The Jacobson Radical of Commutative Semigroup Rings

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In this paper we consider semiprimitive commutative semigroup rings
and related matters. A ring is said to be semiprimitive if the Jacobson
radical of it is equal to zero. This property is one of the most important
in the theory of semigroup rings, and there is a prolific literature pertaining
to the field (see [14]).

All semiprimitive rings are contained in another interesting class of rings.
Let $\mathcal{S}$ denote the class of rings $R$ such that $J(R) = B(R)$, where $J$ and $B$ are
the Jacobson and Baer radicals. Clearly, every semiprimitive ring is in $\mathcal{S}$.
This class appears, for example, in the theory of PI-rings and in commutative algebra. (In particular, every finitely generated PI-ring and every
Hilbert ring are in $\mathcal{S}$.) Therefore, it is of an independent interest.
Meanwhile it is all the more interesting because any characterization of the
semigroup rings in $\mathcal{S}$ will immediately give us a description of semi-
primitive semigroup rings. Indeed, a ring $R$ is semiprimitive if and only if
$R \in \mathcal{S}$ and $R$ is semiprime, i.e., $B(R) = 0$. Semiprime commutative semi-
group rings have been described by Parker and Gilmer [12] and, in other
terms, by Munn [9]. So it suffices to characterize semigroup rings in $\mathcal{S}$.

Semigroup rings of $\mathcal{S}$ were considered by Karpilovsky [5], Munn [6–9],
Okninski [10], and others. In this paper commutative semigroup rings
which are in $\mathcal{S}$ will be described completely.

To this end one should know the structure of the Jacobson radical
$J(R[S])$. In [2] Jespers described $J(R[S])$ under rather weak assumptions
on $R$. They hold, in particular, for every commutative $R$. Here we shall give
another (quite short) description of $J(R[S])$ which does not require any
restriction on $R$. Besides, it is specially fitted for testing whether an element
is in $J(R[S])$, and this is essential for our proofs.
1. Notation and Preliminaries

For details we refer to [1, 4]. Throughout the paper only commutative semigroups will be considered.

Let $p$ be a prime number. A semigroup $S$ is said to be separative ($p$-separative) if for every $s, t \in S$ the equality $s^2 = st = t^2$ ($s^p = t^p$) implies $s = t$. The least separative ($p$-separative) congruence on $S$ is denoted by $\xi(\xi_p)$. Explicitly,

$$\xi = \{(s, t) \mid \exists n: st^n = t^{n+1} \text{ and } s^{n+1}t = s^{n+1}\},$$

$$\xi_p = \{(s, t) \mid \exists n: s^{np} = t^{np}\}.$$

For unification we set $\xi_0 = \xi$.

Let $R$ be a ring, $\rho$ be a congruence on $S$. Then $I(R, S, \rho)$ denotes the ideal $\{\sum r_i(s_i - t_i) \mid r_i \in R, (s_i, t_i) \in \rho\}$ of $R[S]$. Set $\mathcal{B}_n(R) = \{r \in R \mid nr \in \mathcal{B}(R)\}$, where $\mathcal{B}$ is the Baer or the Jacobson radical. Let $\mathbb{P}$ be the set of all prime numbers.

**Proposition 1.** (Munn [9]). Let $R[S]$ be a commutative semigroup ring. Then

$$B(R[S]) = B(R)[S] + I(R, S, \xi) + \sum_{p \in \mathbb{P}} I(B_p(R), S, \xi_p).$$

A semigroup $S$ is said to be Archimedean if for any two elements of $S$, each divides some power of the other.

**Proposition 2.** (Jespers, Krempa, and Wauters [3]). Let $R$ be a commutative ring, $S$ be an Archimedean semigroup. If $S$ is periodic, then

$$J(R[S]) = J(R)[S] + I(R, S, \xi) + \sum_{p \in \mathbb{P}} I(J_p(R), S, \xi_p).$$

Otherwise,

$$J(R[S]) = B(R)[S] + I(R, S, \xi) + \sum_{p \in \mathbb{P}} I(B_p(R), S, \xi_p).$$

Note that in the case of a non-commutative $R$ the results corresponding to Propositions 1 and 2 are proved in [11, 3].

