Optimal control of the system of coupled cylinders

Cite as: AIP Conference Proceedings **2048**, 020007 (2018); https://doi.org/10.1063/1.5082025
Published Online: 11 December 2018

Yu. F. Dolgii, A. A. Petunin, A. N. Sesekin, and O. L. Tashlykov

Optimal Control of the System of Coupled Cylinders

Yu. F. Dolgii[1,2], A. A. Petunin[2], A. N. Sesekin[1,2,a)] and O. L. Tashlykov[2]

[1]*N. N. Krasovskii Institute of Mathematics and Mechanics, UB of RAS, 16 S. Kovalevskaya, Ekaterinburg, 620990, Russia.*
[2]*Ural Federal University, 19 S. Mira, Ekaterinburg, 620002, Russia.*

[a)]Corresponding author: sesekin@list.ru

Abstract. We consider the problem of optimal control of a system consisting of two coupled cylinders. Such a system is a mathematical model of the nuclear fuel transfer mechanism at the nuclear power plant reactor. And also, such models are found in various robotic systems. We have obtained optimal control under certain assumptions on a controllable system.

INTRODUCTION

The time optimal problem for a mechanical system consisting of two coupled cylinders is considered. Such models are found in various technical systems (see [1, 2]). Therefore, the problem under consideration seems to be relevant. We have obtained optimal control for the linear model of the problem. A feature of this optimal control is its nonuniqueness. For a nonlinear model, we propose a variant of its reduction to a quasilinear problem.

MATHEMATICAL MODEL OF SYSTEM COUPLED CYLINDERS

There is a fixed cylinder of radius R_0, inside of which there is a circular cutout of radius R_1 (see Figure 1). In this cutout is placed another cylinder, which can rotate inside a fixed cylinder. At the same time, its geometric center remains stationary. This cylinder has an eccentric circular cutout of radius R_2, inside of which is placed a small cylinder - a disk of radius R_2. When describing the interaction of the cylinders, we neglect the forces of friction. The motion of the mechanical system is flat. The axial moments of inertia of the large and small cylinders are J_1 and J_2, respectively, and their masses are m_1 and m_2. The distance between the geometric centers of the large and small cylinders (eccentricity) is e_1.

To describe the dynamics of the mechanical system with two degrees of freedom, we use the Lagrange equations of the second kind [3]. As generalized coordinates, we select the angle of rotation of the large cylinder — φ_1 and the angle of rotation of the small cylinder — φ_2 with respect to the large cylinder. The generalized forces are the control moments u_1 and u_2 applied to the large and small cylinders, respectively. The kinetic energy of the system is given by

$$T = \frac{1}{2}\Big((J_1 + J_2 + m_2 e_1^2)\dot{\varphi}_1^2 + (J_2 + m_2 e_2^2)\dot{\varphi}_2^2 + 2(J_2 + m_2 e_1 e_2 \cos(\varphi_2))\dot{\varphi}_1\dot{\varphi}_2\Big).$$

Assuming that the geometric center of the small cylinder coincides with its center of mass $e_2 = 0$, we find that the mathematical model of a system of coupled cylinders will be described with the help of the following Lagrange equations of the second kind:

$$(J_1 + J_2 + m_2 e_1^2)\ddot{\varphi}_1 + J_2\ddot{\varphi}_2 = u_1, J_2(\ddot{\varphi}_1 + \ddot{\varphi}_1) = u_2. \tag{1}$$

The control moments applied to the large and small cylinders, respectively, satisfy the constraints

$$|u_1| \le \mu_1, |u_2| \le \mu_2. \tag{2}$$

Proceedings of the 44th International Conference on Applications of Mathematics in Engineering and Economics
AIP Conf. Proc. 2048, 020007-1–020007-6; https://doi.org/10.1063/1.5082025
Published by AIP Publishing. 978-0-7354-1774-8/$30.00

