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On Berezinskii-Kosterlitz-Thouless transition in monoaxial chiral helimagnets

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Abstract. We analyze the possibility of Berezinskii-Kosterlitz-Thouless (BKT) transition in XY-model with in-plane Dzyaloshinskii-Moriya (DM) interaction. It is demonstrated that under standard duality transformation of XY-model into the model of two-dimensional Coulomb gas of magnetic vortices, DM interaction is mapped to the effective electric field perpendicular to the original DM vector. Since the electrostatic energy in the constant electric field becomes dominating over the logarithmic Coulomb attraction at sufficiently large length scales, BKT transition is ruled out by the presence of DM interaction. This behavior is confirmed in the renormalization group analysis, which shows that for finite effective electric field the system always flows towards the high temperature phase.

1. Introduction

The possibility of topological phase transitions in two-dimensional (2D) systems was first anticipated independently by Berezinskii [1], Kosterlitz and Thouless [2]. Since that time, this concept found numerous applications in different fields of condensed matter physics including hydrodynamics, superconductors, and magnetic systems [3]. A classical Berezinskii-Kosterlitz-Thouless (BKT) scenario states that there exists a certain transition temperature such as that below this temperature the topological defects – vortices are bound and hidden from observation, while above it free defects can proliferate into the system. This transition is accompanied by characteristic features in macroscopic properties like, for example, in current-voltage characteristics of semiconductors [3].

During the last decade, a considerable progress has been made in physics of chiral magnets where the existence of asymmetric Dzyaloshinskii-Moriya (DM) exchange interaction leads to a rich variety of magnetic phases (see e. g. [4] and references therein). In monoaxial chiral magnets, an important shift was marked recently by experimental discovery of chiral soliton lattice structure [5] originally predicted by Dzyaloshinskii [6]. With thin films of chiral magnets becoming available, realization of BKT scenario in such materials is an intriguing question.

In this paper we consider BKT transition in 2D monoaxial chiral magnet by mapping the correspondent XY-model into 2D Coulomb gas of magnetic vortices. We found that under such transformation in-plane DM interaction is mapped into an effective electric field in the direction perpendicular to DM vector. Interestingly, similar result was obtained in 2D semiconductors where applied current density is mapped into topological electric field [7]. Note that locked-unlocked transition in the sine-Gordon model, where the misfit parameter plays the role of electric field in the equivalent Coulomb gas model, was also studied in [8].

Our discussion goes along the following line. First, we consider general formulation of the model. After that, the model is subjected to the duality transformation, and for the resulting effective model the renormalization group (RG) equations are derived. Conclusions based on numerical solution of the RG equations are given in the last part. For completeness, some technical details are specified in Appendices.

2. Formulation

We consider 2D chiral helimagnet withing the framework of XY-model on a square lattice with the Hamiltonian given by

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_j - \boldsymbol{D}\cdot\sum_{\langle ij\rangle}\boldsymbol{S}_i\times\boldsymbol{S}_j - \tilde{H}_x\sum_i S_{ix}, \tag{1}$$

where $J>0$ is the ferromagnetic exchange constant, $\boldsymbol{D}=D\boldsymbol{e}_z$ is DM vector taken along the z-axis, and $\tilde{H}_x$ is the magnetic field applied along the x-direction, see Fig. 1. All the classical spin variables $\boldsymbol{S}_i$ are confined in the basal xy-plane and parametrized as $\boldsymbol{S}_i = S(\cos\theta_i, \sin\theta_i, 0)$, where $0\le\theta_i<2\pi$ is the polar angle at the site i. Angular brackets denote summation over the nearest neighboring sites.

