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Trigonometric polynomials of least deviation from zero in measure and related problems[☆]

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Abstract

We give a solution of the problem on trigonometric polynomials f_n with the given leading harmonic $y \cos nt$ that deviate the least from zero in measure, more precisely, with respect to the functional $\mu(f_n) = \mathrm{mes}\{t \in [0, 2\pi] : |f_n(t)| \geq 1\}$. For trigonometric polynomials with a fixed leading harmonic, we consider the least uniform deviation from zero on a compact set and find the minimal value of the deviation over compact subsets of the torus that have a given measure. We give a solution of a similar problem on the unit circle for algebraic polynomials with zeros on the circle.
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1. Statement of the problem and preliminaries

1.1. Introduction

Let $\mathcal{F}_n$ be the set of trigonometric polynomials

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \tag{1.1}$$

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of order $n \geq 0$ with real coefficients; in this paper, depending on the situation, we consider these functions on the whole real line $\mathbb{R}$, on the period, i.e., a segment of length 2π, or on the torus $\mathbb{T}$ which can be interpreted as a segment of length 2π (for example, the segment $[0, 2\pi]$) with identified end-points. On the set $\mathcal{F}_n$, we consider the functional

$$\mu(f_n) = \text{mes}\{t \in \mathbb{T} : |f_n(t)| \geq 1\} \tag{1.2}$$

whose value is the Lebesgue measure of the set of points of the torus at which an absolute value of the polynomial $f_n \in \mathcal{F}_n$ is greater than or equal to 1. For a fixed $y \geq 0$, we introduce the value

$$\sigma_n(y) = \inf\{\mu(y \cos nt - f_{n-1}(t)) : f_{n-1} \in \mathcal{F}_{n-1}\} \tag{1.3}$$

which can be interpreted as the value of the best approximation of the function $y \cos nt$ by the set $\mathcal{F}_{n-1}$ of trigonometric polynomials of order $n-1$ with respect to functional (1.2). Value (1.3) can be written in another form. Let $\mathcal{F}_n(y)$ be the set of trigonometric polynomials of order n of the form

$$f_n(t) = y \cos nt + f_{n-1}(t), \quad f_{n-1} \in \mathcal{F}_{n-1}.$$

Then,

$$\sigma_n(y) = \inf\{\mu(f_n) : f_n \in \mathcal{F}_n(y)\}; \tag{1.4}$$

this is a variant of the problem on polynomials that deviate the least from zero. It is easily seen that problem (1.3)–(1.4) is nontrivial only for $y > 1$. The following assertion is valid for problem (1.4); in this assertion and throughout the paper, we denote by T_n the Chebyshev polynomial of the first kind which is specified by the formula $T_n(x) = \cos(n \arccos x)$ for $x \in [-1, 1]$.

Theorem 1. *For $y > 1$ and $n \geq 1$, the following equality is valid:*

$$\sigma_n(y) = 4 \arccos \frac{1}{y^{\frac{1}{2n}}}. \tag{1.5}$$

Moreover, for $k = 0, 1, \ldots, 2n - 1$, the polynomial

$$f_n(t) = f_{n,k}(t) = (-1)^k T_n \left(y^{\frac{1}{n}} \cos \left(t - \frac{\pi k}{n} \right) - y^{\frac{1}{n}} + 1 \right) \tag{1.6}$$

belongs to the set $\mathcal{F}_n(y)$ and it is an extremal polynomial in problem (1.4) (i.e., the polynomial that deviates the least from zero) and only such polynomials solve problem (1.4).

Mendelev announced this result in 2000 in abstracts of his talk [11]. The proof of Theorem 1 is published in the present paper for the first time. In 1998, Mendelev and Plotnikov [12] proved assertion (1.5) for large values of y or, more precisely, for

$$y \geq \frac{1}{\sin^{2n} \frac{\pi}{2(2n+1)}}.$$

In 1992, Babenko [2] studied the least constant β_n in the inequality

$$\mu(a_n \cos nt + b_n \sin nt) \leq \beta_n \mu(f_n), \quad f_n \in \mathcal{F}_n, \tag{1.7}$$

on the set $\mathcal{F}_n$ of trigonometric polynomials (1.1); he obtained the following estimates for β_n:

$$\sqrt{2n} \leq \beta_n \leq n\sqrt{2}, \quad n \geq 1. \tag{1.8}$$

Below (see Theorem 9), as a consequence of Theorem 1, we find the value β_n; namely, we show that $\beta_n = \sqrt{2n}$. Thus, it turned out that the lower bound in (1.8) holds true.

In this paper, in connection with the investigation of problem (1.4), we discuss several other related extremal problems for trigonometric polynomials on the torus $\mathbb{T}$ and for algebraic polynomials on the unit circle Γ of the complex plane. In particular, we produce the following results.

(1) For algebraic polynomials with zeros on the unit circle of the complex plane and with the unit leading coefficient, we consider the least uniform deviation from zero on a compact set and find the minimal value of the deviation over compact subsets of the circle that have a given measure.
(2) For trigonometric polynomials with a fixed leading harmonic, we consider the least uniform deviation from zero on a compact set and find the minimal value of the deviation over compact subsets of the torus that have a given measure.

The main parts of the results of this paper were stated without proofs in [1].

1.2. A restriction of the class of polynomials

We need certain known facts about trigonometric polynomials (1.1); for further actions, it is sufficient to consider only polynomials whose order is equal to n, i.e., such that $a_n^2 + b_n^2 > 0$. For a trigonometric polynomial f_n of order $n \geq 1$ with real coefficients, the following formula is valid:

$$f_n(t) = e^{-int} P_{2n}(e^{it}), \quad t \in \mathbb{R}, \tag{1.9}$$

where

$$P_{2n}(z) = \sum_{\nu=0}^{2n} u_\nu z^\nu \tag{1.10}$$

is an algebraic polynomial of degree $2n$ whose coefficients have the properties

$$u_{2n-\nu} = \bar{u}_\nu, \quad 0 \leq \nu \leq 2n. \tag{1.11}$$

Conversely, if the coefficients of polynomial (1.10) satisfy conditions (1.11), then formula (1.9) specifies a trigonometric polynomial of order n with real coefficients. In this case, in particular,

$$u_{2n} = \bar{u}_0 = \frac{a_n - ib_n}{2}. \tag{1.12}$$

Condition (1.11) means that the following formula is valid for polynomial (1.10):

$$z^{2n} \overline{P_{2n}\left(\bar{z}^{-1}\right)} = P_{2n}(z), \quad z \in \mathbb{C}, \ z \neq 0. \tag{1.13}$$

Hence, polynomial (1.10) can be written in the form

$$P_{2n}(z) = \frac{a_n - ib_n}{2} \left[\prod_{k=1}^{l} (z - z_k)\left(z - \bar{z}_k^{-1}\right) \right] \prod_{j=2l+1}^{2n} \left(z - e^{i\phi_j}\right). \tag{1.14}$$

In this representation, the first product corresponds to $2l$ complex zeros of the polynomial f_n and $0 < |z_k| < 1$, $1 \leq k \leq l$; if f_n has no complex zeros ($l = 0$), then this representation is absent.

The second product in (1.14) corresponds to $2(n - l)$ real zeros of the polynomial f_n; if all zeros of the polynomial f_n are complex, then the second product in (1.14) is absent.

However, there exist polynomials of the form (1.14) without property (1.13). Substituting expression (1.14) into (1.13), we ascertain that (1.13) holds only in the case where

$$2\theta_n + 2\sum_{k=1}^{l} \varphi_k + \sum_{j=2l+1}^{2n} \phi_j = 2\pi N, \quad N \in \mathbb{Z}, \tag{1.15}$$

where θ_n is an argument of the coefficient $a_n - ib_n$, and φ_k are arguments of the zeros z_k, $1 \le k \le l$, of the polynomial P_{2n}. Thus, relation (1.15) is a necessary and sufficient condition for polynomial (1.14) to have property (1.11), and so to generate, via formula (1.9), a trigonometric polynomial of order n with real coefficients; in addition, the leading harmonic of the polynomial has the form $a_n \cos nt + b_n \sin nt$. More detailed information on the facts presented here can be found in, for example, [13, Sect. VI.2].

Let us discuss the representation of polynomials $f_n \in \mathcal{F}_n(y)$ (for $y > 0$ and $n \ge 1$) in more detail. In this case, $a_n = y > 0$ and $b_n = 0$; therefore, $\theta_n = 0$. Polynomial (1.14) and condition (1.15) take the form

$$P_{2n}(z) = \frac{y}{2} \left[\prod_{k=1}^{l} (z - z_k) \left(z - \overline{z_k}^{-1} \right) \right] \prod_{j=2l+1}^{2n} \left(z - e^{i\phi_j} \right), \tag{1.16}$$

$$2\sum_{k=1}^{l} \varphi_k + \sum_{j=2l+1}^{2n} \phi_j = 2\pi N, \quad N \in \mathbb{Z}. \tag{1.17}$$

By formulas (1.9) and (1.16), the following equality is valid:

$$f_n(t) = e^{-int} \frac{y}{2} \prod_{k=1}^{l} \left(\left(e^{it} - r_k e^{i\varphi_k} \right) \left(e^{it} - \frac{e^{i\varphi_k}}{r_k} \right) \right) \prod_{j=2l+1}^{2n} (e^{it} - e^{i\phi_j}), \tag{1.18}$$

where $r_k = |z_k|$ and $\varphi_k = \arg z_k$ are the modulus and argument of the zero z_k, $1 \le k \le l$, respectively. Let us simplify the right-hand side of representation (1.18). For the multipliers from the second product, we have

$$e^{it} - e^{i\phi_j} = ie^{i\frac{t+\phi_j}{2}} \left(2 \sin \frac{t - \phi_j}{2} \right).$$

Let us transform the multipliers from the first product as follows:

$$\left(e^{it} - r_k e^{i\varphi_k} \right) \left(e^{it} - \frac{e^{i\varphi_k}}{r_k} \right) = e^{2it} - \left(r_k + \frac{1}{r_k} \right) e^{i(t+\varphi_k)} + e^{2i\varphi_k}$$

$$= -2e^{i(t+\varphi_k)}(A_k - \cos(t - \varphi_k)), \quad A_k = \frac{1}{2} \left(r_k + \frac{1}{r_k} \right) > 1.$$

Thus, the following representation is valid for a polynomial $f_n \in \mathcal{F}_n(y)$:

$$f_n(t) = (-1)^{n+N} \frac{y}{2} \prod_{k=1}^{l} (2(A_k - \cos(t - \varphi_k))) \prod_{j=2l+1}^{2n} \left(2 \sin \frac{t - \phi_j}{2} \right), \tag{1.19}$$

where $A_k > 1$ for $1 \leq k \leq l$; $\varphi_k \in \mathbb{R}$ for $1 \leq k \leq l$; and $\phi_j \in \mathbb{R}$ for $2l + 1 \leq j \leq 2n$. We recall that, in addition, condition (1.17) is valid.