2. A Description of the Jacobson Radical

A semigroup $\Gamma$ is called a semilattice if it entirely consists of idempotents. A semigroup $S$ is said to be a semilattice $\Gamma$ of its subsemigroups $S_\alpha (\alpha \in \Gamma)$ if $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, $S_\alpha \cap S_\beta = \emptyset$ when $\alpha \neq \beta$, and $S_\alpha S_\beta \subseteq S_{\alpha \beta}$ for any $\alpha, \beta$. By
Theorem 4.13 in [1] each semigroup can be uniquely represented as a
semilattice of its Archimedean subsemigroups $S$. The semigroups $S$ are
called the Archimedean components of $S$.

Let $R$ be an arbitrary (not necessary commutative) ring, $x \in R[S]$, 
$x = \sum_{t \in S} x_t t$. Set $x_\gamma = \sum_{t \in S_\gamma} x_t t$. The semilattice generated in $\Gamma$ by all $\alpha$
such that $x_\alpha \neq 0$ will be called the support of $x$ and denoted by $\text{supp} (x)$.
(This definition of a support differs from the standard one, cf. [2]. It is the
new concept, that will work in our proofs.) Consider the natural partial
order $\leq$ on $\Gamma$ defined by $\alpha \leq \beta \iff \alpha \beta = \alpha$. Let $\text{max} (x)$ denote the set of
elements in supp $(x)$ maximal with respect to this order. Clearly the sets
$\text{supp} (x)$ and $\text{max} (x)$ are finite. The following lemma was proved in [16]
for the case of a two-element semilattice $\Gamma$.

**Lemma 1.** Let $R$ be an arbitrary ring, $S$ be a commutative semigroup
with Archimedean components $S_\alpha, \alpha \in \Gamma$. The radical $J(R[S])$ is the largest
ideal among ideals $I$ of $R[S]$ such that $x_\mu \in J(R[S_\mu])$ for any $x \in I,$ $\mu \in \text{max} (x)$.

**Proof.** Let $M$ be the set of ideals $I$ of $R[S]$ such that $x_\mu \in J(R[S_\mu])$ for any $x \in I$, $\mu \in \text{max} (x)$. By the proof of Theorem 1 in [15], $J(R[S]) \in M$.

On the other hand, take any $I$ in $M$. We claim that $I$ is quasi-regular (and
so $I \subseteq J(R[S])$). Suppose the contrary and choose $x$ in $I$ which does not
have a right quasi-inverse and $| \text{supp} (x) |$ is minimal. Let $\mu \in \text{max} (x)$. Then
$x_\mu \in J(R[S_\mu])$, and $x_\mu + a + x_\mu a = 0$ for some $a \in J(R[S_\mu])$. Consider the
element $y = -x - xa$. Clearly $y \in I$ and $y_\mu = a$. Further, set $z = x + y + xy$.
Evidently $z \in I$ and supp $(z) \subseteq \text{supp} (x) \setminus \{ \mu \}$. By the choice of $x$ there exists
$u$ such that $z + u + zu = 0$. Then $x + (y + u + yu) + x(y + u + yu) = 0$. So $x$
is quasi-invertible, giving a contradiction. Thus $I \subseteq J(R[S])$. We have
proved that $J(R[S])$ is the largest ideal in $M$. (This also can be proved as
a corollary of Lemma 1.3 in [16].)

Now let us consider a separative semigroup $T$. By Theorem 4.16 in [1]
the Archimedean components $T_\alpha$ of $T$ are cancellative. Denote by $O_\alpha$ the
group of quotients of $T_\alpha$. Let $e_\alpha$ denote the identity element of $O_\alpha$. Set
$Q = \bigcup_{\alpha \in \Gamma} O_\alpha$. The multiplication of $T$ can be easily extended on the whole
$Q$ so that $e_\alpha e_\beta = e_\alpha \beta$. $Q_\mu \ni x \in R[Q_\mu]$, and $A$ be a finite (or empty)
subset of $\mu \Gamma$. Then $(\mu, x, A)$ denotes the product $x \prod_{i \in A} (e_\mu - e_i)$. If
$A = \emptyset$, then $(\mu, x, A) = x$. Following [13] we say that $(\mu, x, A)$ is a
simplest element, if $xe_\alpha \in J(R[Q_\alpha])$ for any $\alpha \mu \Gamma \setminus \mu \Gamma$. Note that
$(\mu, x, A)e_\alpha = 0$ for any $\alpha \in \mu \Gamma$. The set of the simplest elements of $R[Q]$ is
denoted by $Si(R[Q])$. Put $Si(R[T]) = R[T] \cap Si(R[Q])$.

