Application of the Saint-Venant model and the modified Stefan model for modeling the formation of the ice cover at the thermal growth stage

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Abstract. River ice processes are complex phenomena that are affected by many factors, including meteorological conditions, thermal inputs, hydraulic conditions and channel geometry. Because of the complexity of river ice phenomena and the limited current understanding, the development ofa river ice model requires the maximum use of existing theories and mathematical techniques. In the river hydraulics component, the flow condition is often determined by the solution of the Saint-Venant unsteady flow equations. The complex phenomenon of ice formation at the stage of thermal growth in turn is usually described by models based on the Stefan equation. In current paper we consider the opportunity to use the modified Stefan model, which includes the model of mushy layer in the task of ice thermal grows together with the solution of the problem of Saint-Venant.

1. Introduction

The process of river ice formation is a complex and incompletely studied natural phenomenon. It includes the formation, evolution, transport, dissipation and deterioration of various forms of ice. This process depends not only on the geometry of the riverbed, the atmospheric conditions, but also by parameters of the water flow in the river. Thus, the ice formation mechanisms are complex interconnected system, parts of which are not completely understood. However, by using existing theories, an in-depth mathematical models can be proposed [1–6]. Such models allow us to predict a continuous process of ice formation using a limited amount of data. The result of predictions can be used in design and operation of river control works. A mathematical model can also be used by researchers to identify crucial gaps and weaknesses in the current theories of river ice.

The process of ice cover formation is a very complex physical phenomenon that can be divided into a number of essential processes. A detailed schematic description of the process of ice formation is given in [7]. A comprehensive one-dimensional river ice model called RICE is developed in conformity with that scheme and makes maximum use of the existing information, with improvements on analytical and mathematical formulations. Because of using a separate subroutine for the modeling of each ice process, the general model allows modifications of every submodel. In this paper we propose to consider possible the modification for the problem of the ice cover at the thermal growth stage.

2. Modeling of the ice cover at the thermal growth stage

2.1. Unsteady flow modelling

The hydraulics of one-dimensional river flow can be described by Saint Venant equations. These equations represent the conservation ofmass and momentum in the river:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q, \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + \rho g A\left(\frac{\partial h}{\partial x} + S_f\right) = 0, \qquad (1)$$

where :

x – the length of watercourse (m),

t – time (s),

$A\left(x, t\right)$ – net flow cross-sectional area (m^2/s),

$Q\left(x, t\right)$ – water flow (m^3/s),

$q\left(x, t\right)$ – external power per unit length of the watercourse (m^2/s),

g – the gravitational acceleration (m/s^2),

$h\left(x, t\right)$ – depth (m),

$S_f\left(x, t\right)$ – the friction slope flow.

In case of ice cover problem $(S_f\left(x, t\right) = p_b\tau_b + p_i\tau_i)/g\rho\overline{A}$, where:

p_b – wetted perimeter formed by the channel bed,

p_i – wetted perimeters formed by the ice cover,

τ_b – shear stresses at the channel bottom,

τ_i – shear stresses at the ice/water interface,

$\overline{A}$ – mean net flow cross-sectional area.

For such a problem, implicit finite-difference or finite-element numerical methods are usually used. That methods requires the river to be divided into a sufficient number of reaches. Moreover, stability and accuracy are achieved under certain conditions of discretization of the system [7]. We believe that for the calm flow of water in a channel of a rather large width of a river bed, it is possible to use more simple approximations of the Saint-Venant model and, as a consequence, less demanding methods of solution. Such an approach can be the kinematic approximation and its solution proposed in [8]. In this paper, an explicit numerical scheme is proposed that uses analytical solutions within computational cells. This makes it possible to solve the problem of water flow using large cells where possible, but retaining an acceptable level of accuracy due to approximation by analytical methods.