In what follows, the set $\mathcal{F}_n^{\text{real}}(y)$ of trigonometric polynomials from $\mathcal{F}_n(y)$ all zeros of which are real will play an important role. By (1.9) and (1.16), for a polynomial $f_n \in \mathcal{F}_n^{\text{real}}(y)$, we have

$$f_n(t) = e^{-\text{int} \frac{y}{2}} P_{2n}(e^{it}), \qquad P_{2n}(z) = \prod_{j=1}^{2n} \left(z - e^{i\phi_j} \right); \tag{1.20}$$

here, $\phi_j \in \mathbb{R}$, $1 \leq j \leq 2n$, and the following condition holds:

$$\sum_{j=1}^{2n} \phi_j = 2\pi N, \quad N \in \mathbb{Z}. \tag{1.21}$$

Formula (1.20) also implies the representation

$$f_n(t) = (-1)^{n+N} \frac{y}{2} g_{2n}(t), \quad g_{2n}(t) = \prod_{j=1}^{2n} \left(2 \sin \frac{t - \phi_j}{2} \right); \tag{1.22}$$

here, ϕ_j, $1 \leq j \leq 2n$, are (real) zeros of the polynomial f_n. From the above reasoning, it is clear that condition (1.21) is necessary and sufficient for function (1.20) or, what comes to the same, (1.22) to be a polynomial from $\mathcal{F}_n^{\text{real}}(y)$.

Justifying results of [12], Mendelev and Plotnikov obtained the following assertion.

Lemma 1. *For any $n \geq 1$ and $y > 1$, an extremal polynomial exists in problem* (1.4)*; all zeros of the extremal polynomial are real.*

Proof. Assume that a polynomial $f_n \in \mathcal{F}_n(y)$ has at least one complex zero; i.e., representation (1.19) contains at least one multiplier $A_k - \cos(t - \varphi_k)$. Since $A_k > 1$, the following inequality is valid for all t:

$$A_k - \cos(t - \varphi_k) > 1 - \cos(t - \varphi_k) = 2 \sin^2 \frac{t - \varphi_k}{2}.$$

Let us consider the function

$$\widehat{f_n}(t) = e^{-\text{int} \frac{y}{2}} \prod_{k=1}^{l} \left(e^{it} - e^{i\varphi_k} \right)^2 \prod_{j=2l+1}^{2n} \left(e^{it} - e^{i\phi_j} \right), \tag{1.23}$$

which is a trigonometric polynomial of order n. The polynomial f_n satisfies condition (1.21); hence, function (1.23) is a trigonometric polynomial with real zeros; more precisely, $\widehat{f_n} \in \mathcal{F}_n^{\text{real}}(y)$. Absolute values of the polynomials f_n and $\widehat{f_n}$ are connected by the inequality $|\widehat{f_n}(t)| \leq |f_n(t)|$, $t \in \mathbb{R}$; moreover, the strict inequality $|\widehat{f_n}(t)| < |f_n(t)|$ holds at points $t \in \mathbb{R}$ distinct from real zeros $\{\phi_j\}_{j=2l+1}^{2n}$ of the polynomial f_n. Hence, $\mu(\widehat{f_n}) < \mu(f_n)$.

At this stage, we have proved, in particular, that, in (1.4), it is necessary to restrict our attention to polynomials $f_n \in \mathcal{F}_n^{\text{real}}(y)$; consequently, the following equality holds:

$$\sigma_n(y) = \inf\{\mu(f_n) : f_n \in \mathcal{F}_n^{\text{real}}(y)\}. \tag{1.24}$$

To complete the proof of the lemma, it remains to show that the infimum in the right-hand side of (1.24) is reached. For polynomials $f_n \in \mathcal{F}_n^{\text{real}}(y)$, the value $\mu(f_n)$ is a function of $2n$ (real) zeros

of the polynomial $f_n \in \mathcal{F}_n^{\text{real}}(y)$. We will use the same symbol μ to denote this function; thus, $\mu(f_n) = \mu(\phi_1, \ldots, \phi_{2n})$. The zeros $\{\phi_j\}_{j=1}^{2n}$ are related by condition (1.21). We can assume that $0 \le \phi_j \le 2\pi$, $1 \le j \le 2n - 1$, and $\phi_{2n} = -\sum_{j=1}^{2n-1} \phi_j$. This set of points $\phi = \{\phi_j\}_{j=1}^{2n}$ will be denoted by Π_{2n}. Relation (1.24) can be rewritten in the form

$$\sigma_n(y) = \inf\{\mu(\phi_1, \ldots, \phi_{2n}) : \phi = \{\phi_j\}_{j=1}^{2n} \in \Pi_{2n}\}. \tag{1.25}$$

It is easily seen that the function μ continuously depends on the point $\phi = \{\phi_j\}$; in addition, Π_{2n} is a compact subset of the space $\mathbb{R}^{2n}$. Therefore, the infimum in (1.25) or, what comes to the same, in (1.24) is reached. The proof of the lemma is completed. $\square$

1.3. A restatement and expansion of the initial problem

Lemma 1 reduces initial problem (1.4) to a clearer problem of minimization of a (continuous) function of several real variables. Let m be natural; in the sequel, studying problem (1.4), we take $m = 2n$. We denote by $\mathfrak{P}_m(\Gamma)$ the set of algebraic polynomials

$$P_m(z) = \prod_{j=1}^{m} (z - e^{i\phi_j}), \quad z \in \mathbb{C}, \tag{1.26}$$

of order m with the unit leading coefficient, all m zeros of which belong to the unit circle $\Gamma = \{e^{it} : t \in [0, 2\pi]\}$. Every such polynomial is uniquely defined by the point $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m$. On the unit circle, polynomial (1.26) is representable in the form

$$P_m(e^{it}) = e^{i\frac{m}{2}t} e^{i\frac{\Phi}{2}} i^m g_m(t), \quad \Phi = \sum_{j=1}^{m} \phi_j, \tag{1.27}$$

where

$$g_m(t) = g_m(t; \phi) = \prod_{j=1}^{m} \left(2 \sin \frac{t - \phi_j}{2}\right), \quad t \in \mathbb{R}. \tag{1.28}$$

The set of functions (1.28) will be denoted by $\mathcal{G}_m$.

Let us consider the quantity

$$h(m) = \min\{\|g\|_{C_{2\pi}} : g \in \mathcal{G}_m\} = \min\{\|P_m\|_{C(\Gamma)} : P_m \in \mathfrak{P}_m(\Gamma)\} \tag{1.29}$$

of the least value of the uniform norm of polynomials (1.28) on the real line or, what comes to the same, of the uniform norm of polynomials (1.27) on the unit circle. It is well known that

$$h(m) = 2, \quad m \ge 1. \tag{1.30}$$

Besides, this can be easily verified. Indeed, polynomial (1.26) has the form

$$P_m(z) = \sum_{k=0}^{m} c_k z^k; \tag{1.31}$$

here, $c_m = 1$, and $c_0 = e^{i\Psi}$, $\Psi = m\pi + \sum_{j=1}^{m} \phi_j$. For a fixed Ψ, on the set of algebraic polynomials $F_m \in \mathcal{P}_m$ of order $m \ge 1$, let us consider the linear functional

$$\Sigma_m(F_m) = \frac{1}{m} \sum_{l=0}^{m-1} F_m(e^{i\theta_l}), \quad \text{where } \theta_l = \frac{\Psi + 2\pi l}{m}, \ 0 \le l \le m - 1.$$

For the polynomials $p_k(z) = z^k$, we have

$$\Sigma_m(p_k) = \frac{1}{m} \sum_{l=0}^{m-1} e^{ik\theta_l} = e^{i\frac{k\Psi}{m}} \frac{1}{m} \sum_{l=0}^{m-1} e^{i\frac{2\pi kl}{m}}.$$

Hence, we see that $\Sigma_m(p_k) = 0$, $1 \le k \le m - 1$; $\Sigma_m(1) = 1$; $\Sigma_m(p_m) = e^{i\Psi}$. Therefore, $\Sigma_m(P_m) = 2e^{i\Psi}$ for polynomial (1.26). On the other hand, the estimate $|\Sigma_m(P_m)| \le \|P_m\|_{C(\Gamma)}$ is valid. Consequently, $h(m) \ge 2$. The polynomial $P_m(z) = z^m + 1$ provides the inverse estimate. Thus, assertion (1.30) really holds.

For a parameter h, $0 \le h \le 2 = h(m)$, we set

$$\delta_m(h) = \inf\{\text{mes}\{t \in \mathbb{T} : |P_m(e^{it})| \ge h\} : P_m \in \mathfrak{P}_m(\Gamma)\}. \tag{1.32}$$

Relations (1.27) and (1.28) imply also that

$$\delta_m(h) = \inf\{\text{mes}\{t \in \mathbb{T} : |g_m(t)| \ge h\} : g_m \in \mathcal{G}_m\} \tag{1.33}$$

$$= \inf\{\chi_m(\phi; h) : \phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m\}, \tag{1.34}$$

where

$$\chi_m(\phi; h) = \chi_m(g_m; h) = \text{mes}\{t \in [0, 2\pi] : |g_m(t; \phi)| \ge h\} \tag{1.35}$$

is a function of the point $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m$. Representation (1.34) means that (for a fixed $h \in [0, 2]$) the value $\delta_m(h)$ can be interpreted as the minimum of a (continuous) function $\chi_m(\phi) = \chi_m(\phi; h)$ of m variables. A considerable part of this paper is devoted to studying the value $\delta_m(h)$. In the sequel, depending on the situation, it will be convenient for us to use one of the three representation forms, (1.32)–(1.34), for the value $\delta_m(h)$.