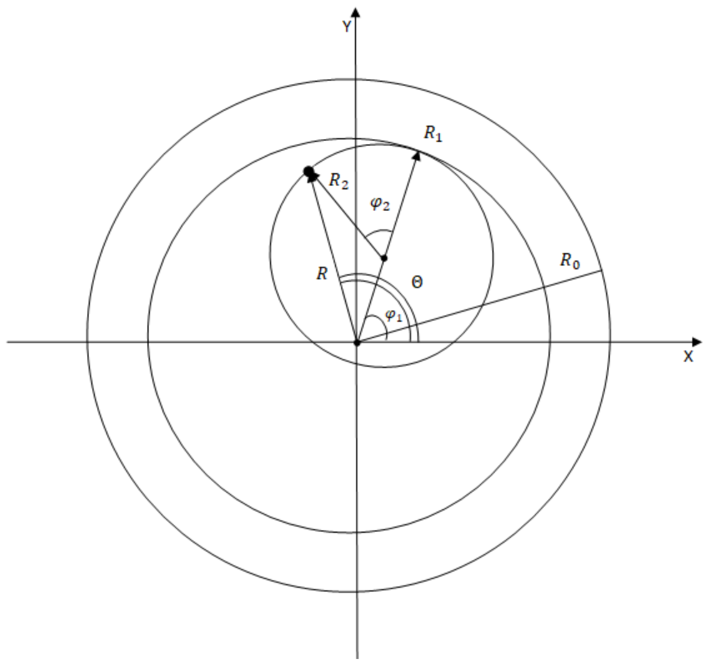

FIGURE 1. Scheme of coupled cylinders.

We solve the equations of motion relative to the second derivatives φ_1 and φ_2. As a result, we obtained

$$\begin{cases} \ddot{\varphi}_1 &=& a_1(u_1 - u_2), \\ \ddot{\varphi}_2 &=& -a_1 u_1 + a_2 u_2, \end{cases} \tag{3}$$

where

$$a_1 = \frac{1}{J_1 + m_2 e_1^2}, \quad a_2 = \frac{J_1 + J_2 + m_2 e_1^2}{J_2(J_1 + m_2 e_1^2)}. \tag{4}$$

We write the system (3) in the normal form:

$$\begin{cases} \dot{\varphi}_1 &=& \omega_1, \\ \dot{\omega}_1 &=& a_1(u_1 - u_2), \\ \dot{\varphi}_2 &=& \omega_2, \\ \dot{\omega}_2 &=& -a_1 u_1 + a_2 u_2. \end{cases} \tag{5}$$

The coordinates of the point in the fixed coordinate system r and θ are related to the variables φ_1 and φ_2 using formulas

$$R = \sqrt{(R_1 - R_2)^2 + R_2^2 - 2(R_1 - R_2)R_2 \cos(\pi - \varphi_2)}, \tag{6}$$

$$\theta = \varphi_1 + \arccos \frac{R^2 + (R_1 - R_2)^2 - R_2^2}{2(R_1 - R_2)R} \operatorname{sign}\varphi_2. \tag{7}$$

The following problem will be considered below. Let the initial point have coordinates $x^0 = (\varphi_1^0, 0, \varphi_2^0, 0)^\top$ and the endpoint — $x^f = (\varphi_1^f, 0, \varphi_2^f, 0)^\top$. Phase vector $x(t) = (\varphi_1(t), \omega_1(t), \varphi_2(t), \omega_2(t))^\top$ is described by the system of equations (5). It is required to move the phase point from the initial state to the final state in the shortest time, subject to the constraints (2).

TIME OPTIMAL PROBLEM

First we consider the auxiliary problem. We introduce a new variable $\varphi = \varphi_1 + \varphi_2$. It follows from (3), (4) that the new variable satisfies equation

$$\ddot{\varphi} = (a_2 - a_1)u_2, \tag{8}$$

or system

$$\begin{cases} \dot{\varphi} &= w, \\ \dot{w} &= (a_2 - a_1)u_2. \end{cases} \tag{9}$$

We consider the problem of the fastest moving from the initial position $(\varphi(0), 0)^\top$ to position $(\varphi(\vartheta), 0)^\top$, where $\varphi(0) = \varphi_1^0 + \varphi_2^0,\ \ \varphi(\vartheta) = \varphi_1^f + \varphi_2^f$, for the shortest time ϑ.