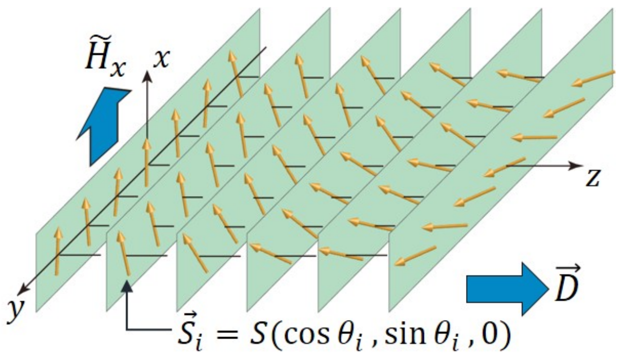

Figure 1. Schematic picture of the monoaxial 2D chiral helimagnet. The classical spins $\boldsymbol{S}_i$ rotate in the basal xy-plane, the DM vector is along the z-axis, and in-plane magnetic field $\tilde{\boldsymbol{H}}$ is applied along the x-direction.

The thermodynamic properties of the model with the Hamiltonian in (1) are described by the partition function

$$\mathcal{Z} = \int\prod_i d\theta_i \exp\left[\beta\tilde{J}\sum_{\langle ij\rangle}\cos\left(\theta_i-\theta_j-\alpha_{ij}\right) + \beta h\sum_i\cos\theta_i\right], \tag{2}$$

where $\beta=(k_BT)^{-1}$ is the inverse temperature, $\tilde{J}=S^2\sqrt{J^2+D^2}$ is the effective exchange parameter, $h=\tilde{H}_xS$, and $\alpha_{ij}=\tan^{-1}(D/J)$ for the nearest neighbor ij-bond along the z-direction and zero otherwise.

2.1. *Effective model*

In the low temperature limit, the spin-wave contribution to the partition function (2) can be integrated out using the duality mapping technique [9]. The essential steps of the duality transformation are summarized in Appendix A. The resulting effective model takes into account vortex configurations where θ_i field, while still being slowly varying form site to site over a closed loop, acquires total contribution proportional to the multiply of 2π. After the transformation partition function takes the form

$$\mathcal{Z} \propto \sum_{\{l_\mu(\boldsymbol{r})\}} \exp\left[-\sum_{\boldsymbol{r}\mu}\left(\frac{l_\mu^2(\boldsymbol{r})}{2\beta\tilde{J}} + i\alpha_\mu l_\mu(\boldsymbol{r})\right)\right] \times \prod_{\boldsymbol{r}}\left\{\sum_{\kappa=-\infty}^{\infty} I_\kappa(\beta h)\delta\left[\sum_\mu \left(l_\mu(\boldsymbol{r}) - l_\mu(\boldsymbol{r}-\boldsymbol{a}_\mu)\right) - \kappa(\boldsymbol{r})\right]\right\}, \quad (3)$$

where a particular configuration is expressed by an integer vector field $l_\mu(\boldsymbol{r})$ defined on the bond between two lattice sites with radius vectors $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{a}_\mu$ where $\boldsymbol{a}_\mu$ ($\mu = y, z$) is the lattice vector, and $\alpha_\mu = \alpha\delta_{\mu z}$ with $\alpha = \tan^{-1}(D/J)$. The effect of the magnetic field h is to generate magnetic charges $\kappa(\boldsymbol{r})$ at the center of each lattice cell with the amplitudes of such contributions proportional to the modified Bessel functions $I_\kappa(\beta h)$. The total summation is over all possible configurations with $l_\mu(\boldsymbol{r})$ running form $-\infty$ to ∞ on each bond.