Proposition 1 shows that $I(R, S, \xi) \subseteq J(R[S])$. Clearly $R[S]/I(R, S, \xi) \cong R[S/\xi]$. Therefore it suffices to describe the Jacobson radical for the semigroup $T = S/\xi$. In this case we state
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THEOREM 1. Let \( x \in R[T] \), \( \mu = \max (x) \), \( A \) be the set of maximal elements in the finite set \( \mu \supp (x) \backslash \{ \mu \} \), \( y = (\mu, x_\mu, \Lambda) \). Then

1. \( x \in J(R[T]) \iff x \in R[T] \cap J(R[\mathcal{Q}]) \);
2. \( x \in J(R[\mathcal{Q}]) \iff y, x - y \in J(R[\mathcal{Q}]) \);
3. \( y \in J(R[\mathcal{Q}]) \iff y \in Si(R[\mathcal{Q}]). \)

Assertions (1) and (2) reduce the inclusion \( x \in J(R[T]) \) to \( y, x - y \in J(R[\mathcal{Q}]) \). Since \( |\supp (x-y)| < |\supp (x)| \), applying (2) several times one can reduce \( x \in J(R[T]) \) to some inclusions of the form \( y \in J(R[\mathcal{Q}]) \), which can be checked with (3). Note that \( Si(R[\mathcal{Q}]) \) is defined in terms of the radicals of the components \( R[\mathcal{Q}_x] \).

Proof of Theorem 1. (1) Take \( x \in R[T] \cap J(R[\mathcal{Q}]), \mu = \max (x) \). Since \( x \in J(R[\mathcal{Q}]), \) Lemma 1 yields \( x_\mu \in J(R[\mathcal{Q}_\mu]) \). By Proposition 2 we get \( x_\mu \in J(R[T_\mu]) \), for \( \mathcal{Q}_\mu \) and \( T_\mu \) are Archimedean. Then Lemma 1 implies \( J(R[T]) \supseteq R[T] \cap J(R[\mathcal{Q}]). \)

Now take \( x \in J(R[T]) \). Denote by \( I \) the ideal generated by \( x \) in \( R[\mathcal{Q}] \). Choose \( z \in I \). Then \( z = \sum a_i x b_i \), where \( a_i, b_i \in R[\mathcal{Q}] \). Let \( \mu = \max (z) \), \( t \in T_\mu \). Evidently \( xt \in J(R[T]) \). By Lemma 1 and Proposition 2, \( (xt)_\mu \in J(R[T_\mu]) \subseteq J(R[\mathcal{Q}_\mu]) \). Therefore \( z_\mu = z_\mu e_\mu = \sum (a_i t) (b_i t)^{-1} \in J(R[\mathcal{Q}_\mu]). \) Then Lemma 1 implies \( I \subseteq J(R[\mathcal{Q}]) \), completing the proof of (1).

(2) Let \( x \in J(R[\mathcal{Q}]) \). Take any nonzero element \( z \) of the ideal generated in \( R[\mathcal{Q}] \) by \( y \). Say \( z = \sum a_i y b_i \), where \( a_i, b_i \in R[\mathcal{Q}] \), and set \( u = \sum a_i x b_i \). We may assume that each product \( a_i x b_i \) is a homogeneous element, i.e., \( a_i x \mu, b_i \in R[\mathcal{Q}_x] \), for some \( \alpha_i \in \Gamma \) (otherwise we would split \( a_i \) or \( b_i \) into several summands). Then

\[
z_x = \left( \sum_{x, \geq x} a_i y b_i \right) = \left[ \left( \sum a_i x b_i \right) \left( \prod_{\lambda \in \Lambda} (e_\mu - e_\lambda) \right) \right].
\]

