The lower domination parameters in inflation of graphs of radius 1

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Abstract

The inflation $G_I$ of a graph $G$ is the line graph of the subdivision of $G$. If $G$ is a complete graph the equality $\gamma(G_I) = \gamma(G)$ was proved by Favaron in 1998. We conjectured that the equality holds when $G$ is any graph of radius 1. But it turned out that it is not true. Moreover, we proved that for the class of radius 1 graphs there does not exist a better upper bound for the relation $\gamma(G_I)/\gamma(G_I)$ then $\frac{\gamma}{\gamma}$. We found also a sufficient condition for the equality $\gamma(G_I) = \gamma(G_I)$.

Keywords: Lower domination parameters; Inflations; Claw-free graphs

We consider only simple graphs. Let $G = (V(G), E(G))$ be a graph of order $n(G)$ and diameter $\text{diam}(G)$. The neighbourhood and closed neighbourhood of a vertex $x$ of $G$ are, respectively, denoted by $N(x)$ and $N[x]$ (with $N[x] = N(x) \cup \{x\}$). If $X \subseteq V(G)$, $G[X]$ is the subgraph induced in $G$ by $X$. Let $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. A set $X$ of vertices of $G$ is dominating if $N[X] = V(G)$. The minimum cardinality of a dominating set is denoted by $\gamma(G)$.

If $x$ is a vertex of a subset $X$ of $V(G)$, the set $PN(x) = N[x] \setminus N[X \setminus \{x\}]$ is called the $X$-private neighbourhood of $x$ and its elements are the $X$-private neighbours of $x$. The vertex $x$ of $X$ is irredundant in $X$ if its $X$-private neighbourhood is not empty, redundant otherwise. The set $X$ is irredundant in $G$ if all its vertices are irredundant. If an irredundant set $X$ is maximal for the inclusion, then for any vertex $u$ which is not dominated by $X$ there exists a non-isolated vertex $y$ of $X$ which is redundant in $X \cup \{u\}$. That means that $u$ dominates every $X$-private neighbour of $y$. We say that $u$ annihilates $y$. The minimum cardinality of a maximal irredundant set of $G$ is denoted

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by \(\text{ir}(G)\). Since any minimal dominating set is a maximal irredundant set, we have for any graph \(G\)

\[
\text{ir}(G) \leq \gamma(G).
\]

The inflation \(G_1\) of a graph \(G\) is the line graph of the subdivision of \(G\). The subdivision of \(G\) is the graph obtained from \(G\) by replacing each edge by a path of length 2. We denote by \(X\) the clique in \(G_1\) corresponding to vertex \(x\) in \(G\). If \(xy\) is an edge in \(G\) then we denote \((xy)\) and \((yx)\) the vertices in \(X\) and \(Y\) corresponding to this edge. An edge \((xy)(yx)\) will be a blue edge while \((xy)\) will be a blue neighbour of \((yx)\), the other vertices adjacent to \((xy)\) will be the red neighbours (that are vertices in \(X\)).

Dunbar and Haynes [1] proved that for every graph \(H\), \(\text{diam}(H) \leq 2\) if and only if \(\gamma(H_1) = n(H) - 1\).

If \(G\) is a complete graph the equality \(\text{ir}(G_1) = \gamma(G_1)\) was proved in [2] by Favaron. She conjectured that the same equality holds in the case when \(G\) is a tree, which was proved by Puech [4].

It is interesting to find other classes of graphs for which this equality holds.

Let \(G\) be a graph of radius 1 that is \(\gamma(G) = 1\). Let \(c\) be a vertex which is adjacent to all other vertices of \(G\). Then in \(G_1\) the vertex \(c\) is replaced by a clique \(C\) that contains \(n - 1\) vertices.

We conjectured that the equality holds when \(G\) is any graph of radius 1. But it turned out that it is not true. Moreover, we proved that for the class of radius 1 graphs there does not exist a better upper bound for the relation \(\gamma(G_1)/\text{ir}(G_1)\) than \(\frac{3}{2}\). It was shown in [3] that the bound \(\frac{3}{2}\) is the best possible in the class of claw-free graphs and

\[
\frac{\gamma(G_1)}{\text{ir}(G_1)} < \frac{3}{2}
\]

in the class of inflations.

**Theorem 1.** There exists a series \(\{G^k\}\) of graphs of radius 1, such that

\[
\lim_{k \to \infty} \frac{\gamma(G^k)}{\text{ir}(G^k)} = \frac{3}{2}.
\]

**Proof.** For \(k \geq 2\) graph, \(G^k\) consists of \(k\) paths \((a_i, b_i, c_i)\) for \(1 \leq i \leq k\), vertices of different paths are not adjacent, and a vertex \(c\) adjacent to all vertices of the paths, mentioned. It is clear from the definition that every \(G^k\) is a graph of radius 1. It is obvious that \(\text{diam}(H) \leq 2\) for every graph \(H\) of radius 1, hence \(\gamma(G^k_1) = n(G^k) - 1 = 3k\) by Dunbar and Haynes [1]. Consider the following set \(W = \{(cb_1), (b_1a_1), (b_1c_1), \ldots, (b_k a_k), (b_k c_k)\}\) in \(G^k_1\), for \(k \geq 2\). \(W\) is irredundant, for \((cb_1)\) is isolated in \(W\), and the \(W\)-private neighbours of the other vertices are their blue neighbours. For an arbitrary vertex \(v \in V \setminus W\) their exists of the three possibilities:

1. \(v \in C \setminus \{(cb_1)\}\), then \(v\) dominates \((cb_1)\) and its private neighbourhood;
2. \(v \in B_i\), for some \(i, 1 \leq i \leq k\), in this case \(N[x] \subset N[W]\);
(3) \( v \in A_i \) (resp. \( C_i \)), for \( 1 \leq i \leq k \), then either \( v \) is a unique \( W \)-private neighbour of \((b, a_i)\) (resp. \((b, c_i)\)) or \( v \) dominates this private neighbour.