The physical meaning of the constraint imposed by δ-function in equation (3) can be elucidated through the splitting of $l_\mu(\boldsymbol{r})$ into the longitudinal and transverse parts [10]

$$l_\mu(\boldsymbol{r}) = m(\boldsymbol{r}) - m(\boldsymbol{r}+\boldsymbol{a}_\mu) + \sigma(\boldsymbol{r}) - \sigma(\boldsymbol{r}+\boldsymbol{a}_\mu) + \varepsilon_{\mu\nu}\left[n(\boldsymbol{r}) - n(\boldsymbol{r}-\boldsymbol{a}_\nu)\right], \quad (4)$$

where $m(\boldsymbol{r})$ and $n(\boldsymbol{r})$ are integers, $|\sigma(\boldsymbol{r})| < 1$ and $\varepsilon_{\mu\nu}$ denotes the second rank Levi-Civita tensor. The longitudinal part satisfies lattice form of the Poisson equation

$$\sum_\mu \left[2m(\boldsymbol{r}) + 2\sigma(\boldsymbol{r}) - m(\boldsymbol{r}+\boldsymbol{a}_\mu) - \sigma(\boldsymbol{r}+\boldsymbol{a}_\mu) - m(\boldsymbol{r}-\boldsymbol{a}_\mu) - \sigma(\boldsymbol{r}-\boldsymbol{a}_\mu)\right] = \kappa(\boldsymbol{r}), \quad (5)$$

while the transverse part is a pure rotation over the unit cell with zero lattice divergence

$$\sum_\mu \varepsilon_{\mu\nu}\left[n(\boldsymbol{r}) - n(\boldsymbol{r}-\boldsymbol{a}_\nu)\right] = 0. \quad (6)$$

By substituting (4) in (3) and applying the Poisson formula to the summation over $n(\boldsymbol{r})$, we obtain

$$\mathcal{Z} \propto \sum_{\{\kappa(\boldsymbol{r})\}}\sum_{\{q(\boldsymbol{r})\}}\int\prod_{\boldsymbol{r}}\varphi(\boldsymbol{r}) I_{\kappa(\boldsymbol{r})}(\beta h) \times \exp\left\{-\sum_{\boldsymbol{r}\mu}\left[\frac{\left(\hat{\Delta}_\mu\varphi(\boldsymbol{r})\right)^2}{2\beta\tilde{J}} + i\alpha_\mu\hat{\Delta}_\mu\varphi(\boldsymbol{r})\right] + 2\pi i\sum_{\boldsymbol{r}} q(\boldsymbol{r})\varphi(\boldsymbol{r})\right\}, \quad (7)$$

where

$$\hat{\Delta}_\mu\varphi(\boldsymbol{r}) = \varphi(\boldsymbol{r}+\boldsymbol{a}_\mu) - \varphi(\boldsymbol{r}) + \varepsilon_{\mu\nu}\left[m(\boldsymbol{r}+\boldsymbol{a}_\nu) + \sigma(\boldsymbol{r}+\boldsymbol{a}_\nu) - m(\boldsymbol{r}) - \sigma(\boldsymbol{r})\right]. \quad (8)$$

In what follows, we only consider the case of vanishing magnetic fields, since in this paper we are mainly focused on the effect of DM interaction. The model that includes magnetic charges will be analyzed in further publications. In the $\beta h \to 0$ limit, $I_\kappa(\beta h)$ is replaced by $\delta_{\kappa 0}$, and equation (5) has a trivial solution $m(\boldsymbol{r}) + \sigma(\boldsymbol{r}) = 0$. Taking into account that $\alpha_\mu = \alpha\delta_{\mu z}$ and making the shift of variables $\varphi(\boldsymbol{r}) \to \varphi(\boldsymbol{r}) + i\alpha\beta\tilde{J}y$, we obtain an effective model of a 2D Coulomb gas of magnetic vortices with integer charges $q(\boldsymbol{r})$ in the external electric field along the y-axis proportional to the strength of DM interaction

$$\mathcal{Z}_{\text{eff}} \propto \sum_{\{q(\boldsymbol{r})\}} \exp\left\{-2\pi^2\beta\tilde{J}\sum_{\boldsymbol{r}\boldsymbol{r}'} q(\boldsymbol{r})G(\boldsymbol{r}-\boldsymbol{r}')q(\boldsymbol{r}') + 2\pi\alpha\beta\tilde{J}\sum_{\boldsymbol{r}} y\, q(\boldsymbol{r})\right\}, \tag{9}$$

where

$$G(\boldsymbol{r}-\boldsymbol{r}') = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{4-2\cos k_x - 2\cos k_y} \approx -\frac{1}{2\pi}\log|\boldsymbol{r}-\boldsymbol{r}'| + G(0), \tag{10}$$

is 2D Coulomb potential, and $G(0)$ contains the divergent part of the interaction potential that ensures 'electroneutrality' of the system (hereafter we take the lattice constant $a = 1$).