Take any \( \alpha \in \max (z) \). Evidently \( \alpha \in \mu \Gamma \), since \( \supp (y) \subseteq \mu \Gamma \). If \( \alpha \in \Lambda \Gamma \), then the support of the sum \( s = \sum_{x, \geq x} a_i x b_i \) is contained in \( \Lambda \) because of the maximality of \( \alpha \). Hence \( s \prod_{\lambda \in \Lambda} (e_\mu - e_\lambda) = 0 \) yielding \( z_x = 0 \), a contradiction. Thus \( \alpha \) is not in \( \Lambda \Gamma \). Clearly \( z_\beta = \sum_{x, \geq x} a_i x b_i = u_\beta \) for any \( \beta \in \mu \Gamma \backslash \Lambda \Gamma \), and so \( \alpha \in \max (u) \). Besides \( z_x = u_x \in J(R[\mathcal{Q}_x]) \), since \( x \in J(R[\mathcal{Q}]) \). By Lemma 1, \( y \in J(R[\mathcal{Q}]) \), and so does \( x - y \). The converse is trivial.

(3) Let \( y \in Si(R[\mathcal{Q}]) \). Take any element \( z \) of the ideal generated by \( y \) in \( R[\mathcal{Q}] \), say \( z = \sum a_i y b_i \neq 0 \), where \( a_i, b_i \in R[\mathcal{Q}] \). Let \( \alpha \in \max (z) \). If \( \alpha \in \Lambda \Gamma \) then \( ye_x = 0 \), and so \( z_x = (ze_x)_x = 0 \). Therefore \( \alpha \in \mu \Gamma \backslash \Lambda \Gamma \). Evidently \( y \) may be written as \( y = x + y' \) where \( \supp (y') \subseteq \Lambda \Gamma \). Then \( \supp (\sum a_i y' b_i) \subseteq \Lambda \Gamma \) and so \( ye_x = xe_x \). Since \( y \) is simplest, \( ye_x = xe_x \in J(R[\mathcal{Q}_x]) \). So \( z_x = \sum_i (a_i e_x)(ye_x)(b_i e_x) \in J(R[\mathcal{Q}_x]) \), implying \( y \in J(R[\mathcal{Q}]) \).
by Lemma 1. Conversely, let \( y \in J(R[Q]) \), \( \alpha \in \mu f \setminus \mathcal{A} \). Then \( xe_\alpha = (ye_\alpha)_\alpha \in J(R[Q_\alpha]) \), since \( \alpha = \max (ye_\alpha) \).

**Corollary [13].** \( J(R[Q]) \) is the additive group generated by \( Si(R[Q]) \).

**Proof.** Take any \( z \in J(R[Q]) \) and set \( n = |\text{supp} (z)| \). If \( n = 1 \), then Lemma 1 shows that \( z \in Si(R[Q]) \). If \( n > 1 \), then Theorem 1 and induction on \( n \) give the result.

### 3. Main Result and Corollaries

We need a few definitions. Let \( G \) be a finite subgroup of a semigroup \( T \), \( I \) be an ideal generated in \( T \) by a finite (or empty) set of idempotents which does not contain \( G \). Put down all subgroups \( H_1, \ldots, H_n \) of \( G \) such that \( H_i = \{ h \in G \mid ht_i = et_i \} \) for a non-periodic element \( t_i \in GI \), where \( e \) is the identity of \( G \). Numerate the elements of \( G = \{ g_1, \ldots, g_m \} \). The matrix of the conjugacy relation of \( G \) by \( H_i \) is the \((m \times m)\)-matrix \( D_i = [d_{jk}] \) such that

\[
    d_{jk} = \begin{cases} 
        1 & \text{when } g_j \in H_i g_k, \\
        0 & \text{otherwise.}
    \end{cases}
\]

Set \( D_i(G) = [D_1 \mid D_2 \mid \cdots \mid D_n] \). If \( n = 0 \) (i.e., \( G \) has no subgroup with the property mentioned or, equivalently, there is not any non-periodic element in \( GT \setminus I \)), then set \( D_i(G) = [0] \).

For a ring \( R \) denote by \( \pi(R) \) the set of all \( q \) such that \( q \) is prime or zero and \( J(R)/B(R) \) has a nonzero element with an additive period \( q \). (Here an element with an additive period 0 is a non-periodic element.) We say that \( G \) is \( q \)-complete in \( T \), if \( q \) divides \( |G| \) or \( q \) does not divide the determinant of an \((m \times m)\)-submatrix of \( D_i(G) \) (for any \( I \)).