In the case $m = 1$, function (1.28) takes the form

$$g_1(t) = 2 \sin \frac{t - \phi_1}{2}. \tag{1.36}$$

For any such function, $\text{mes}\{t \in \mathbb{T} : |g_1(t)| \ge h\} = 4 \arccos(h/2)$. Consequently, the following formula holds for $m = 1$:

$$\delta_1(h) = 4 \arccos\left(\frac{h}{2}\right), \quad 0 \le h \le 2. \tag{1.37}$$

Moreover, any polynomial (1.36) is extremal in (1.33); so, any polynomial $P_1(z) = z - \zeta$ whose zero ζ satisfies the condition $|\zeta| = 1$ is extremal in (1.32).

Lemma 2. *For $m \ge 1$, the following assertions are valid:*

(1) *for any h, $0 \le h \le 2$, there exists an extremal point $\phi = \phi^* \in \mathbb{R}^m$, at which an infimum in (1.34) is reached;*

(2) *for the extreme values of h, we have $\delta_m(0) = 2\pi$, $\delta_m(2) = 0$;*

(3) *the value $\delta_m(h)$ (strictly) decreases with respect to $h \in [0, 2]$.*

Proof. It is sufficient to consider $m \ge 2$. If $h = 0$, then, for any function (1.28), value (1.35) is equal to 2π; so $\delta_m(0) = 2\pi$. Let us discuss the case $h = 2$. For example, the polynomial $P_m(z) = z^m + 1$ belongs to the set $\mathfrak{P}_m(\Gamma)$; for this polynomial, the set $\{t \in \mathbb{T} : |P_m(e^{it})| \ge 2\}$ consists of m points; thus, its measure is zero. Therefore, $\delta_m(2) = 0$.

The existence of an extremal point for $0 < h < 2$ in (1.34) (and so, of extremal functions in (1.32) and (1.33)) can be easily justified with the help of the arguments used in the proof of Lemma 1.

Finally, let us prove the monotonicity of the value $\delta_m(h)$ with respect to $h \in [0, 2]$. Let $0 \leq h_1 < h_2 \leq 2$. We denote by $g_m^{(1)}$ the polynomial from $\mathcal{G}_m$ on which the infimum in (1.33) is reached for $h = h_1$. The strict inequality $\text{mes}\{t \in \mathbb{T} : |g_m^{(1)}(t)| \geq h_2\} < \text{mes}\{t \in \mathbb{T} : |g_m^{(1)}(t)| \geq h_1\}$ holds. This implies that $\delta_m(h_2) < \delta_m(h_1), 0 \leq h_1 < h_2 \leq 2$. The lemma is proved. $\qquad \square$

We denote by $\mathbb{H} = \mathbb{H}_m$ the hyperplane of points $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m$ satisfying the condition

$$\sum_{j=1}^{m} \phi_j = 0. \tag{1.38}$$

This hyperplane is orthogonal to the vector $\Theta = \Theta_m = (1, 1, \ldots, 1) \in \mathbb{R}^m$. Let us ascertain that, in (1.34), we can restrict our attention to points $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{H}_m$; more precisely, that the following formula holds:

$$\delta_m(h) = \inf\{\chi_m(\phi; h) : \phi = (\phi_1, \ldots, \phi_m) \in \mathbb{H}_m\}, \quad 0 \leq h \leq 2. \tag{1.39}$$

Indeed, let $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m$. We set $\phi_0 = (\phi_1 + \cdots + \phi_m)/m$. Then, the point $\overline{\phi} = \phi - \phi_0 \Theta = (\phi_1 - \phi_0, \ldots, \phi_m - \phi_0)$ belongs to the hyperplane $\mathbb{H}_m$. It is easily seen that the equality $\chi_m(\phi, h) = \chi_m(\overline{\phi}, h)$ holds. Hence, relation (1.39) follows.

By Lemma 1 and formula (1.22), the following assertion is valid.

Corollary 1. *For $n \geq 1$ and $y > 1$, values (1.4) and (1.32) are related as follows:*

$$\sigma_n(y) = \delta_{2n}(h), \quad h = \frac{2}{y}. \tag{1.40}$$

2. A problem equivalent to problem (1.34) and its investigation

2.1. An equivalent problem

In this section, we will study a problem equivalent to problem (1.34). For natural m and real $h \geq 0$, we introduce the set

$$\mathbb{V} = \mathbb{V}(h) = \left\{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \left| \prod_{j=1}^{m} s(x_j) \right| \geq h \right\} \subset \mathbb{R}^m; \tag{2.1}$$

here and subsequently,

$$s(t) = 2 \sin \frac{t}{2}, \quad t \in \mathbb{R}.$$

Set (2.1) is nonempty if and only if $h \leq 2^m$. However, as will be seen below, we are interested only in values $0 < h < 2$. To a number $a \in \mathbb{R}$ and point $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m$ we assign the set

$$\Upsilon_m(\phi, h; a) = [\phi + a\Theta, \phi + (a + 2\pi)\Theta] \cap \mathbb{V}(h) \subset \mathbb{R}^m, \tag{2.2}$$

which is the intersection of the segment $[\phi + a\Theta, \phi + (a + 2\pi)\Theta] \subset \mathbb{R}^m$ with set (2.1); here, $\Theta = (1, 1, \ldots, 1) \in \mathbb{R}^m$. We are interested in the linear measure $\text{mes}_1(\Upsilon_m(\phi, h; a))$ of this set. The

subsequent considerations will show that this measure is independent of the parameter a. We set

$$\Delta_m(h) = \inf\{\text{mes}_1\left([\phi + a\Theta, \phi + (a + 2\pi)\Theta] \cap \mathbb{V}(h)\right) : \phi \in \mathbb{R}^m\}. \tag{2.3}$$

As will be shown below (see Lemma 4), problems (2.3) and (1.34) are closely interrelated; namely, the following equality holds:

$$\Delta_m(h) = \sqrt{m} \cdot \delta_m(h), \quad 0 < h < 2. \tag{2.4}$$

To make sure of this, let us compare the linear measure $\text{mes}_1(\Upsilon_m(\phi, h; a))$ of set (2.2) and the measure $\text{mes}(\upsilon_m(\phi, h; a))$ of the set

$$\upsilon_m(\phi, h; a) = \{t \in [a, a + 2\pi] : |g_m(t; \phi)| \geq h\} \subset \mathbb{R} \tag{2.5}$$

constructed from the function

$$g_m(t) = g_m(t; \phi) = \prod_{j=1}^{m} \left(2 \sin \frac{t + \phi_j}{2}\right) = \prod_{j=1}^{m} s(t + \phi_j). \tag{2.6}$$

Note that, here, in comparison with the previous section, the zeros ϕ_j have reversed signs.

Lemma 3. *For a point $\phi = (\phi_1, \ldots, \phi_m) \in \mathbb{R}^m$ and parameter $h, 0 < h < 2$, the following assertions are valid:* (1) *there holds the equality*

$$\begin{aligned}\text{mes}_1&([\phi + a\Theta, \phi + (a + 2\pi)\Theta] \cap \mathbb{V}(h)) \\ &= \sqrt{m} \cdot \text{mes}\{t \in [a, a + 2\pi] : |g_m(t; \phi)| \geq h\};\end{aligned} \tag{2.7}$$

(2) *the two sets (2.5) and (2.2) consist of the same number of segments (probably, degenerating to a point) whose lengths are directly proportional to the coefficient $\sqrt{m}$;* (3) *the measures of sets (2.5) and (2.2) are independent of the parameter $a \in \mathbb{R}$.*

Proof. Let us consider the linear vector function $\alpha(t) = \phi + t\Theta = (\alpha_1(t), \ldots, \alpha_m(t))$, where $\alpha_j(t) = \phi_j + t, j = 1, \ldots, m, t \in [a, a + 2\pi]$. This function is a bijection of the segment $[a, a + 2\pi]$ onto the segment $[\phi + a\Theta, \phi + (a + 2\pi)\Theta]$. An interval $X \subset [a, a + 2\pi]$ is mapped by the function α onto an interval $\mathcal{X} = \alpha(X) \subset [\phi + a\Theta, \phi + (a + 2\pi)\Theta]$ of the same type; moreover, it is easily seen that the measure $\text{mes}(X)$ of an interval $X \subset [a, a + 2\pi]$ and the linear measure $\text{mes}_1(\mathcal{X})$ of the interval $\mathcal{X} = \alpha(X)$ are related by the equality $\text{mes}_1(\mathcal{X}) = \sqrt{m} \cdot \text{mes}(X)$. It easily follows that, for any measurable subset $X \subset [a, a + 2\pi]$, its image $\mathcal{X} = \alpha(X) \subset [\phi, \phi + 2\pi\Theta]$ is also measurable and the (linear) measures of these sets are related by the equality $\text{mes}_1(\mathcal{X}) = \sqrt{m} \cdot \text{mes}(X)$.

Let us ascertain that the set (2.2) is the image of set (2.5) under the mapping α; i.e.,

$$\Upsilon_m(\phi, h; a) = \alpha(\upsilon_m(\phi, h; a)). \tag{2.8}$$

The fact that the point $\alpha(t) = \phi + t\Theta$ belongs to set (2.2) means that this point lies on the segment $[\phi + a\Theta, \phi + (a + 2\pi)\Theta]$ and in the set $\mathbb{V}(h)$ simultaneously. The first fact means that $t \in [a, a + 2\pi]$. By definition (2.1), the fact that the point $\alpha(t)$ belongs to the set $\mathbb{V}(h)$ means that $\prod_{k=1}^{m} |s(\alpha_k(t))| \geq h$. However,

$$\prod_{j=1}^{m} s(\alpha_j(t)) = \prod_{j=1}^{m} s(\phi_j + t) = \prod_{j=1}^{m} \left(2 \sin \frac{t + \phi_j}{2}\right) = g_m(t; \phi).$$

Thus, $\alpha(t) \in \Upsilon_m(\phi, h; a)$ if and only if $t \in \upsilon_m(\phi, h; a)$. Assertion (2.8) is proved.

Set (2.5) consists of a finite number of segments (some of which can degenerate to a point). Set (2.2) has the same structure. Both of the sets are measurable and their measures are related by equality (2.7).