Equation (9) is the starting point for the RG analysis [11, 12]. It can be rewritten in a more convenient form [9, 11]

$$\mathcal{Z}_{\text{eff}} \propto \sum_{\{q(\boldsymbol{r})\}} \exp\left[2\pi K \sum_{\boldsymbol{r}\neq\boldsymbol{r}'} q(\boldsymbol{r})\log|\boldsymbol{r}-\boldsymbol{r}'|q(\boldsymbol{r}') - 2\pi\sum_{\boldsymbol{r}} \epsilon_y(\boldsymbol{r})\, q(\boldsymbol{r})\right] y_0^{\sum_{\boldsymbol{r}} q^2(\boldsymbol{r})}, \tag{11}$$

where $K = \beta\tilde{J}$, $\boldsymbol{\epsilon}(\boldsymbol{r}) = -\alpha\beta\tilde{J}\boldsymbol{r}$, and $y_0 = \exp(-\pi^2\beta\tilde{J}/2)$ is so called fugacity of the Coulomb gas. Since we consider the case $\beta\tilde{J} \gg 1$, all the contribution with large $|q(\boldsymbol{r})|$ are exponentially small, and we can restrict the summation in (11) to $q(\boldsymbol{r}) = 0, \pm 1$.

2.2. RG equations

The partition function in equation (11) can be expanded in terms of the total number of vortex pairs $N = \frac{1}{2}\sum_{\boldsymbol{r}} q^2(\boldsymbol{r})$

$$\mathcal{Z}_{\text{eff}} = \sum_{N=0}^{\infty} \frac{y_0^{2N}}{(N!)^2} \int \prod_{i=1}^{2N} d^2 s_i \exp\left[2\pi K \sum_{i<j} q(\boldsymbol{s}_i)q(\boldsymbol{s}_j)\log|\boldsymbol{s}_i - \boldsymbol{s}_j| - 2\pi\sum_i \epsilon_y(\boldsymbol{s}_i)q(\boldsymbol{s}_i)\right], \tag{12}$$

where we take heed of $|q(\boldsymbol{r})| = 0, 1$. The factor $(N!)^2$ arises from independent permutations of all the vortices and antivortices. In the lowest order in y_0, the system contains only one vortex-antivortex pair with ± 1 charges at the positions $\boldsymbol{s}$ and $\boldsymbol{s}'$

$$\mathcal{Z}_{\text{eff}}^{(1)} = 1 + y_0^2 \int d^2 s \int d^2 s' \exp\left[-2\pi K\log|\boldsymbol{s}-\boldsymbol{s}'| - 2\pi\epsilon_y(\boldsymbol{s}-\boldsymbol{s}')\right] + O(y_0^4). \tag{13}$$

In order to derive RG equations for the model parameters, we calculate effective interaction energy for the pair of ± 1 external charges, placed at the positions $\boldsymbol{r}$ and $\boldsymbol{r}'$, which includes the screening effects from the internal charges at $\boldsymbol{s}$ and $\boldsymbol{s}'$ [11]. In the lowest order expansion, the effective interaction reads

$$e^{\mathcal{S}_{\text{eff}}} = \frac{1}{\mathcal{Z}_{\text{eff}}^{(1)}}\left[1 + y_0^2\int d^2 s\int d^2 s' e^{\mathcal{S}_{\text{int}}(\boldsymbol{r},\boldsymbol{r}';\boldsymbol{s},\boldsymbol{s}')}\right] + O(y_0^4), \tag{14}$$