**Theorem 2.** Let \( R[S] \) be a commutative semigroup ring, \( \xi \) the least separative congruence on \( S \), and \( T = S/\xi \). The Jacobson radical \( J(R[S]) \) is nil if and only if for any \( q \in \pi(R) \) every finite subgroup \( G \) of \( T \) is \( q \)-complete in \( T \).

Theorem 2 and Proposition 1 give us a description of semiprimitive commutative semigroup rings.

**Corollary 1.** A commutative semigroup ring \( R[S] \) is semiprimitive if and only if \( R \) is semiprime, \( S \) is separative, and \( p \)-separative for every prime \( p \in \pi(R) \), each finite subgroup \( G \) in \( S \) is \( q \)-complete in \( S \) for any \( q \in \pi(R) \).
Note that when $R$ is a field a description of semiprimitive $R[S]$ was given in [6].

Now we show that all the previous results on commutative semigroup rings of the class $\mathcal{S}$ are in fact partial cases of Theorem 2. The previous results are listed in Corollaries 2–5.

**Corollary 2 [9].** If $J(R)$ is nil, then $J(R[S])$ is nil.

This follows from Theorem 2 because $J(R) = B(R)$ if and only if $\pi(R) = \emptyset$; that is, there are no $q$ in $\pi(R)$.

**Corollary 3 [7].** If $S$ has no idempotent elements, then $J(R[S])$ is nil.

This is clear because if $S$ has no idempotents, then $T$ does not have any subgroup.

**Corollary 4 [9].** Let $S$ be a periodic semigroup. Then $J(R[S])$ is nil if and only if $J(R)$ is nil.

Indeed, a periodic $S$ does not have a non-periodic element. Therefore all $D_q[G]$ are equal to $[0]$, and so every finite subgroup is not $q$-complete in $S$ for each $q$. So $J(R[S])$ is nil if and only if $\pi(R) = 0$, which is equivalent to $J(R)$ is nil.

**Corollary 5 [7].** Let $S$ be a semilattice of cancellative and non-periodic $S_\alpha$, $\alpha \in \Gamma$. Then $J(R[S])$ is nil.

Indeed, let us take a finite subgroup $G$ in $S$. There is $\alpha$ such that $G \subseteq S_\alpha$. Fix a non-periodic element $t$ in $S_\alpha$. Then $H = \{h \in G \mid ht = et\} = \{e\}$, for $S$ is cancellative. Hence the matrix of the conjugacy relation of $G$ by $H$ is the identity matrix. Its determinant is equal to 1, and $q$ does not divide 1. Therefore $G$ is $q$-complete in $S$ for every $q$, not only for $q \in \pi(R)$.

4. Proof of the Main Theorem

**Lemma 2.** Let $F = R/B(R)$, $T = S/\xi$. The radical $J(R[S])$ is nil if and only if $J(F[T])$ is nil.

**Proof.** This easily follows from Proposition 1 and the isomorphisms $R[S]/I(R, S, \xi) \cong R[T]$, $R[T]/B(R)[T] \cong F[T]$.

Recall that $T = \bigcup_{\alpha \in \Gamma} T_\alpha$, $Q_\alpha$ denotes the group of quotients of $T_\alpha$, $e_\alpha$ is the identity of $Q_\alpha$, and $Q = \bigcup_{\alpha \in \Gamma} Q_\alpha$. Say that a subgroup $G$ of $T$ is $q$-incomplete in $T$ if $G$ is not $q$-complete in $T$. Note that $\pi(R) = \pi(F)$. In view of Lemma 2, Theorem 2 is equivalent to the following
Lemma 3. \( J(F[T]) \) has a non-nilpotent element if and only if \( T \) has a \( q \)-incomplete finite subgroup for some \( q \in \pi(F) \).