2.2. Thermal growth of ice covers

The thickness of an ice cover can be changed by thermal growth as a result of heat exchange with the environment and the water. The ice cover thickness can influent on the flow cross-sectional area and by this way on the velocity of the flow. It's important in determining the stability of an ice cover and its break-up. Many methods have been used to model the thickness of ice covers. Many of them are basically based on the degree-day method or the Stefan model [1, 9, 10]. But as rule the model of planar front is used under consideration. We propose to consider a modification of the Stefan model which includes the model of mushy layer – regions of mixed phase (solid and liquid) in which the solid forms a rigid matrix and the liquid fills its interstices. In classical degree-day method the ice thickness h is given as

$$h = \alpha_h\sqrt{S}, \qquad (2)$$

$S = \int_{t_0}^{t} (T_m - T_a)\mathrm{dt}$ – cumulative freezing degree-days of air temperature since the formation of the cover at time t_0.

t – time from the formation of the ice cover

T_m – freezing temperature of water

T_a – air temperature

α_h – empirical degree-day constant. In case of planar front this constant can be expressed as $\sqrt{2k_i/L_v}$, where L_v is the latent heat released as the ice fraction increases and k_i is the thermal conductivity.

In [11] the modified equation of the Stefan model is presented taking into account the mushy layer approach. The scheme of the difference between classical model of planar front and mushy-layer model is presented in Figure 1. This non-linear problem under consideration is solved
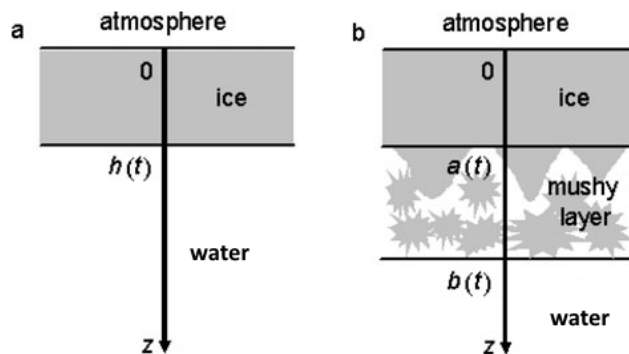

Figure 1. A scheme illustrating the models which are taking into consideretion: (a) planar front and (b) mushy layer.

analytically in [11]. Solutions represented by the following equations:

$$a(t) = -\frac{D_w L_v \varphi_b}{T_o \Phi} b(t), \tag{3}$$

$$b(t) = \alpha_h \sqrt{S}. \tag{4}$$

In a result the degree-day constant is modified and expressed by $\sqrt{2/I}$, where:

$$I = \frac{L_v \varphi_b}{\Phi}[1 - \frac{D_w L_v \varphi_b}{T_o \Phi}(\frac{L_v D_w}{k_i T_o} + 1 - K)(\varphi_b - 1)]. \tag{5}$$

Here where φ_b is the solid fraction and T_o is the constant temperature of the water determined on the interface mushy-layer/water. $\Phi = k_i \varphi_b + k_w(1 - \varphi_b)$. k_w and D_w are the thermal conductivity and diffusion coefficient, respectively, of the water. $K = \frac{k_w}{k_i}$. [11] shows that analytical solutions for the mushy layer transform to their analogues for the planar front. But in contrast to it, the modified model allows for taking into account the mushy layer, which can have a significant effect on the hydraulic effects in water flow under certain conditions.

3. Conclusions

The paper considers the complex phenomenon of ice formation on the rivers. Mathematical modeling of such a phenomenon is an extremely difficult task. One of the successful projects to create a comprehensive mathematical model of ice formation is the RICE system. This

system includes sub-models of individual physical processes. This makes it possible to modify the general model by changing of components. The authors consider a submodel of the thermal growth, which can be described by various techniques. However, all these techniques are based on the degree-day Stefan model, taking into account only planar front of ice formation. The authors propose to consider the possibility of using a modified model that takes into account the mushy layer of ice formation. This model allows to evaluate not only the ice-water interface, but also the mushy zone in which the ice formation is not yet complete, but the hydraulic processes of the water flow can have significant features. In addition, the authors considered the main hydraulic model of Saint-Venant. For it, in appropriate cases, it is suggested to use a simplified model for which there is a numerical solution that uses the analytical approximation in the computational cells. Such an approach can simplify the formulation of the problem for data deficit cases.

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