where

$$\begin{aligned}\mathcal{S}_{\text{int}}(\boldsymbol{r},\boldsymbol{r}';\boldsymbol{s},\boldsymbol{s}') = -2\pi K\big(&\log|\boldsymbol{r}-\boldsymbol{r}'| + \log|\boldsymbol{s}-\boldsymbol{s}'| + \log|\boldsymbol{r}-\boldsymbol{s}'| + \log|\boldsymbol{r}'-\boldsymbol{s}| \\ &- \log|\boldsymbol{r}-\boldsymbol{s}| - \log|\boldsymbol{r}'-\boldsymbol{s}'|\big) - 2\pi\epsilon_y(\boldsymbol{r}+\boldsymbol{s}-\boldsymbol{r}'-\boldsymbol{s}').\end{aligned} \tag{15}$$

Direct calculation of $\mathcal{S}_{\text{eff}}$ in equation (14) results in

$$\mathcal{S}_{\text{eff}} = -2\pi K_{\text{eff}}\log|\boldsymbol{r}-\boldsymbol{r}'| - 2\pi\epsilon_{\text{eff}}(\boldsymbol{y}-\boldsymbol{y}'), \tag{16}$$

where the screening from internal charges is absorbed in the effective model parameters

$$K_{\text{eff}} = K - 4\pi^3 K^2 y_0^2 \int_1^\infty dx x^{3-2\pi K} + O(\epsilon^2, y_0^4), \tag{17}$$

$$\epsilon_{\text{eff}} = \epsilon - 4\pi^3 K \epsilon y_0^2 \int_1^\infty dx x^{3-2\pi K} + O(\epsilon^2, y_0^4). \tag{18}$$

Here, $\epsilon = \alpha\beta\tilde{J}$ is the effective electric field strength. Some technical details of the derivation are reproduced in Appendix B. Giving the cut-off parameter $b = e^\ell$ in the integration over x, we eventually obtain the following system of RG equations [7]

$$\frac{dK}{d\ell} = -4\pi^3 y_0^2 K^2, \tag{19}$$

$$\frac{dy_0}{d\ell} = (2 - \pi K + \pi\epsilon) y_0, \tag{20}$$

$$\frac{d\epsilon}{d\ell} = \epsilon + 4\pi^3 y_0^2 K \epsilon. \tag{21}$$

3. Discussion and summary

Below, we discuss the system of RG equations (19)–(21). For $\epsilon = 0$, these equations describe BKT transition at the temperature $T_{\text{C}} = \pi\tilde{J}/2$ between the underscreened and overscreened regimes. In the underscreened regime realized at $T < T_{\text{C}}$ and $\pi^2 y_0 < |K^{-1} - \pi/2|$, all the vortices in the system are coupled into dipoles resulting in zero effective screening. In RG language, it means that $K(\ell)$ and $y_0(\ell)$ flow towards the $y_0 = 0$ line that corresponds to $K_{\text{eff}} = K$ according to equation (17). In contrast, in the overscreened phase, $T > T_{\text{C}}$ or $\pi^2 y_0 > |K^{-1} - \pi/2|$, the dipoles disperse into free charges and the screening appears. In this scenario, the model parameters flow towards the strong coupling fixed point at $K = 0$ and $y_0 = \infty$ [11].