Proof. First we prove the "only if" part. Choose in \( J(F[T]) \) a non-nilpotent element \( x \) with minimal \( \| \text{supp}(x) \| \). Let \( \mu = \max(x) \). Then \( \text{supp}(x^n) = \text{supp}(x) \) for each \( n \), and so \( x_\mu \) is not nilpotent. Further, the element \( y = x_\mu x \) is not nilpotent, for \( y_{\mu} = x_\mu^2 \). Hence \( \text{supp}(y) = \text{supp}(x) \), that is \( \mu \text{supp}(x) = \text{supp}(x) \). Therefore \( \max(x) = \{ \mu \} \). Let \( A \) be the set of maximal elements of \( \text{supp}(x) \) \( \setminus \{ \mu \} \), \( y = (\mu, x_\mu, A) \). By Theorem 1, \( y \in S_\mu(F(Q)) \). We are to prove that \( y \in F[T] \).

To this end we first prove that \( e_\lambda \in T \) for every \( \lambda \in A \). Suppose the contrary. Then \( T_\lambda \) does not have any idempotent, and so all elements in \( T_\lambda \) are non-periodic. Denote by \( P \) (and \( N \)) the set of periodic (non-periodic) elements of \( Q_\lambda \). Then \( T_\lambda \subseteq N \). The definition of a simplest element implies \( y_\lambda \in J(F(Q_\mu)) \). Hence \( J(F(Q_\mu)) \) is not nil. This and Propositions 1, 2 show that \( Q_\mu \) is a periodic group. Therefore \( y = e_\lambda y_\mu \in F[P] \). On the other hand, \( x_\lambda \in F[T_\lambda] \subseteq F[N] \), implying \( x_\lambda \neq y_\lambda \). Consider \( z = x - y \). Clearly \( \lambda = \max(z) \). Since \( x, y \in J(F(Q)) \), Lemma 1 shows that \( z_\lambda \in J(F(Q_\lambda)) \). By Proposition 2, \( J(F(Q_\lambda)) = \sum_{\rho \in P} l(B_\rho(F), Q_\lambda, \xi_\rho) \), since \( Q_\lambda \) is not periodic. Evidently, \( \xi_\rho \) does not join a periodic element with a non-periodic one. Therefore \( y_\lambda \in F[P] \), \( x_\lambda \in F[N] \), and \( x_\lambda - y_\lambda \in J(F(Q_\lambda)) \) yield \( x_\lambda, y_\lambda \in J(F(Q_\lambda)) \). By Propositions 1, 2 \( J(F(Q_\lambda)) \) is nil, and so \( x_\lambda \) is nilpotent. Hence \( w = x - x_\lambda \) is in \( J(F[T]) \). Meanwhile \( w \) is not nilpotent, for \( w_\mu = x_\mu \). However, \( \| \text{supp}(w) \| < \| \text{supp}(x) \| \) contradicting the choice of \( x \). We have shown that \( e_\lambda \in T_\lambda \) for any \( \lambda \in A \).

Now take any \( y \in \text{supp}(y) \setminus \{ \mu \} \). There are \( \lambda_1, ..., \lambda_n \) such that \( y = \lambda_1 \cdots \lambda_n \). Further, \( y_\gamma = k x_\mu e_{\lambda_1} \cdots e_{\lambda_n} \) for an integer \( k \). Since \( x_\mu \in F[T] \) and all \( e_{\lambda_\lambda} \in F[T] \) we get \( y_\gamma \in F[T] \). Therefore \( y \in F[T] \).

Propositions 1 and 2 show that \( J(F(T_\mu)) \) is nil modulo \( J(F)(T_\mu) \). Hence \( y_\mu \in J(F(T_\mu)) \). Since \( y_\mu^{-1} y = (\mu, x_\mu, A) \) we may for simplicity of notation assume that \( y_\mu \in J(F)(T_\mu) \). Further, \( y_\mu = (\mu, x_\mu, A) \) because \( (\prod_{\lambda \in A} (e_{\mu} - e_\lambda)) \) is an idempotent. Denote by \( p(y_\mu) \) the additive period of \( y_\mu \). Obviously \( p(y_\mu) \) divides \( p(y_\mu^{n+1}) \). If there is a periodic element among \( y, y^2, y^3, ... \) then we choose \( m \) such that \( p(y_\mu) \) is the smallest possible period. For simplicity of notation assume that \( m = 1 \). Then \( p(y) = p(y^2) = ... \). If all \( y, y^2, ... \) are non-periodic then \( 0 = p(y) = p(y^2) = ... \). Thus we may assume that from the very beginning all the elements \( y_\mu, y^2_\mu, ... \) are of same additive period. Denote it by \( d \). Let \( F_d = \{ f \in F \mid df = 0 \} \). Since \( F_d \) is an ideal of \( F \), we get \( y \in J(F_d[T]) \). To simplify the notation, assume that \( F = F_d \). If \( d = 0 \), then we denote by \( I \) the set of periodic elements of \( F \) and put \( q = 0 \). If \( d \neq 0 \), then \( d \) can be written as \( d = qr \) for a prime number \( q \), and we set \( I = F_r \). Let \( K = F/I \) and \( y \) denote also the image of \( y \) in \( K[T] \). Then in both the cases \( q \in \pi(K) \), for
Evidently \( y \) is a non-nilpotent simplest element of \( K[T] \), and \( K \) is a ring of characteristic \( q \).