We apply the Pontryagin maximum principle to this problem [4]. Let

$$H(\psi_1, \psi_2, \varphi, w, u_2) = \psi_1 w + \psi_2(a_2 - a_1)u_2.$$

The conjugate system will have the form

$$\begin{cases} \dot{\psi}_1 &= 0, \\ \dot{\psi}_2 &= -\psi_1. \end{cases} \tag{10}$$

Consequently $\psi_1 \equiv \psi_{10},\ \psi_2 = -\psi_{10}t + \psi_{20}$. If $\psi_{10} = 0\ \ \psi_2(t) \equiv \psi_{20}$, than the control u_2 according to the maximum principle will have the form

$$u_2 = \mu_2 \mathrm{sign}[(a_2 - a_1)\psi_2(t)]. \tag{11}$$

We see from (11), that the control u_2 will not have any switches. Otherwise, we will not be able to provide the boundary conditions $\dot{\varphi}(0) = w(0) = \dot{\varphi}(\vartheta) = w(\vartheta) = 0$. Hence $\psi_{10} \neq 0$ and the optimal control will have one switch. According to (10) and (11)

$$w(t) = \mu_2 \int_0^t \mathrm{sign}[(a_2 - a_1)\psi_2(s)]\, ds.$$

Let ϑ^* be the switching point. Then for $t \in [0, \vartheta^*)$

$$w(t) = \mu_2 \int_0^t \mathrm{sign}[(a_2 - a_1)\psi_{20}]\, ds = \mu_2 \mathrm{sign}[(a_2 - a_1)\psi_{20}]t,$$

and for $t \in (\vartheta^*, \vartheta]$

$$w(t) = \mu_2 \mathrm{sign}[(a_2 - a_1)\psi_{20}](2\vartheta^* - t).$$

From the condition $w(0) = w(\vartheta) = 0$ it follows that

$$w(\vartheta) = \mu_2 \mathrm{sign}[(a_2 - a_1)\psi_{20}](2\vartheta^* - \vartheta) = 0.$$

As a result, we have

$$\vartheta^* = \frac{\vartheta}{2}.$$

After integrating the system (9) with control (11), we get that

$$\varphi(t) = \varphi(0) + \mu_2 \mathrm{sign}[(a_2 - a_1)\varphi_{20}]\frac{t^2}{2}$$

for $t \in [0, \frac{\vartheta}{2})$ and

$$\varphi(t) = \varphi(0) + \mu_2 \mathrm{sign}[(a_2 - a_1)\varphi_{20}]\frac{\vartheta^2}{8} - \mu_2 \mathrm{sign}[(a_2 - a_1)\varphi_{20}]\frac{(t - \vartheta)^2}{2}$$

for $t \in (\frac{\vartheta}{2}, \theta]$. Therefore

$$\varphi(\vartheta) = \varphi(0) + \mu_2 \mathrm{sign}[(a_2 - a_1)\varphi_{20}]\frac{\vartheta^2}{8}.$$

Then

$$\varphi(\vartheta) - \varphi(0) = \mu_2 \text{sign}[(a_2 - a_1)\varphi_{20}]\frac{\vartheta^2}{8}.$$

Consequently $\text{sign}(\varphi(\vartheta) - \varphi(0)) = \text{sign}[(a_2 - a_1)\varphi_{20}]$

$$\vartheta_2 = \sqrt{\frac{|\varphi(\vartheta) - \varphi(0)|}{\mu_2}}. \tag{12}$$

As a result, optimal control will take the form

$$u_2(t) = \mu_2 \,\text{sign}\,(\varphi(\vartheta) - \varphi(0))\ \text{sign}\left(t - \frac{\vartheta}{2}\right). \tag{13}$$