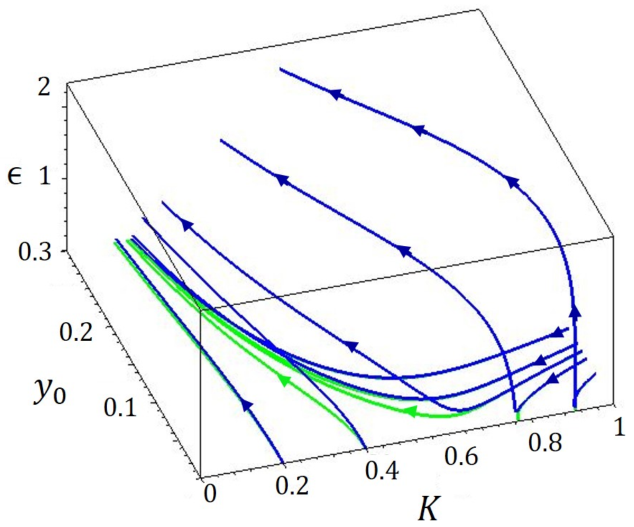

Figure 2. RG flow for the Coulomb gas in finite electric field. The flow in $\epsilon = 0$ plane is shown by green color. The BKT transition point is at the position $(2/\pi, 0, 0)$.

The presence of the electric field drastically changes the above picture. Even infinitesimal ϵ lifts BKT transition in favor of the overscreened scenario. Figure 2 show RG flow in the axes K, y_0, and ϵ calculated numerically from equation (19)–(21). In the $\epsilon = 0$ plane, RG flow is consistent with BKT transition behavior at the point $(2/\pi, 0, 0)$. However, above the transition point the electric field pulls positive and negative charges apart breaking the dipoles and redirecting the flow towards the strong screening fixed point, $K \to 0$ and $y_0 \to \infty$.

The absence of BKT transition in the system with finite ϵ can be understood from the following simple argument. In 2D system, the electrostatic energy in a constant electric field being linear in $\boldsymbol{r}$ becomes dominant over the logarithmic Coulomb attraction at sufficiently large distances. Therefore, to gain the total energy, it is always favorable to break the dipole pairs apart for any values of K and y_0.

By noting that in our model the effective electric field is proportional to the strength of DM interaction, we make the conclusion that DM interaction is relevant. The presence of in-plane DM exchange in magnetic thin film should remove BKT transition and assist the observation of free magnetic vortices.

In summary, we considered duality transformation and RG analysis in 2D chiral magnets with in-plane DM interaction. It was demonstrated that this system can be mapped into the model of 2D Coulomb gas of magnetic vortices with DM interaction playing the role of the effective electric field directed perpendicularly to the original DM vector. By applying RG analysis, we showed that the effective electric field greatly effects on the vortex-antivortex pairs, prevents the formation of magnetic dipoles, and destroys BKT transition.

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Appendix A. Duality transformation

In the low temperature limit $\beta\tilde{J} \gg 1$, one can approximate periodic function in equation (2) defined on the bond between ith and jth sites by a series of its expansions near each minimum labeled by the integer m_{ij}

$$e^{\beta\tilde{J}(\cos\Theta_{ij}-1)} \approx \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\beta\tilde{J}}{2}\left(\Theta_{ij}-2\pi m_{ij}\right)^2\right], \tag{A.1}$$

where $\Theta_{ij} = \theta_i - \theta_j - \alpha_{ij}$. The summation over m_{ij} is transformed into the integration over ϕ_{ij}-field using the Poisson formula

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\, f(\phi) e^{2\pi i l\phi}, \tag{A.2}$$

and after integrating out ϕ, we arrive to the following expression for the partition function

$$\mathcal{Z} \propto \sum_{\{l_{ij}\}} \int \prod_i d\theta_i \exp\left[-\sum_{\langle ij\rangle}\left(\frac{l_{ij}^2}{2\beta\tilde{J}} - i l_{ij}\Theta_{ij}\right) + \beta h \sum_i \cos\theta_i\right]. \tag{A.3}$$