Clearly \( y' \) is of the form \( y' = \sum_{i=1}^{k} a_i s_i \), where \( 0 \neq a_i \in K \), \( s_i \in T \). Denote by \( G \) or \( G(y) \) the subsemigroup generated in \( T \) by \( s_1, ..., s_k \). Since \( T \) is periodic, \( G \) is a finite group. We may assume that from the very beginning \( y \) is chosen so that the cardinality of \( G \) is minimal. Now we shall prove that \( G \) is \( q \)-incomplete in \( T \).

First we show that \( q \) does not divide \( |G| \). Suppose the contrary and represent \( G \) as a direct product \( H \times E \), where \( H \) is the largest \( q \)-subgroup of \( G \). Then \( |E| < |G| \). Write \( s_i \) as \( s_i = (h_i, b_i) \), where \( h_i \in H, b_i \in E \). Set \( z = \sum_{i=1}^{k} a_i (h_i, b_i) - a_i (e_i, b_i) \). The elements \((h_i, b_i)\) and \((e_i, b_i)\) are in the relation \( \xi_q \) with each other, since \( H \) is a \( q \)-group. By Proposition 1, \( z \in B(K[T]) \). Put \( c = y - z \), \( d = (\mu, c, A) \). Evidently \( d - y = (\mu, z, A) \in B(K[T]) \), and so \( d \in \text{Si}(K[T]) \) by Theorem 1. Further, \( d \) is not nilpotent and \( G(d) \subseteq E \subseteq G(y) \), a contradiction with the minimality of \( G(y) \). Thus \( q \) does not divide \( |G| \).

Let \( I \) be the ideal generated in \( T \) by all \( e_\lambda, \lambda \in \Lambda \). Put down all subgroups \( H_1, ..., H_n \) of \( G \) such that \( H_i = \{ h \in G \mid h t_i = e \} \) for a non-periodic element \( t_i \) of \( G \). Denote by \( D_i \) the matrix of the relation of \( G \) by \( H_i \) and set \( D_j(G) = [D_1 | \cdots | D_n] \). We are to prove that \( q \) divides every \((m \times m)\)-minor of \( D_j(G) \).

Since \( \text{char} K = q \), it suffices to prove the equality \( (a_1, ..., a_m) D_j(G) = 0 \), where \( y = \sum_{i=1}^{m} a_i g \), \( G = \{ g_1, ..., g_m \} \). This is equivalent to equalities \( (a_1, ..., a_m) D_i = 0 \), \( i = 1, ..., n \). Let \( (a_1, ..., a_m) \) \( D_i = (b_1, ..., b_m) \). We claim that \( b_j = 0 \).

The definition of \( D_i \) shows that \( b_j = \sum_{k \in H_j} a_k \). Take \( \alpha \in \Gamma \) such that \( t_i \in T_\alpha \). Since \( t_i \in GT \backslash I \), we get \( \alpha \in \mu \Gamma \backslash \Lambda \Gamma \), implying \( y e \in J(K[T_\alpha]) \). In view of the fact that \( T_\alpha \) is not periodic, Proposition 2 yields \( y e \in I(K, T_\alpha, \xi) \). Further, \( y e \in K[G e] \) and \( q \) does not divide the order of the group \( G e \). Therefore \( I(K, G e, \xi) = 0 \), implying \( y e = 0 \). Hence \( y t_i = 0 \), and so \( \sum_{k=1}^{m} a_k t_i = 0 \). Therefore \( \sum_{k=1}^{m} a_k t_i = 0 \). The equality \( g_j t_i = g_j t_i \) is equivalent to \( g_j^{-1} g_k H_i \) by the definition of \( H_i \). Hence \( b_j = \sum_{k \in H_i} a_k \) \( g_j t_i = \sum_{k \in H_i} a_k \) \( g_j t_i = 0 \), yielding \( (a_1, ..., a_m) D_i(G) = 0 \). Thus \( G \) is \( q \)-incomplete in \( T \) as required.

