can be justified. For this class of equations, the corresponding leading systems (7) are P-systems.

The following lemmas are valid.

Lemma 1. *Let* $f_1(x, u, \dots, \frac{\partial^m u}{\partial x^m})$ *and* $f_2(x, u, \dots, \frac{\partial^m u}{\partial x^m})$ *are P-functions.*
Then $f = \alpha_1 f_1 + \alpha_2 f_2$*, where* $\alpha_1, \alpha_2 = \text{const}$ *is a P-function.*

Lemma 2. *For any natural number* $l \ge 1$ *functon* $f = \alpha_1 u^2 \sin lx + \alpha_2 u^3 \cos lx$ *is a P-function.*

Lemma 3. *For any natural number* $l \ge 1$ *functon* $f = u_x^l u_{xx}$ *is a P-function.*

The following theorems are valid.

Theorem 1. *Let nonlinear function* f *is a polynomial of the form*

$$f = \sum_{k=1}^{K} [a_k(x) u^{2k} + b_k(x) u^{2k+1} + c_k(u_x^k)_x + d_k(u_{xx}^{2k+1})_{xx},$$

here $a_k(x)$ *are the continuous even functions,* $b_k(x)$ *are the continuous odd functions,* c_k, d_k *are constant.*
Then the corresponding leading systems (7) *is a P-system.*

Theorem 2. *Let the conditions of Theorem 1 are satisfied for function* f*, the initial data* $\alpha_{\nu i}$ *satisfy the conditions*

$$|\alpha_{0i}| \le \frac{M}{\omega_i^{2(n+1)}}, \ |\alpha_{1i}| \le \frac{M}{i^{2+n}}, \ i = \overline{1, N}, \ M \ge 0$$

and

$$|a_k(x)| + |b_k(x)| + |c_k| + |d_k| \le M_1, \quad M_1 \ge 0.$$

Then the solutions of the corresponding leading systems (7) *are bounded for all* $t \ge 0$ *when* $|\varepsilon| \le \varepsilon_0(f, M_1)$.

If we use the first Lyapunov stability theorem [17], we can prove that the existence of the function (15) with property (16) is sufficient for the solutions of the leading system to be bounded for all $t \ge 0$. To prove the Theorem 2 it is sufficiently to check the relations (14). We can estimate the functions z_i, if $|\varepsilon| \le \varepsilon_0(f, M_1)$

$$|z_i| \le \frac{M}{\omega_i^{2(n+1)}}, \quad |\dot z_i| \le \frac{M}{i^{2+n}}, \quad i = \overline{1, N}, \quad t \ge 0. \tag{23}$$

Theorem 3. *Let the conditions of Theorem 2 are satisfied.*

Then series (13) *uniformly converges to the solution of the initial-boundary value problem for all* $0 \leq x \leq 1$ *and* $0 \leq t \leq T$ *(*$T \sim \varepsilon^{-1}$*).*

If we use the method of mathematical induction and (23), we can estimate the functions q_{ij} as follows

$$|q_{ij}| \leq \frac{M^i t^i}{N^2 i^2 \omega_j^{2(n+1)}}, \quad i \geq 1, \quad j = \overline{1, m_i}. \tag{24}$$

The estimates (23), (24) allow us to prove that the series (13) converges to the solution of the initial-boundary value problem (1)–(3) for all $0 \leq x \leq 1$ and $0 \leq t \leq T$ $(T = (M\varepsilon)^{-1})$.

Remark 1. *The additional function* v *is estimated as follows*

$$|v(\varepsilon, t, x, N)| \leq \frac{1}{N^2}\varepsilon t\Big(\frac{1}{N^{2(n-1)}} + \varepsilon t\Big)C, \quad C = \text{Const}. \tag{25}$$

Remark 2. *In contrast to [14] in this work a class of nonlinear equations for which the solutions of leading system are bounded for all* $t \geq 0$ *is more completely described.*

Remark 3. *Small parameter* ε_0 *is independent on* N, *and the estimates* (25) *allow us to prove that the additional function* v *tends to zero when* N *is increasing.*

CONCLUSION

Thus for the defined class of nonlinear equations in Theorem 1 the applicability of Fourier method is justified for all $0 \leq x \leq 1$ and $0 \leq t \leq T$ $(T \sim \varepsilon^{-1})$. We can also construct a solution of problem (1)–(3) in the form of special series (22) with recurrently calculation coefficients. Numerical calculations for problem (17)–(19) have shown that domain of convergence of series (22) will be less than the domain of convergence of the series (13).

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