We replace $\theta_i \to \theta(\boldsymbol{r})$ and $l_{ij} \to l_\mu(\boldsymbol{r})$ where $\mu = y, z$, and take the advantage of infinite lattice

$$-i\sum_{\langle ij\rangle} l_{ij}\left(\theta_i - \theta_i\right) \to -i\sum_{\boldsymbol{r}\mu} l_\mu(r)\left[\theta(\boldsymbol{r}) - \theta(\boldsymbol{r}+\boldsymbol{a}_\mu)\right] = -i\sum_{\boldsymbol{r}\mu}\left[l(\boldsymbol{r}) - l(\boldsymbol{r}-\boldsymbol{a}_\mu)\right]\theta(\boldsymbol{r}), \tag{A.4}$$

which gives

$$\begin{aligned}\mathcal{Z} \propto & \sum_{\{l_\mu(\boldsymbol{r})\}} \int \prod_{\boldsymbol{r}} d\theta(\boldsymbol{r}) \\ & \times \exp\left[-\sum_{\boldsymbol{r}\mu}\left(\frac{l_\mu^2(\boldsymbol{r})}{2\beta\tilde{J}} - i\left[l(\boldsymbol{r}) - l(\boldsymbol{r}-\boldsymbol{a}_\mu)\right]\theta(\boldsymbol{r}) + i\alpha_\mu l_\mu(\boldsymbol{r})\right)\right] \exp\left[\beta h \sum_{\boldsymbol{r}} \cos\theta(\boldsymbol{r})\right].\end{aligned} \tag{A.5}$$

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Now it is possible to integrate over θ from 0 to 2π using the expansion

$$e^{z\cos\theta} = I_0(z) + 2\sum_{k=1}^{\infty} I_k(z)\cos(k\theta), \tag{A.6}$$

that gives equation (3).

Appendix B. Effective model parameters

Here we highlight crucial steps in the derivation of RG equation following the discussion in [11]. First, we rewrite (14) in the following form

$$\begin{aligned} e^{\mathcal{S}_{\rm eff}} &= e^{-2\pi K\log|\boldsymbol{r}-\boldsymbol{r}'|-2\pi\epsilon_y(\boldsymbol{r}-\boldsymbol{r}')} \\ &\times\left[1+y_0^2\int d^2s\int d^2s' e^{-2\pi K\log|\boldsymbol{s}-\boldsymbol{s}'|-2\pi\epsilon_y(\boldsymbol{s}-\boldsymbol{s}')}\left(e^{2\pi K D(\boldsymbol{r},\boldsymbol{r}';\boldsymbol{s},\boldsymbol{s}')}-1\right)\right]+O(y_0^4), \end{aligned}\tag{B.1}$$

where

$$D(\boldsymbol{r},\boldsymbol{r}';\boldsymbol{s},\boldsymbol{s}') = \log|\boldsymbol{r}-\boldsymbol{s}'| + \log|\boldsymbol{r}'-\boldsymbol{s}| - \log|\boldsymbol{r}-\boldsymbol{s}| + \log|\boldsymbol{r}'-\boldsymbol{s}'|, \tag{B.2}$$

and change to the center of mass variables $\boldsymbol{X} = (\boldsymbol{s}+\boldsymbol{s}')/2$ and $\boldsymbol{x} = \boldsymbol{s}-\boldsymbol{s}'$. Second, we expand over $|\boldsymbol{x}|$ keeping only terms up x^2

$$e^{2\pi K D(\boldsymbol{r},\boldsymbol{r}';\boldsymbol{s},\boldsymbol{s}')} - 1 = 2\pi K\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} f(\boldsymbol{X}) + 2\pi^2K^2\left(\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} f(\boldsymbol{X})\right)^2 + O(x^3), \tag{B.3}$$

$$e^{-2\pi\epsilon_y(\boldsymbol{s}-\boldsymbol{s}')} = 1 - 2\pi\epsilon\left(\boldsymbol{x}\cdot\boldsymbol{e}_y\right) + O(x^2), \tag{B.4}$$

where $f(\boldsymbol{X}) = \log|\boldsymbol{r}-\boldsymbol{X}| - \log|\boldsymbol{r}'-\boldsymbol{X}|$. Substitution of these equations in (B.1) gives