Now we will prove the "if" part. Let \( q \in \pi(F) \) and \( T \) contains a \( q \)-incomplete subgroup \( G \). It is well known that a cancellative Archimedean semigroup is a group if it contains an idempotent. Therefore \( T_\mu \) is a group.

Suppose that \( T_\mu \) has a non-periodic element \( t \) and consider the group \( H = \{ h \in G \mid h t = e \} \). Clearly \( H = \{ e \} \). Then the matrix \( D \) of the relation of \( G \) by \( H \) is the identity matrix. Therefore \( q \) does not divide \( \det(D) = 1 \), and \( D \) lies in the matrix \( D_0(G) \). The contradiction with \( q \)-incompleteness of \( G \) shows that \( T_\mu \) is a periodic group.

Let \( G = \{ g_1, ..., g_m \} \). Since \( G \) is \( q \)-incomplete, \( q \) does not divide \( m \) and
there is an ideal \( I \) of \( T \) generated by idempotents \( e_1, ..., e_k \) and such that \( q \) divides the determinant of every \((m \times m)\)-matrix of \( D_1(G) \). Then \( e_i \in T_{\lambda_i} \) for some \( \lambda_i \in \Gamma \). We may assume that \( \lambda_i \leq \mu \), because otherwise one could substitute \( ee_i \) for \( e_i \) and \( \lambda_i \mu \) for \( \lambda_i \) without changing the set of non-periodic elements in \( GT \setminus I \). Write down all the groups \( H_1, ..., H_n \) such that \( H_i = \{ h \in G \mid ht = e_{\mu}t_i \} \) for non-periodic \( t_i \in GT \setminus I \). Denote by \( D_i \) the matrix of the conjugacy relation of \( G \) by \( H_i \) and set \( D_2(G) = [D_1 | \cdots | D_n] \). Then \( q \) divides the determinant of each \((m \times m)\)-submatrix of \( D_2(G) \). Therefore the \( q \)-element field \( GF(q) \) (or the field of rational numbers, if \( q = 0 \)) contains elements \( u_1, ..., u_m \) such that \( (\mu, u_1, ..., u_m) D_2(G) = 0 \), \( (u_1, ..., u_m) \neq 0 \). Since \( \pi(R) = \pi(F) \), by the choice of \( q \) and \( F \) there exists a nonzero \( r \in F \) such that \( qr = 0 \). Set \( x = u_1 r g_1 t + \cdots + u_m r g_m t \). Since \( q \) does not divide \( G \) and \( r \notin B(F) = 0 \), Proposition 2 shows that \( x \) is not nilpotent. Put \( A = \{ \lambda_1, ..., \lambda_k \} \), \( y = (\mu, x, A) \). We claim that \( y \in Si(F[T]) \), i.e., \( x e \in J(F[Q]) \) for any \( \lambda \in \mu \Gamma \setminus \Lambda \).

Indeed, if \( T_{\lambda} \) is periodic then the claim follows from Proposition 2 and \( r \in J(F) \). Now consider the case where \( T_{\lambda} \) has a non-periodic element \( t \). Then \( t \notin I \) implying \( \{ h \in G \mid ht = e_{\mu}t \} = H_i \) for some \( i \). Write \( xt = u_1 r g_1 t + \cdots + u_m r g_m t \). Here \( g_j t \) coincides with \( g_k t \) if and only if \( g_j \) and \( g_k \) lie in the same class of the conjugacy relation of \( G \) by \( H_i \). This and \( (u_1, ..., u_m) D_2 = 0 \) yield \( xt = 0 \). Therefore \( x e = x t^{-1} = 0 \), and so \( y \in Si(F[T]) \). By Theorem 1, \( J(F[T]) \) contains \( y \), which was proved to be non-nilpotent. This proves the result.

References