The measure of set (2.5) and, by equality (2.7), the measure of set (2.2) are independent of the number a. The lemma is proved. $\quad \square$

Lemma 4. *For $m \geq 1$ and $0 < h < 2$, the following assertions are valid for problems* (1.34) *and* (2.3)*: (1) equality* (2.4) *holds; (2) there exists an extremal point $\phi = \phi^* \in \mathbb{R}$ at which infima in* (1.34) *and* (2.3) *are reached; this point has property* (2) *from Lemma* 3*; (3) value* (2.3) *is independent of the parameter $a \in \mathbb{R}$.*

Proof. Equality (2.4) is a consequence of the previous lemma. By Lemma 3, it is also sufficient to justify the existence of an extremal point in (1.34); this has been done in Lemma 2. Lemma 4 is proved. $\quad \square$

Our immediate aim is to ascertain that an extremal set in problem (2.3) (i.e., set (2.2) for the extremal point $\phi = \phi^*$ of problem (2.3)) consists of one segment and $m - 1$ points. This and Lemmas 3 and 4 will imply that the set $\{t \in \mathbb{T} : |g_m(t)| \geq h\}$ for the extremal polynomial g_m in (1.33) or, what comes to the same, the set $\{t \in \mathbb{T} : |P_m(e^{it})| \geq h\}$ for the extremal polynomial P_m in (1.32) also consists of one segment and $m - 1$ points.

2.2. Properties of the set $\mathbb{V}$

Starting with the set $\mathbb{V} = \mathbb{V}(h)$ defined by (2.1), we introduce the set $\mathbb{V}_0 = \mathbb{V}_0(h) = \mathbb{V}(h) \cap (0, 2\pi)^m$. The function s has the property $s(2l\pi) = 0$ for $l \in \mathbb{Z}$; therefore,

$$\mathbb{V}_0(h) = \mathbb{V}(h) \cap (0, 2\pi)^m = \mathbb{V}(h) \cap [0, 2\pi]^m, \quad h > 0. \tag{2.9}$$

Evidently, for any $h, 0 \leq h \leq 2^m$, the set $\mathbb{V}_0(h)$ is nonempty; and, for $0 \leq h < 2^m$, this set consists of more than one point. In addition, the set $\mathbb{V}_0(h)$ decreases with respect to h; more precisely,

$$\mathbb{V}_0(h_2) \subset \mathbb{V}_0(h_1), \quad 0 \leq h_1 \leq h_2 \leq 2^m. \tag{2.10}$$

We consider, along with $\mathbb{V}_0$, the sets $\mathbb{V}_k = \mathbb{V}_k(h) = \mathbb{V}_0 + 2\pi k = \{x + 2\pi k : x \in \mathbb{V}_0\}$ that are shifts of set (2.9) by the vectors $2\pi k, k \in \mathbb{Z}^m$.

Lemma 5. *For $m \geq 1$ and $0 < h < 2^m$, the following assertions are valid:*

the sets $\{\mathbb{V}_k : k \in \mathbb{Z}^m\}$ *are pairwise disjoint,* $\tag{2.11}$

$$\mathbb{V} = \bigcup_{k \in \mathbb{Z}^m} \mathbb{V}_k, \tag{2.12}$$

$\mathbb{V}_0$ *is compact,* $\tag{2.13}$

$\mathbb{V}_0$ *is strictly convex,* $\tag{2.14}$

$\mathbb{V}_0$ *has a nonempty interior; i.e., it is a body in $\mathbb{R}^m$.* $\tag{2.15}$

Proof. Property (2.11) is evident.

The set $\mathbb{V}$ (see definition (2.1)) can be written in the form

$$\mathbb{V} = \left\{ x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : w(x) \geq h \right\},$$

where

$$w(x) = w(x_1, \ldots, x_m) = \prod_{j=1}^{m} |s(x_j)|, \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m. \tag{2.16}$$

The function $|s(t)| = 2|\sin(t/2)|$ is continuous and 2π-periodic on the real line. Therefore, function (2.16) is continuous on $\mathbb{R}^m$ and 2π-periodic with respect to every variable; more precisely, $w(x + 2\pi k) = w(x)$ for all $x \in \mathbb{R}^m$ and $k \in \mathbb{Z}^m$. This, in particular, implies property (2.12).

Since function (2.16) is continuous on $\mathbb{R}^m$, the set $\mathbb{V}$ is closed. By representation (2.9), the set $\mathbb{V}_0$ is also closed and bounded, i.e., compact. Property (2.13) is checked.

Let us prove property (2.14); moreover, property (2.15) will be proved simultaneously. Let us take points $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m) \in \mathbb{V}_0$, i.e., points with coordinates $x_j, y_j \in (0, 2\pi)$, $1 \leq j \leq m$, satisfying the conditions

$$\prod_{j=1}^{m} s(x_j) \geq h, \qquad \prod_{j=1}^{m} s(y_j) \geq h. \tag{2.17}$$

Let us prove that $\frac{x+y}{2} \in \mathbb{V}_0$; i.e., $\prod_{j=1}^{m} s\left(\frac{x_j+y_j}{2}\right) \geq h$. We will prove the stronger inequality

$$\prod_{j=1}^{m} s\left(\frac{x_j + y_j}{2}\right) \geq \sqrt{\prod_{j=1}^{m} s(x_j) \prod_{j=1}^{m} s(y_j)}. \tag{2.18}$$

To do this, let us find the logarithms of the left-hand and right-hand sides of (2.18); we obtain the equivalent inequality

$$\sum_{j=1}^{m} \ln s\left(\frac{x_j + y_j}{2}\right) \geq \sum_{j=1}^{m} \frac{\ln s(x_j) + \ln s(y_j)}{2}. \tag{2.19}$$

The function $\ln s(t) = \ln\left(2\sin\frac{t}{2}\right)$ is strictly convex upwards on $(0, 2\pi)$; therefore, the following inequalities are valid:

$$\ln s\left(\frac{x_j + y_j}{2}\right) \geq \frac{\ln s(x_j) + \ln s(y_j)}{2}, \quad j = 1, 2, \ldots, m. \tag{2.20}$$

Consequently, inequality (2.19) and so inequality (2.18) are valid. Thus, we have proved that if $x, y \in \mathbb{V}_0$, then $\frac{x+y}{2} \in \mathbb{V}_0$. Since $\mathbb{V}_0$ is closed, we can conclude that the set $\mathbb{V}_0$ is convex.

Let us ascertain that, in fact, the set $\mathbb{V}_0$ is strictly convex. Let us prove that if $x, y \in \mathbb{V}_0$ and $x \neq y$, then the following strict inequality holds:

$$w\left(\frac{x + y}{2}\right) = \prod_{j=1}^{m} s\left(\frac{x_j + y_j}{2}\right) > h. \tag{2.21}$$

If $x \neq y$, then $x_j \neq y_j$ at least for one index j. By the strict convexity of the function $\ln s(t) = \ln\left(2\sin\frac{t}{2}\right)$ on the interval $(0, 2\pi)$, corresponding inequality (2.20) is strict; but then, inequality (2.18) is also strict; consequently, (2.21) holds. The function w defined by (2.16) is continuous everywhere in $\mathbb{R}^m$; therefore, there exists a neighborhood $\mathcal{O}$ of the point $(x + y)/2$ (situated in $(0, 2\pi)^m$) in which the inequality $w(z) > h, z \in \mathcal{O}$, is valid. Consequently, this neighborhood lies in the set $\mathbb{V}_0(h)$.

Since, for $0 \leq h < 2^m$, the set $\mathbb{V}_0(h)$ consists of more than one point, the previous reasonings imply that $\mathbb{V}_0(h), 0 < h < 2^m$, is strictly convex; in particular, its interior is nonempty; i.e., this set is a body. Properties (2.14) and (2.15) are checked. The proof of Lemma 5 is complete.   □

2.3. The intersection of a line with cubes

We assign to a point $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ the m-dimensional cube

$$\mathbb{U}_k = \times_{i=1}^m [2\pi k_i, 2\pi(k_i + 1))$$

in the space $\mathbb{R}^m, m \geq 2$. Evidently, sets from the family $\{\mathbb{U}_k : k \in \mathbb{Z}^m\}$ are disjoint and $\mathbb{R}^m = \cup\{\mathbb{U}_k : k \in \mathbb{Z}^m\}$. Let $\mathbb{Z}_0^m = \{0\} \times \mathbb{Z}^{m-1}$ be the set of points $k = (0, k_2, \ldots, k_m) \in \mathbb{Z}^m$ with integer coordinates, the first of which is zero. For $x \in \mathbb{R}^m$, we denote by $\ell(x)$ the line with the direction vector $\Theta = (1, \ldots, 1)$ (i.e., the line orthogonal to the plane $\mathbb{H} = \mathbb{H}_m$) passing through the point x. In the following lemma, we study the intersection of lines $\ell(x)$ with cubes $\mathbb{U}_k, k \in \mathbb{Z}_0^m$. We denote by $e_1, e_2, \ldots, e_m$ the unit orts of the space $\mathbb{R}^m$.

Lemma 6. *The positional relationship of lines $\ell = \ell(x), x \in \mathbb{R}^m$, and cubes $\mathbb{U}_k, k \in \mathbb{Z}_0^m$, has the following properties.*

(1) *Any line ℓ intersects at most m cubes*

$$\{\mathbb{U}_{k^{(p)}}, k^{(p)} \in \mathbb{Z}_0^m\}_{p=1}^{\overline{p}}, \quad 1 \leq \overline{p} \leq m.$$

(2) *For $\overline{p} \geq 2$ and points $k^{(p)} \in \mathbb{Z}_0^m, 2 \leq p \leq \overline{p}$, the following recurrence formula is valid:*

$$k^{(p)} = k^{(p-1)} + \sum \{e_i : i \in I(p)\}, \quad 2 \leq p \leq \overline{p}, \tag{2.22}$$

where $k^{(1)} \in \mathbb{Z}_0^m$ and $\{I(p)\}_{p=2}^{\overline{p}}$ is a family of nonintersecting subsets that form a decomposition of the set of natural numbers $\{2, \ldots, m\}$.

(3) *If a line ℓ intersects exactly m cubes $\{\mathbb{U}_{k^{(p)}}, k^{(p)} \in \mathbb{Z}_0^m\}_{p=1}^m$, then*

$$k^{(p)} = k^{(1)} + \sum_{q=2}^p e_{i(q)}, \quad 2 \leq p \leq m,$$

where $\{i(q)\}_{q=2}^m$ is a permutation of the set $\{2, \ldots, m\}$.