$$\begin{aligned} e^{\mathcal{S}_{\rm eff}} &= e^{-2\pi K\log|\boldsymbol{r}-\boldsymbol{r}'|-2\pi\epsilon_y(\boldsymbol{r}-\boldsymbol{r}')} \\ &\times\left[1-4\pi^2K\epsilon y_0^2\int d^2x\int d^2X e^{-2\pi K\log|x|}(\boldsymbol{x}\cdot\boldsymbol{e}_y)\,\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} f(\boldsymbol{X})\right. \\ &\left.+2\pi^2K^2\int d^2x\int d^2X e^{-2\pi K\log|x|}\left(\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} f(\boldsymbol{X})\right)^2\right]+O(\epsilon^2,y_0^4). \end{aligned}\tag{B.5}$$

In the first integral we perform integration over the angular part of $\boldsymbol{x}$ and use the identity $\boldsymbol{e}_y = \boldsymbol{\nabla}_{\boldsymbol{X}}(\boldsymbol{X}\cdot\boldsymbol{e}_y)$ that provides the answer

$$\int d^2x\int d^2X e^{-2\pi K\log|x|}(\boldsymbol{x}\cdot\boldsymbol{e}_y)\,\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} f(\boldsymbol{X}) = -2\pi^2\boldsymbol{e}_y\cdot(\boldsymbol{r}-\boldsymbol{r}')\int_1^{\infty} dx x^{3-2\pi K}, \tag{B.6}$$

where we introduced an infrared cut-off at the lattice spacing distances. The second integral is calculated using 2D form of the Poisson equation

$$\nabla^2_{\boldsymbol{X}}\log|\boldsymbol{r}-\boldsymbol{X}| = 2\pi\delta(\boldsymbol{r}-\boldsymbol{X}), \tag{B.7}$$

that provides

$$\int d^2x\int d^2X e^{-2\pi K\log|x|}\left(\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} f(\boldsymbol{X})\right)^2 = (2\pi)^2\log|\boldsymbol{r}-\boldsymbol{r}'|\int_1^{\infty} dx x^{3-2\pi K}. \tag{B.8}$$

After substitution of (B.6) and (B.8) in (B.5), we obtain equations (17, 18).

References

[1] Berezinskii V 1971 *Sov. Phys. JETP* **32** 493

[2] Kosterlitz J M and Thouless D J 1973 *Journal of Physics C: Solid State Physics* **6** 1181

[3] José J V 2013 *40 years of Berezinskii-Kosterlitz-Thouless theory* (World Scientific)

[4] Kishine J and Ovchinnikov A S 2015 Theory of monoaxial chiral helimagnet (*Solid State Physics* vol 66) ed Camley R E and Stamps R L (Academic Press) pp 1 – 130

[5] Togawa Y, Koyama T, Takayanagi K, Mori S, Kousaka Y, Akimitsu J, Nishihara S, Inoue K, Ovchinnikov A S and Kishine J 2012 *Phys. Rev. Lett.* **108**(10) 107202

[6] Dzyaloshinskii I 1964 *Sov. Phys. JETP* **19** 960

[7] Sujani S, Chattopadhyay B and Shenoy S R 1994 *Phys. Rev. B* **50**(22) 16668–16678

[8] Puga M W, Simanek E and Beck H 1982 *Phys. Rev. B* **26**(5) 2673–2676

[9] Nagaosa N 1999 *Quantum Field Theory in Condensed Matter Physics* Texts and monographs in physics (Springer)

[10] Polyakov A M 1987 *Gauge Fields and Strings* Contemporary concepts in physics (Taylor & Francis)

[11] Altland A and Simons B 2006 *Condensed Matter Field Theory* (Cambridge University Press)

[12] Chaikin P and Lubensky T 2000 *Principles of Condensed Matter Physics* (Cambridge University Press)