In all cases \( W \cup \{x\} \) is redundant, therefore \( W \) is a maximal irredundant set. So
\[
\ir(G_i^k) \leq |W| = 2k + 1.
\]

Then using (2) we get
\[
\frac{3k}{1 + 2k} \leq \frac{\gamma(G_i^k)}{\ir(G_i^k)} < \frac{3}{2}
\]
and
\[
\lim_{k \to \infty} \frac{\gamma(G_i^k)}{\ir(G_i^k)} = \frac{3}{2}.
\]

\begin{proof}
We use the result of Dunbar and Haynes [1] as in Theorem 1 to get \( \gamma(G_1) = n(G) - 1 \). An obvious consequence of the last equality and inequality (1) is \( \ir(G_1) \leq n(G) - 1 \). Also note that \( |C| = n(G) - 1 \).

Assume first \( |W \cap C| = 0 \). We want to prove that \( |W| \geq n(G) - 1 \). If \( v \in C \) then it is either a private neighbour of some vertex in \( W \), or a vertex in \( V \setminus N[W] \). Let \( C_1 \) be a set of vertices with the first property, \( C_2 = C \setminus C_1 \).

Next, we prove that there exists an injective mapping \( \phi \) from \( C \) into \( W \). For \( v \in C_1 \) let \( \phi(v) \) be the blue neighbour of \( v \). Consider \( v \in C_2 \). Let \( B \) be the clique containing the blue neighbour of \( v \). The vertex \((bc)\) must annihilate some vertex in \( W \), otherwise \( W \cup \{(bc)\} \) is redundant, because in this case \( v \) would be a \((W \cup \{(bc)\})\)-private neighbour for \((bc)\). Let \( \phi(v) \) be one of these vertices annihilated by \((bc)\). \( PN(\phi(v)) \subseteq B \), otherwise \( \phi(v) \) is not annihilated by \((bc)\).

Let \( v_1, v_2 \in C_1, v_1 \neq v_2 \) then \( \phi(v_1) \neq \phi(v_2) \), otherwise \( w = \phi(v_1) = \phi(v_2) \) would have two blue neighbours, that contradicts the simpleness of \( G \). Now let \( v_1 \in C_1, v_2 \in C_2 \). The vertex \( v_1 \) is a \( W \)-private neighbour of \( \phi(v_1) \), hence \( W \)-private neighbourhood of \( \phi(v_1) \) intersects \( C \), and \( W \)-private neighbourhood of \( \phi(v_2) \) as noted above does not intersect \( C \), thus \( \phi(v_1) \neq \phi(v_2) \). This time let \( v_1, v_2 \in C_2, v_1 \neq v_2 \). \( B_1 \) (resp. \( B_2 \)) is the clique containing the blue neighbour of \( v_1 \) (resp. \( v_2 \)). If \( \phi(v_1) \) is isolated in \( W \) and \( \phi(v_2) \) is not isolated in \( W \), then \( \phi(v_1) \neq \phi(v_2) \). If \( \phi(v_1) \) and \( \phi(v_2) \) are both isolated or both not isolated in \( W \) their \( W \)-private neighbourhoods are contained in \( B_1 \) and \( B_2 \), respectively, hence are different. Again we get \( \phi(v_1) \neq \phi(v_2) \). We proved that for arbitrary \( v_1, v_2 \in C, v_1 \neq v_2 \) implies \( \phi(v_1) \neq \phi(v_2) \). That is \( \phi \) is an injective mapping. Hence \( |\ir(G_1)| = |W| \geq |\phi(C)| = n(G) - 1 \).

Assume now \( |W \cap C| \geq 2 \). Let \( v \in C \setminus W \). As \( |W \cap C| \geq 2 \) this vertex is not contained in the \( W \)-private neighbourhoods of vertices in \( C \setminus W \). Let \( B \) be the clique containing the blue neighbour of \( v \). Then \( B \) must also contain a vertex from \( W \), otherwise \( W \cup \{v\} \) is irredundant.
Again we can define $\varphi$ as a mapping from $C$ into $W$:

(i) for $v \in C \cap W$ we define $\varphi(v) = v$;
(ii) for $v \in C \setminus W$ we define $\varphi(v)$ as one vertex in $B \cap W$.

So if $v_1, v_2 \in C \setminus W$, with $v_1 \neq v_2$, then $\varphi(v_1)$ and $\varphi(v_2)$ lie in different red cliques. Hence $\varphi(v_1) \neq \varphi(v_2)$. As well as above $\varphi$ is an injection and we have $\text{ir}(G_I) = |W| \geq |\varphi(C)| = n(G) - 1$. The theorem is proved. \qed

References