Proof. We can consider the line $\ell = \ell(x)$ parallel to the vector $\Theta = (1, \ldots, 1)$ and passing through the point $x = (x_1, \ldots, x_m)$ in the parametric form

$$\lambda(t) = (t, t + x_2 - x_1, \ldots, t + x_m - x_1) = t\Theta + A, \quad t \in \mathbb{R}, \tag{2.23}$$

where $A = \lambda(0) = (0, x_2 - x_1, \ldots, x_m - x_1)$; in particular, $\lambda(x_1) = x$. We denote by $\Pi = \Pi(m)$ the band of points $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ satisfying the condition $0 \leq z_1 < 2\pi$; we have $\Pi = \cup_{k \in \mathbb{Z}_0^m} \mathbb{U}_k$. The line ℓ intersects the boundary hyperplanes $z_1 = 0$ and $z_1 = 2\pi$ of the band Π at the points A and $B = \lambda(2\pi) = (2\pi, 2\pi + x_2 - x_1, \ldots, 2\pi + x_m - x_1)$; the distance between these points is equal to $\sqrt{m}2\pi$. Moreover, the line ℓ intersects the band Π exactly by the half-open interval $[A, B)$. Now we have to find cubes that are intersected by this half-open interval.

The fact that the point $\lambda(t) = (t, t + x_2 - x_1, \ldots, t + x_m - x_1)$ belongs to the cube $\mathbb{U}_k, k = (0, k_2, \ldots, k_m) \in \mathbb{Z}_0^m$, means that the following m relations are valid:

$$0 \leq t < 2\pi; \quad 2\pi k_i \leq t + x_i - x_1 < 2\pi(k_i + 1), \quad 2 \leq i \leq m. \tag{2.24}$$

First, we construct a cube $\mathbb{U}_{k^{(1)}}, k^{(1)} \in \mathbb{Z}_0^m$, containing the point $A = \lambda(0)$; since the cubes $\{\mathbb{U}_k\}$ do not intersect, such a cube is unique. Let us choose integers $k_i^{(1)}, 2 \le i \le m$, satisfying the condition $2\pi k_i^{(1)} \le x_i - x_1 < 2\pi(k_i^{(1)} + 1), 2 \le i \le m$, and let us set $k^{(1)} = (k_1^{(1)}, k_2^{(1)}, \ldots, k_m^{(1)})$, where $k_1^{(1)} = 0$. It is clear that $A \in \mathbb{U}_{k^{(1)}}$. Let us consider the numbers $r_i = 2\pi(k_i^{(1)} + 1) - (x_i - x_1), 1 \le i \le m$; they satisfy the condition $0 < r_i \le 2\pi$; moreover, $r_1 = 2\pi$. Let T' be the set of distinct numbers $\{r_i\}_{i=1}^m$; we denote by $\overline{p}, 1 \le \overline{p} \le m$, the number of elements of this set. Let us arrange elements of the set T' in order of magnitude and let us number them by index $p, 2 \le p \le \overline{p} + 1$; as a result, we obtain the representation $T' = \{t_p\}_{p=2}^{\overline{p}+1}$. In this case, we have $t_{\overline{p}+1} = 2\pi$. Let us specify the set $T = \{t_p\}_{p=1}^{\overline{p}}$, where $t_1 = 0$; elements of this set are distinct and situated on the half-open interval $[0, 2\pi)$. We set $A_p = \lambda(t_p), 1 \le p \le \overline{p} + 1$. We have $A_1 = \lambda(0) = A$ and $A_{\overline{p}+1} = \lambda(2\pi) = B$. For $1 \le p \le \overline{p}$, points A_p lie in the band Π.

First, we assume that $\overline{p} > 1$. For every $p, 2 \le p \le \overline{p} + 1$, we denote by $I(p)$ the set of (all) indices i with the property $r_i = t_p$. The sets $I(p)$ are disjoint; their union composes the set $\{1, \ldots, m\}$ of the first m natural numbers. With the help of recurrence relation (2.22), we define a family of $\overline{p} - 1$ integral points; these points can be also written in the form

$$k^{(p)} = k^{(1)} + \sum \{e_i : i \in \mathcal{I}(p)\}, \quad \mathcal{I}(p) = \bigcup_{q=2}^p I(q), \ 2 \le p \le \overline{p}.$$

For $2 \le p \le \overline{p}$, the sets $I(p)$ do not contain the number 1; therefore, the constructed points $k^{(p)}, 1 \le p \le \overline{p}$, belong to the set $\mathbb{Z}_0^m$.

Let us ascertain that the line ℓ intersects exactly the cubes $\mathbb{U}_{k^{(p)}}, 1 \le p \le \overline{p}$. Let us verify that, in fact, the following stronger assertion is valid:

$$[A_p, A_{p+1}) = \ell \cap \mathbb{U}_{k^{(p)}}, \quad 1 \le p \le \overline{p}. \tag{2.25}$$

On the basis of the choice of values of the parameter $\{t_p\}_{p=1}^{\overline{p}+1}$, the definitions of the points $\{A_p\}_{p=1}^{\overline{p}}$, and the integral points $\{k^{(p)}\}_{p=1}^{\overline{p}}$, it is easily seen that the following embeddings are valid:

$$[A_p, A_{p+1}) \subset \mathbb{U}_{k^{(p)}}, \quad 1 \le p \le \overline{p}. \tag{2.26}$$

Since, in addition,

$$\bigcup_{p=1}^{\overline{p}} [A_p, A_{p+1}) = [A, B] = \ell \cap \Pi,$$

all embeddings (2.26) turn into equalities (2.25). Thus, in fact, the line ℓ intersects only the cubes $\mathbb{U}_{k^{(p)}}, 1 \le p \le \overline{p}$.

If $\overline{p} = 1$, then, as is easily seen, the line ℓ intersects only one cube $\mathbb{U}_{k^{(1)}}$.

Let us discuss the situation when the line ℓ intersects exactly m cubes; i.e., $\overline{p} = m$. This will be in the case if, for any $p, 2 \le p \le m + 1$, the sets $I(p)$ consist only of one number $i(p)$; as a consequence, the numbers $\{i(p)\}_{p=1}^m$ form the set $\{1, \ldots, m\}$ and $i(m + 1) = 1$. The lemma is proved. □

2.4. Decomposition of the measure into a sum of strictly convex functions

By assertion 3 of Lemma 3, the measure of the set $[x + a\Theta, x + (a + 2\pi)\Theta) \cap \mathbb{V}(h), 0 < h < 2$, is independent of the numbers $a \in \mathbb{R}$. In the sequel, it is convenient for us to take $a = -x_1$ for

every point $x = (x_1, \ldots, x_m) \in \mathbb{H}$. We set $\mathrm{I}(x) = [x - x_1 \Theta, x + (-x_1 + 2\pi)\Theta)$. Let us consider the function

$$\tau(x) = \tau(x; h) = \mathrm{mes}_1\{[x - x_1 \Theta, x + (-x_1 + 2\pi)\Theta) \cap \mathbb{V}(h)\} = \mathrm{mes}_1\{\mathrm{I}(x) \cap \mathbb{V}\} \quad (2.27)$$

of variable $x \in \mathbb{H}$. With this notation, we can write problem (2.3) in the form

$$\Delta_m(h) = \min\{\tau(x; h) : x \in \mathbb{H}\}. \quad (2.28)$$

Let us decompose function (2.27) into a sum of simpler functions. To this end, let us represent the set $\mathrm{I}(x) \cap \mathbb{V}$ in a special form. The half-open interval $\mathrm{I}(x) = [x - x_1 \Theta, x + (-x_1 + 2\pi)\Theta)$ is the intersection $\mathrm{I}(x) = \ell(x) \cap \Pi$ of the line $\ell(x)$ and the band $\Pi = \{z = (z_1, \ldots, z_m) \in \mathbb{R}^m : 0 \leq z_1 < 2\pi\} = [0, 2\pi) \times \mathbb{R}^{m-1}$. Now, using property (2.12) of the set $\mathbb{V}$, we represent the set $\mathrm{I}(x) \cap \mathbb{V}$ as a union of disjoint subsets:

$$\mathrm{I}(x) \cap \mathbb{V} = \ell(x) \cap \Pi \cap \bigcup_{k \in \mathbb{Z}^m} \mathbb{V}_k = \ell(x) \cap \bigcup_{k \in \mathbb{Z}^m} (\mathbb{V}_k \cap \Pi) = \ell(x) \cap \bigcup_{k \in \mathbb{Z}_0^m} \mathbb{V}_k$$
$$= \bigcup_{k \in \mathbb{Z}_0^m} (\ell(x) \cap \mathbb{V}_k).$$

We recall that $\mathbb{Z}_0^m = \{0\} \times \mathbb{Z}^{m-1}$. Thus, we obtain the representation

$$\mathrm{I}(x) \cap \mathbb{V} = \bigcup \{\ell(x) \cap \mathbb{V}_k : k \in \mathbb{Z}_0^m\}. \quad (2.29)$$

The sets $\mathbb{V}_k$ are strictly convex (see property (2.14)) and pairwise disjoint (see property (2.11)); $\ell(x)$ is a line. Consequently, each of the sets $\{\ell(x) \cap \mathbb{V}_k\}$ can be either the empty set or a point, or a segment; in addition, the sets $\{\ell(x) \cap \mathbb{V}_k\}$ are pairwise disjoint. Therefore, the following decomposition is valid:

$$\tau(x) = \sum_{k \in \mathbb{Z}_0^m} \tau_k(x), \quad \text{where } \tau_k(x) = \tau_k(x; h) = \mathrm{mes}_1\{\ell(x) \cap \mathbb{V}_k\}. \quad (2.30)$$

Let $\mathbb{K}(x)$ be the set of those $k \in \mathbb{Z}_0^m$ for which $\ell(x) \cap \mathbb{V}_k \neq \varnothing$. By Lemma 6, for any point $x \in \mathbb{H}$, the set $\mathbb{K}(x)$ consists of at most m elements. Consequently, for any $x \in \mathbb{H}$, at most m summands are nonzero in sum (2.30). In the following lemma, we study properties of the functions τ_k, in particular, properties of their supports $\mathrm{supp}\, \tau_k \subset \mathbb{H}$.