We introduce a new function $\chi = \varphi_1 + \frac{a_1}{a_2}\varphi_2$. It follows from (3) that the function $\chi(t)$ satisfies the equation

$$\ddot{\chi} = \frac{a_1}{a_2}(a_2 - a_1)\, u_1. \tag{14}$$

We write this equation in the form of a system

$$\begin{cases} \dot{\chi} &= \eta, \\ \dot{\eta} &= \frac{a_1}{a_2}(a_2 - a_1)\, u_1. \end{cases} \tag{15}$$

To find the control, we consider the boundary value problem

$$\chi(0) = \varphi_1^0 + \frac{a_1}{a_2}\varphi_2^0,\ \ \chi(\vartheta) = \varphi_1^f + \frac{a_1}{a_2}\varphi_2^f,\ \ \eta(0) = \eta(\vartheta) = 0. \tag{16}$$

From the system (15) and the boundary condition (16) it follows that

$$\int_0^\vartheta u_1(s)\, ds = 0,\ \ \chi(\vartheta) = \chi(0) + \int_0^\vartheta (\vartheta - s) u_1(s)\, ds. \tag{17}$$

In the last expression ϑ can be replaced by $\frac{\vartheta}{2}$, because $\int_0^\vartheta u_1(s)\, ds = 0$. As a result, from (17) we have

$$\chi(\vartheta) - \chi(0) = \int_0^\vartheta \left(\frac{\vartheta}{2} - s\right) u_1(s)\, ds. \tag{18}$$

We look for control u_1 in the form of the following relay control

$$u_1 = \beta_1 \text{sign}\left(t - \frac{\vartheta}{2}\right), \quad t \in [0, \vartheta]. \tag{19}$$

After substituting (19) into (18), we get $\chi(\vartheta) - \chi(0) = -\beta_1 \frac{\vartheta^2}{8}$. From here

$$\beta_1 = \frac{8\,(\chi(0) - \chi(\vartheta))}{\vartheta^2}. \tag{20}$$

So we got control

$$u_1 = \frac{8(\varphi_1^0 - \varphi_1^f + \frac{a_1}{a_2}(\varphi_2^0 - \varphi_2^f))}{\vartheta^2} \text{sign}\left(t - \frac{\vartheta}{2}\right), \tag{21}$$

which solves the above boundary-value problem (16).

If $|\beta_1| > \mu_1$, then we must first solve the time optimal problem for the system (15) under the boundary conditions (16) and the control constraint $|u_1| \le \mu_1$.

It is not difficult to show that in this case

$$\vartheta_1 = \sqrt{\frac{a_2|\chi(\vartheta_1) - \chi(0)|}{a_1\mu_2}}. \tag{22}$$

and

$$u_1(t) = \mu_1 \operatorname{sign}(\chi(\vartheta_1) - \chi(0)) \operatorname{sign}\left(t - \frac{\vartheta}{2}\right). \tag{23}$$

The control u_2 will be sought from the boundary value problem

$$\varphi(0) = \varphi_1^0 + \varphi_2^0, \quad \varphi(\vartheta) = \varphi_1^f + \varphi_2^f$$

for the system

$$\dot{\varphi} = w, \dot{w} = (a_2 - a_1)u_2. \tag{24}$$

Making similar calculations we obtain that

$$u_2(t) = \frac{8(\varphi_1^0 - \varphi_1^f + (\varphi_2^0 - \varphi_2^f))}{\vartheta_1^2} \operatorname{sign}\left(t - \frac{\vartheta_1}{2}\right), \tag{25}$$

Thus, we have constructed the optimal control for the original problem. It should be noted that the optimal control in this problem is not unique. This follows from the fact that the boundary-value problem has a non-unique solution. We have found relay controls that solve the boundary problem. Obviously, there are other solutions.