Lemma 7. *For $m \geq 2$, $0 < h < 2^m$, and $k \in \mathbb{Z}_0^m$, the functions $\tau_k(x)$, $x \in \mathbb{H}$, have the following properties:*

$\mathrm{supp}\, \tau_k$ *is a strictly convex compact body in the hyperplane $\mathbb{H}$;* $\quad (2.31)$

τ_k *is a strictly convex (upwards) nonnegative function on $\mathrm{supp}\, \tau_k$;* $\quad (2.32)$

τ_k *is a continuous function on the hyperplane $\mathbb{H}$.* $\quad (2.33)$

Proof. For a subset $X \in \mathbb{R}^m$, we denote by $\mathrm{Pr}_{\mathbb{H}}(X)$ the orthogonal projection of the set X to the hyperplane $\mathbb{H} = \mathbb{H}_m$; i.e.,

$$\mathrm{Pr}_{\mathbb{H}}(X) = \{x \in \mathbb{H} : \ell(x) \cap X \neq \varnothing\}.$$

In these terms, we have $\mathrm{supp}\, \tau_k = \mathrm{Pr}_{\mathbb{H}}(\mathbb{V}_k)$. By Lemma 5, the sets $\mathbb{V}_k$ are strictly convex and compact bodies in $\mathbb{R}^m$. Therefore, each of the sets $\mathrm{supp}\, \tau_k$ is also a strictly convex and compact $(m-1)$-dimensional body in the hyperplane $\mathbb{H}$. The function τ_k is nonnegative and strictly

convex (upwards) on its support supp τ_k, since, for $x \in$ supp τ_k, its value is the measure of the intersection of the strictly convex set $\mathbb{V}_k$ and the line $\ell(x)$. The function τ_k, as a strictly convex function, is continuous in the interior of the set supp τ_k. It is easily understood that this function is also continuous and has zero values at points of the boundary of the set supp τ_k. Hence, the function τ_k is continuous on the hyperplane $\mathbb{H}$. The lemma is proved.   □

2.5. The structure of an extremal set

Lemma 8. *Let* $m \geq 2$ *and let* $0 < h < 2$. *If a point* $x^* \in \mathbb{H}$ *is extremal in* $\Delta_m(h)$—*more precisely, if the infimum in* (2.28) *is reached at this point—then the set* $\mathbb{K}(x^*)$ *consists of* m *points and only one of the functions* $\tau_k, k \in \mathbb{K}(x^*)$, *is different from the identical zero.*

Proof. Let L be a linear or, more generally, affine subspace of $\mathbb{R}^m$ of dimension l, $1 \leq l \leq m-1$. In the sequel, by the *l-neighborhood* of a point $z \in L$ we mean the open ball $O(z) = O_r(z) = \{x \in L : |x - z| < r\}$ in L centered at the point z of certain radius $r > 0$; here, $|\cdot|$ is the Euclidean norm of the space $\mathbb{R}^m$.

Let us pay attention to formula (2.30) for the point $x^* \in \mathbb{H}$ at which an infimum in (2.28) is reached:

$$\tau(x^*) = \sum_{k \in \mathbb{K}(x^*)} \tau_k(x^*). \tag{2.34}$$

We recall that, here, $\mathbb{K}(x^*)$ is the set of those $k \in \mathbb{Z}_0^m$ for which $\ell(x^*) \cap \mathbb{V}_k \neq \varnothing$; this set contains at most m elements. The set

$$\mathbb{W}(x^*) = \bigcup \{\mathbb{V}_k : k \in \mathbb{Z}_0^m \setminus \mathbb{K}(x^*)\},$$

being the union of all the sets $\mathbb{V}_k$ over $k \in \mathbb{Z}_0^m \setminus \mathbb{K}(x^*)$, is closed (in $\mathbb{R}^m$) and does not intersect the line $\ell(x^*)$. It is easily understood that a distance $\rho(\ell(x^*), \mathbb{W}(x^*))$ between these sets (in the space $\mathbb{R}^m$) is positive. Let us fix an $(m-1)$-neighborhood $O_\rho(x^*) = \{x \in \mathbb{H} : |x - x^*| < \rho\}$ of the point x^* of radius $\rho = \rho(\ell(x^*), \mathbb{W}(x^*))$ in the hyperplane $\mathbb{H}$. For each $k \in \mathbb{Z}_0^m \setminus \mathbb{K}(x^*)$, the projection

$$\omega_k = \mathrm{Pr}_{\mathbb{H}}(\mathbb{V}_k) = \mathrm{supp}\, \tau_k$$

of the set $\mathbb{V}_k$ to the hyperplane $\mathbb{H}$ and the neighborhood $O_\rho(x^*)$ of the point x^* are disjoint. Therefore, in fact, representation (2.34) is valid in the whole neighborhood $O_\rho(x^*)$; i.e., the following formula is valid:

$$\tau(x) = \sum_{k \in \mathbb{K}(x^*)} \tau_k(x), \quad x \in O_\rho(x^*); \tag{2.35}$$

though some of the summands in (2.35) can be zero.

Points $k \in \mathbb{K}(x^*)$ necessarily have one of the following two properties: (1) x^* is an interior point of the set $\omega_k = \mathrm{supp}\, \tau_k$; (2) x^* is a boundary point of the set $\omega_k = \mathrm{supp}\, \tau_k$. Let us number the points $k \in \mathbb{K}(x^*)$ (or, what comes to the same, the functions $\tau_k, k \in \mathbb{K}(x^*)$) in a special way. First, note that there exists a point $k \in \mathbb{K}(x^*)$ with property (1). Indeed, otherwise, representation (2.34) implies that $\tau(x^*) = 0$; consequently, $\Delta_m(h) = 0$. However, this contradicts Lemmas 2 and 4. Thus, let us number points $k \in \mathbb{K}(x^*)$ with property (1) (with upper indices) from 1 to p. We number points $k \in \mathbb{K}(x^*)$ with property (2), if they exist, from $p + 1$ to q.

Assume that $m \geq 3$. For every $i = 1, \ldots, p$ there exists an $(m-1)$-neighborhood $O^{(i)}(x^*)$ of the point x^* in the hyperplane $\mathbb{H}$ such that $O^{(i)}(x^*) \subset \mathrm{supp}\, \tau_{k^{(i)}}$. For each of the numbers $i = (p + 1), \ldots, q$, in hyperplane $\mathbb{H}$, there exists an (affine) subspace $L^{(i)}(x^*)$ (of dimension

$m - 2$) tangent to $\operatorname{supp} \tau_{k(i)}$ at the point x^*. Since the set $\operatorname{supp} \tau_{k(i)}$ is strictly convex, we can assert that this subspace has only one point in common with the set $\operatorname{supp} \tau_{k(i)}$, namely, the point x^*. In each of the subspaces $L^{(i)}(x^*)$, we choose an $(m - 2)$-neighborhood $\mathrm{O}^{(i)}(x^*)$ of the point x^* (though we can take the subspaces $L^{(i)}(x^*)$ themselves). Note that the dimension of each of these neighborhoods is equal to $m - 2$. We need to prove that $p = 1$ and $q = m$. Let us argue by contradiction. Assume that $p > 1$ or $q < m$. We consider the intersection

$$\widetilde{\mathrm{O}}(x^*) = \left(\bigcap_{i=1}^q \mathrm{O}^{(i)}(x^*) \right) \bigcap \mathrm{O}_\rho(x^*)$$

of the neighborhoods constructed. Let us find the dimension $l = \dim \widetilde{\mathrm{O}}(x^*)$ of this set. If $p = q$ (i.e., there are no points of the second type), then $l = m - 1$. Now, let $p < q$. Then, $q - p$ tangent planes intersect at the point x^* and

$$l = \dim \widetilde{\mathrm{O}}(x^*) = \dim \bigcap_{i=p+1}^q \mathrm{O}^{(i)}(x^*) \geq m - 1 - (q - p) = m - q + p - 1.$$

By assumption, at least one of the inequalities $p \geq 1$ or $q < m$ holds. Therefore, $m - q + p - 1 \geq 1$. Thus, $l = \dim \bigcap_{i=p+1}^q \mathrm{O}^{(i)}(x^*) \geq 1$; i.e., $\widetilde{\mathrm{O}}(x^*)$ is an l-neighborhood of the point x^* of dimension $l \geq 1$. By (2.35), the following representation holds at points $x \in \widetilde{\mathrm{O}}(x^*)$:

$$\tau(x) = \sum_{i=1}^p \tau_{k(i)}(x).$$

The function τ, as a sum of strictly convex functions, is strictly convex on the set $\widetilde{\mathrm{O}}(x^*)$ and has a minimum on this set (at the point x^*); the set $\widetilde{\mathrm{O}}(x^*)$ is open in an affine subspace of dimension $l \geq 1$ (passing through the point x^*). However, this is impossible; a contradiction. In the case $m \geq 3$, the lemma is proved.

In the case $m = 2$, we have to prove that $p = 1$ and $q = 2$. The negative fact means that either $p = q = 1$ or $p = q = 2$. Using the arguments above, we can again ascertain that both of the situations are impossible. Thus, Lemma 8 is proved. $\square$

As a consequence of Lemma 8, the following assertion is valid, which will be important in the sequel.

Theorem 2. *For $m \geq 1, 0 < h < 2$, and the extremal polynomial $P_m \in \mathfrak{P}_m(\Gamma)$ of problem (1.32), the set $\{t \in \mathbb{T} : |P_m(e^{it})(t)| \geq 1\}$ consists of one segment and $m - 1$ points.*

For $m \geq 2$, this assertion follows from Lemma 8. The case $m = 1$ is trivial and discussed in the first section (see, in particular, (1.37)).

By relation (1.40) between problems (1.4) and (1.32), the following assertion is valid as a special case of Theorem 2.

Corollary 2. *For $n \geq 1, y > 1$, and the extremal polynomial $f_n(t) = y \cos nt + f_{n-1}(t)$ of problem (1.4), the set $\{t \in \mathbb{T} : |f_n(t)| \geq 1\}$ consists of one segment and $2n - 1$ points.*

3. On Chebyshev polynomials on compact sets of the unit circle; the completion of studying problem (1.32)

In this section, we discuss several close problems on polynomials with a fixed leading coefficient, all zeros of which are situated on the unit circle, and that deviate the least from

zero on compact sets of the circle. Problems of this type are related to problem (1.32). Using this relationship, Theorem 2, and a result of Maergoiz and Rybakova [8–10], we will, in particular, complete the study of problem (1.32).

3.1. Chebyshev polynomials that deviate the least from zero on compact sets of the unit circle

We recall that, in this paper, we denote by $\mathfrak{P}_m(\Gamma)$ the set of algebraic polynomials

$$P_m(z) = \prod_{j=1}^{m}(z - e^{i\phi_j}), \quad \{\phi_j\}_{j=1}^{m} \subset \mathbb{R}, \tag{3.1}$$

of degree m with the unit leading coefficient, all m zeros of which are situated on the unit circle $\Gamma = \{e^{it} : t \in [0, 2\pi]\}$ of the complex plane $\mathbb{C}$. To the parameter $\theta \in \mathbb{R}$ and polynomial (3.1), we assign the polynomial

$$\prod_{j=1}^{m}(z - e^{i(\phi_j+\theta)}) = e^{im\theta} P_m(ze^{-i\theta}), \tag{3.2}$$

whose zeros are obtained by a rotation of zeros of polynomial (3.1) by the angle θ around the origin of the complex plane $\mathbb{C}$. Polynomial (3.2) also belongs to the set $\mathfrak{P}_m(\Gamma)$; we will say that it is obtained by a rotation of polynomial (3.1) (by the angle θ around the origin of the complex plane).

For a compact subset Q of the circle Γ, we define the value

$$E_m(Q) = \inf\{\|P_m\|_{C(Q)} : P_m \in \mathfrak{P}_m(\Gamma)\} \tag{3.3}$$

of the best uniform deviation from zero of polynomials from the set $\mathfrak{P}_m(\Gamma)$ on Q. For $0 < \alpha < \pi$, we denote by $\mathcal{Q} = \mathcal{Q}(2\alpha)$ the set of all compact sets $Q \subset \Gamma$ whose (linear) measure $|Q|$ is equal to the number $2\alpha : |Q| = 2\alpha$. We are interested is the least value

$$E_m(2\alpha) = E_m(\mathcal{Q}(2\alpha)) = \inf\{E_m(Q) : |Q| = 2\alpha\} \tag{3.4}$$

of (3.3) over all compact sets $Q \in \mathcal{Q}(2\alpha)$. The following assertion is valid.

Theorem 3. *Problems (1.32) and (3.4) are related as follows:*

$$E_m(2\alpha) = h, \qquad \delta_m(h) = 2\pi - 2\alpha, \quad 0 < h < 2, \ 0 < \alpha < \pi. \tag{3.5}$$

This assertion seems to be rather natural; let us present some arguments. Let $Q \subset \Gamma$ be a compact set of measure $|Q| = 2\alpha, 0 < \alpha < \pi$, and let P_m be a polynomial from $\mathfrak{P}_m(\Gamma)$. For $h = \|P_m\|_{C(Q)}$, the measure of the set $\{t \in \mathbb{T} : |P_m(e^{it})| \geq h\}$ is at most $2\pi - 2\alpha$. A fortiori, the following inequality is valid:

$$\delta_m(h) \leq 2\pi - 2\alpha, \quad h = \|P_m\|_{C(Q)}.$$

In the sequel, we will not return to Theorem 3 since Theorems 6 and 7 proved below contain stronger assertions in comparison with (3.5).

Chebyshev polynomials (that deviate the least from zero with the unit leading coefficient) on compact sets of the complex plane have been studied by many mathematicians; they have numerous applications (see, for example, [14]). Let us describe in more detail the results of Chebyshev and Polya on algebraic polynomials that deviate the least from zero. Let $\mathfrak{P}_m = \mathfrak{P}_m(\mathbb{C})$,

$m \geq 1$, be the set of algebraic polynomials

$$P_m(x) = x^m + \sum_{k=0}^{m-1} c_k x^k$$

with the unit leading coefficient and, generally speaking, with complex other coefficients. Chebyshev [4] found the least deviation from zero

$$e_m([-1, 1]) = \inf\{\|P_m\|_{C[-1,1]} : P_m \in \mathfrak{P}_m\} \tag{3.6}$$

on the segment $[-1, 1]$ of polynomials from the class $\mathfrak{P}_m$. Namely, he showed that

$$e_m([-1, 1]) = \frac{1}{2^{m-1}}, \quad m \geq 1,$$

and the polynomial

$$P_m^*(x) = \frac{1}{2^{m-1}} T_m(x), \quad T_m(x) = \cos(m \arccos x), \ x \in [-1, 1]$$

is extremal. Using a linear change of variable, it is easily checked that, for any segment $I = [a, b]$ of length $|I| = b - a = 2\rho, \rho > 0$, the quantity

$$e_m(I) = \inf\{\|P_m\|_{C(I)} : P_m \in \mathfrak{P}_m\}$$

has the value $e_m(I) = 2 \left(\frac{\rho}{2}\right)^m$ and an extremal polynomial can be found accordingly. For a closed set $Q \subset \mathbb{R}$, we set

$$e_m(Q) = \inf\{\|P_m\|_{C(Q)} : P_m \in \mathfrak{P}_m\}. \tag{3.7}$$

Polya studied the least value

$$e_m(2\rho) = \inf\{e_m(Q) : Q \in \mathcal{Q}(2\rho)\} \tag{3.8}$$

of (3.7) over the family $\mathcal{Q} = \mathcal{Q}(2\rho)$ of all compact subsets $Q \subset \mathbb{R}$ of the real line whose measure is equal to a fixed number $2\rho, \rho > 0$. He proved the following assertion (see, for example, [3, p. 23]).

Theorem 4. *For any $\rho > 0$ and any set $Q \in \mathcal{Q}(2\rho)$, the following inequality is valid:*

$$e_m(Q) \geq 2 \left(\frac{\rho}{2}\right)^m;$$

it turns into an equality only in the case where Q is a segment (of length 2ρ). As a consequence,

$$e_m(2\rho) = 2 \left(\frac{\rho}{2}\right)^m.$$

Problem (3.4) can be considered to be an analog of problem (3.8). However, to study problem (3.4), we need other arguments in comparison with the proof of Theorem 4.

3.2. Chebyshev polynomials on an arc of the unit circle

For a segment $I = [a, b]$ of the real line, we define the arc $\Gamma(I) = e^{iI} = \{e^{it} : t \in I\}$ of length $|I|$ of the unit circle $\Gamma = \{e^{it} : t \in [0, 2\pi]\}$. Let $\mathfrak{P}_m(\Gamma(I))$ be the set of algebraic polynomials (3.1) with the unit leading coefficient, all zeros of which are situated on $\Gamma(I)$. We set

$$\varepsilon_m(I) = \min\{\|P_m\|_{C(\Gamma(I))} : P_m \in \mathfrak{P}_m(\Gamma(I))\}; \tag{3.9}$$

this is one of the variants of the problem of finding polynomials that deviate the least from zero. Value (3.9) is invariant with respect to shifts of the segment I (i.e., with respect to rotations of the arc $\Gamma(I)$). Therefore, this value depends only on the length of the segment I. Let us fix the lengths of segments: $|I| = 2\alpha$, $0 < \alpha < \pi$, and let us set

$$\varepsilon_m(2\alpha) = \varepsilon_m(I), \quad |I| = 2\alpha. \tag{3.10}$$

Maergoiz and Rybakova [8–10] obtained a solution of problem (3.10). Earlier, Tyshkevich [16] solved problem (3.10) for at least two arcs of the circle. His solution is expressed in terms of harmonic measure. Lukashov and Tyshkevich also discuss problem (3.9) for several arcs of the circle in their recent paper [7]. As a special case, [7] contains a solution of problem (3.10) (for one arc). Note that methods of [16,7] are different from that of [8–10]; in fact, [16,7] continue investigations by Lukashov [6]. Problem (3.9) on an arc of the circle without any restrictions on the arrangement of zeros was studied earlier in [15]. The following assertion is contained in [8–10].

Theorem 5. *For $m \geq 1$ and $0 < \alpha < \pi$, the following formula is valid:*

$$\varepsilon_m(2\alpha) = 2 \sin^m \frac{\alpha}{2}. \tag{3.11}$$

Moreover, with the notation

$$x_{km} = \cos \frac{\pi(2k-1)}{2m}, \qquad a_{km}(\alpha) = 1 - 2x_{km}^2 \cdot \sin^2 \frac{\alpha}{2}, \quad k = 1, \ldots, n,$$

for an arc $\Gamma(\alpha) = \{z = e^{it} : t \in \mathbb{R}, |t| \leq \alpha\}$ and $m = 2n$ or $m = 2n + 1$, the polynomials

$$P_m(z) = S_n(z), \quad S_n(z) = \prod_{k=1}^{n} (z^2 - 2a_{km}(\alpha)z + 1), \ m = 2n, \ n \geq 1; \tag{3.12}$$

$$P_m(z) = (z - 1)S_n(z), \quad m = 2n + 1, \ n \geq 0, \tag{3.13}$$

are the unique extremal polynomials in problem (3.10).

For easy reference in the sequel, we observe some properties of polynomials (3.12) and (3.13) on the unit circle. The following two relations are valid:

$$|P_m(e^{it})| \leq 2 \sin^m \frac{\alpha}{2}, \quad t \in [-\alpha, \alpha], \tag{3.14}$$

$$|P_m(e^{it})| > 2 \sin^m \frac{\alpha}{2}, \quad t \in (\alpha, 2\pi - \alpha). \tag{3.15}$$

Starting with polynomials (3.12) and (3.13), we define the polynomials

$$g_m(t) = e^{-int} P_m(e^{it}), \quad m = 2n, \ n \geq 1; \tag{3.16}$$

$$g_m(t) = -ie^{-i\frac{m}{2}t} P_m(e^{it}), \quad m = 2n + 1, \ n \geq 0. \tag{3.17}$$

Let us introduce the notation

$$\lambda = \left(\sin \frac{\alpha}{2}\right)^{-2}.$$

It is easily checked that polynomial (3.16) has the following structure:

$$g_{2n}(t) = \lambda^{-n} 2T_n(\lambda \cos t - (\lambda - 1)), \quad t \in \mathbb{R}, \tag{3.18}$$

where T_n is the Chebyshev polynomial (of the first kind) of order n. Polynomial (3.17) can be represented in the form

$$g_{2n+1}(t) = 2 \sin\left(\frac{t}{2}\right) D_n(\lambda \cos t - (\lambda - 1)), \quad t \in \mathbb{R}, \tag{3.19}$$

where D_n is an algebraic polynomial of order n which is expressed in terms of the Dirichlet kernel on the segment $[-1, 1]$ by the formula

$$D_n(\cos t) = 2\mathcal{D}_n(t), \quad \mathcal{D}_n(t) = \frac{\sin\left(\frac{2n+1}{2}t\right)}{2 \sin\left(\frac{1}{2}t\right)}, \ t \neq 2k\pi, \ k \in \mathbb{Z}.$$

Note that polynomial (3.18) arose in investigations by Chebyshev [5]; before [8–10], it was used, in particular, in papers by Mendelev and Plotnikov [12], Mendelev [11], and Babenko [2].