NON-LINEAR VERSION OF THE PROBLEM

Equations of motion (1) are obtained under the assumption that $c_2 = 0$. This assumption is permissible in the first approximation. But in real systems this value is small, but still different from zero. Under the assumption that $c_2 \neq 0$, the Lagrange equations of the second kind will have the form

$$\left(J_1 + J_2 + m_2 e_1^2\right)\ddot{\varphi}_1 + (J_2 + m_2 e_1 e_2 \cos(\varphi_2))\,\ddot{\varphi}_2 - m_2 e_1 e_2 \sin(\varphi_2)(\dot{\varphi}_2)^2 = u_1,$$

$$(J_2 + m_2 e_1 e_2 \cos(\varphi_2))\,\ddot{\varphi}_1 + \left(J_2 + m_2 e_2^2\right)\ddot{\varphi}_2 = u_2. \tag{26}$$

All the coefficients included in these equations were described earlier.

Solving the system (26) with respect to $\ddot{\varphi}_1$ and $\ddot{\varphi}_2$, we obtain

$$\ddot{\varphi}_1 = \frac{1}{\Delta}\left((J_2 + m_2 e_2^2) m_2 e_1 e_2 \sin(\varphi_2)(\dot{\varphi}_2)^2 + (J_2 + m_2 e_2^2)u_1 - (J_2 + m_2 e_1 e_2 \cos(\varphi_2))u_2\right),$$

$$\ddot{\varphi}_2 = \frac{1}{\Delta}\left((J_1 + J_2 + m_2 e_1^2)u_2 - (J_2 + m_2 e_1 e_2 \cos(\varphi_2))u_1 - m_2 e_1 e_2 \sin(\varphi_2)(J_2 + m_2 e_1 e_2 \cos(\varphi_2))(\dot{\varphi}_2)^2\right), \tag{27}$$

where

$$\Delta = (J_1 + J_2 + m_2 e_1^2)(J_2 + m_2 e_2^2) - (J_2 + m_2 e_1 e_2 \cos(\varphi_2))^2.$$

We expand the right-hand side of the equations (27) with respect to e_2 and preserve only terms not exceeding the first order in e_2. As a result, we get

$$\ddot{\varphi}_1 = a_1(u_1 - u_2) + \frac{m_2 e_1 \sin(\varphi_2)(\dot{\varphi}_2)^2}{J_1 + m_2 e_1^2} e_2 - \frac{m_2 e_1 \cos(\varphi_2)}{J_2(J_1 + m_2 e_1^2)} e_2 u_2.$$

$$\ddot{\varphi}_2 = -a_1 u_1 + a_2 u_2 + \frac{m_2 e_1 \cos(\varphi_2)}{J_2(J_1 + m_2 e_1^2)} e_2 u_1 - \frac{m_2 e_1 \sin(\varphi_2)}{J_1 + m_2 e_1^2} e_2. \tag{28}$$

We obtained a controllable system of differential equations with a small parameter e_2. Thus, a quasilinear model is constructed for the control object. The speed problem for an object described by the equations (28) can be solved, for example, using the methods proposed in [5].

CONCLUSIONS

For the linear model of the optimal control problem for the system of two coupled cylinders, optimal control is found. The peculiarity of the optimal control found is its nonuniqueness. For a nonlinear model, a variant of reducing such a model to a quasilinear model with a small parameter is proposed.

ACKNOWLEDGMENTS

The research was supported by Russian Foundation for Basic Research, project no. 17-08-01385.

REFERENCES

[1] K. Kozowski. *Modelling and Identification in Robotics.* Springer, London, UK, 1998.

[2] M. W. Spong, S. A. Hutchinson, and M. Vidyasagar. *Robot Modeling and Control.* John Wiley and Sons, 2006.

[3] R. A. Becker. *Introduction to theoretical mechanics*. WcGrao-Hill. 1954.

[4] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko. *The mathematical theory of optimal processes*. Pergamon Press. 1964.

[5] Ya. O. Grudo, A. I. Kalinuin *Asymptotic optimization method for quasilinear system with multidimensional control.* Differential equations, Vol. 42, 12, 2006, pp. 1674-1681.