3.3. Solution of problem (1.32)

The following assertion is valid for problem (1.32).

Theorem 6. *For $m \geq 1$ and $0 < h < 2$, the following equality holds:*

$$\delta_m(h) = 4 \arccos\left(\frac{h}{2}\right)^{\frac{1}{m}}. \tag{3.20}$$

Moreover, polynomials (3.12) and (3.13) are the unique (to within an arbitrary rotation) extremal polynomials in problem (1.32) for even and odd m, respectively; here, the parameters h and α are related as follows:

$$h = 2 \sin^m \frac{\alpha}{2}, \quad 0 < \alpha < \pi. \tag{3.21}$$

Proof. Let

$$P_m(z) = \prod_{j=1}^{m} \left(z - e^{i\phi_j}\right) \tag{3.22}$$

be an extremal polynomial of problem (1.32). By Theorem 2, the set

$$\{t \in \mathbb{T} : |P_m(e^{it})| \geq h\}$$

consists of $m - 1$ points and a segment; we can assume that this segment is symmetrical with respect to the point π; more precisely, it has the form $[a, 2\pi - a], 0 < a < \pi$. On the complementary segment $I^* = [-a, a]$, the inequality $|P_m(e^{it})(t)| \leq h, t \in I^* = [-a, a]$, holds or, what comes to the same, the value of the uniform norm of polynomial (3.22) on the arc $\Gamma(I^*) = e^{iI^*} = \{e^{it} : t \in I^*\}$ is equal to h:

$$\|P_m\|_{C(\Gamma(I^*))} = h. \tag{3.23}$$

Note that, in addition, all m zeros of polynomial (3.22) belong to the arc $\Gamma(I^*)$. By definition (3.10), assertion (3.11), and relation (3.23), we have

$$\varepsilon_m(2a) = 2 \sin^m \frac{a}{2} \leq h. \tag{3.24}$$

Hence, we obtain the following upper estimate for the length of the segment I^*:

$$|I^*| = 2a \leq 4 \arcsin\left(\frac{h}{2}\right)^{\frac{1}{m}}.$$

The equality $\delta_m(h) = 2\pi - |I^*|$ provides now the estimate

$$\delta_m(h) = 2\pi - 2a \geq 2\pi - 4\arcsin\left(\frac{h}{2}\right)^{\frac{1}{m}} = 4\arccos\left(\frac{h}{2}\right)^{\frac{1}{m}}. \tag{3.25}$$

Using (3.21), we represent the parameter h in terms of $\alpha \in (0, \pi)$. By properties (3.14) and (3.15), polynomials (3.12) and (3.13) provide the inverse estimate. Thus, assertion (3.20) and the property of polynomials (3.12) and (3.13) to be extremal are proved. It is seen from the proof that $a = \alpha$ and polynomial (3.22) solves problem (3.10). By Theorem 5, polynomial (3.22) coincides with (3.12) or (3.13) depending on the evenness of the number m. Theorem 6 is proved. □

3.4. The investigation of problem (3.4)

Let us return to approximation problem (3.4). Evidently, values (3.4) and (3.10) are related by the inequality

$$E_m(2\alpha) \leq \varepsilon_m(\Gamma(\alpha)) = \varepsilon_m(2\alpha) = 2\sin^m\frac{\alpha}{2}, \quad 0 < \alpha < \pi. \tag{3.26}$$

Now, we will see that, in fact, they coincide.

Theorem 7. *For any $\alpha, 0 < \alpha < \pi$, and any set $Q \in \mathcal{Q}(2\alpha)$, the following inequality is valid:*

$$E_m(Q) \geq 2\sin^m\frac{\alpha}{2};$$

it turns into an equality only in the case where the set Q is an arc (of length 2α). As a consequence,

$$E_m(2\alpha) = \varepsilon_m(2\alpha) = 2\sin^m\frac{\alpha}{2}.$$

Proof. Assume that, for a compact subset $Q \subset \Gamma$ of measure $|Q| = 2\alpha, 0 < \alpha < \pi$, the following inequality is valid:

$$E_m(Q) \leq h, \quad \text{where } h = 2\sin^m\frac{\alpha}{2}; \tag{3.27}$$

by definitions (3.3) and (3.9), an arbitrary arc of the unit circle of length 2α has this property *a fortiori*. Let us prove that, then, the set Q is certainly an arc of length 2α and inequality (3.27) turns into an equality. Thus, Theorem 7 will be proved.

Let $P_m \in \mathfrak{P}_m(\Gamma)$ be the polynomial on which an infimum in (3.3) is reached for the set Q under consideration. We use the notation $h' = \|P_m\|_{C(Q)}$; assumption (3.27) means that $h' \leq h$. Let us ascertain that the following estimate is valid for the measures of the set $\{z \in \Gamma : |P_m(z)| \geq h'\}$:

$$|\{z \in \Gamma : |P_m(z)| \geq h'\}| \leq 2\pi - 2\alpha. \tag{3.28}$$

Indeed,

$$\{z \in \Gamma : |P_m(z)| \geq h'\} = \{z \in \Gamma : |P_m(z)| > h'\} \cup \{z \in \Gamma : |P_m(z)| = h'\}.$$

The embedding $\{z \in \Gamma : |P_m(z)| > h'\} \subset \Gamma \setminus Q$ is valid; therefore,

$$|\{z \in \Gamma : |P_m(z)| > h'\}| \le 2\pi - 2\alpha.$$

Using, for instance, representation (1.27)–(1.28) of the polynomial P_m, we can easily check that the set $\{z \in \Gamma : |P_m(z)| = h'\}$ is finite and so has the measure zero. Therefore, estimate (3.28) is really valid.

By definition (1.32) and inequality (3.28), we have

$$\delta_m(h') \le |\{z \in \Gamma : |P_m(z)| \ge h'\}| \le 2\pi - 2\alpha. \tag{3.29}$$

Using (3.20), we can easily check that if $h = 2\sin^m \alpha/2$, then $2\pi - 2\alpha = \delta_m(h)$. Therefore, inequality (3.29) can be written in the form

$$\delta_m(h') \le \delta_m(h). \tag{3.30}$$

By Theorem 6, value (1.32) decreases with respect to its argument; therefore, (3.30) implies that $h' \ge h$. Taking into account the property $h' \le h$, we conclude that $h' = h$ and

$$\delta_m(h) = |\{z \in \Gamma : |P_m(z)| \ge h\}|.$$

This fact means that the polynomial P_m is extremal in problem (1.32). By Theorem 6, the polynomial P_m coincides to within a rotation with (3.12) or (3.13) depending on the evenness of the number m. Consequently, the set

$$\{z \in \Gamma : |P_m(z)| \le h\}, \quad h = 2\sin^m \frac{\alpha}{2}, \tag{3.31}$$

is an arc of length 2α. The set Q is compact; its measure is also equal to 2α; this set belongs to arc (3.31). Therefore, the set Q coincides with arc (3.31). The arguments above contain the equality $h = \|P_m\|_{C(Q)}$, which means that (3.27) turns into an equality for the set or, more precisely, the arc Q. Theorem 7 is proved. $\square$

4. The completion of the proof of Theorem 1; trigonometric polynomials deviating the least from zero with respect to the uniform norm on compact subsets of the torus that have a given measure

4.1. Proof of Theorem 1

Assertion (1.5) follows from Corollary 1 and Theorem 6, more precisely, from equalities (1.40) and (3.20). It remains to describe the set of extremal polynomials of problem (1.4). By Lemma 1, an extremal polynomial has only real roots; i.e., it belongs to the set $\mathcal{F}_n^{\text{real}}(y)$. According to the results of the first section (see (1.20) and (1.21)), a polynomial $f_n \in \mathcal{F}_n^{\text{real}}(y)$ has the representation

$$f_n(t) = \frac{y}{2} e^{-int} P_{2n}(e^{it}), \tag{4.1}$$

where

$$P_{2n}(z) = \prod_{j=1}^{2n} (z - e^{i\phi_j}) \tag{4.2}$$

is a polynomial from the set $\mathfrak{P}_{2n}(\Gamma)$ with the property

$$\Phi = \sum_{j=1}^{2n} \phi_j = 2\pi k, \quad k \in \mathbb{Z}. \tag{4.3}$$

By Corollary 1, polynomial (4.1) is extremal in problem (1.4) if and only if polynomial (4.2) is extremal in problem (1.32) for $m = 2n$ and $h = 2/y$. According to Theorem 6, such a polynomial coincides to within a rotation with polynomial (3.12) if the parameters satisfy relations (3.21).

Polynomial (3.12) has n pairs of complex-conjugate roots $e^{\pm i\phi_j}$, where $0 < \phi_j < \alpha < \pi, 1 \leq j \leq n$. For this polynomial, sum (4.3) is equal to zero; i.e., $k = 0$. Hence, on the basis of formulas (3.16), (3.18) and (4.1), we conclude that the polynomial

$$f_{n,0}(t) = \frac{y}{2} e^{-int} S_n(e^{it}) = T_n \left(y^{\frac{1}{n}} \cos t - y^{\frac{1}{n}} + 1 \right) \tag{4.4}$$

belongs to the set $\mathcal{P}_{2n}(y)$ and is extremal in problem (1.4).

The procedure of rotation (3.2) of polynomial (3.12) by a value $\theta \in \mathbb{R}$ gives the polynomial

$$P_{2n}(z) = e^{i2n\theta} S_n(ze^{-i\theta}), \tag{4.5}$$

for which sum (4.3) is equal to the number $2n\theta$. By (4.3), for the respective polynomial (4.1) to be extremal it is necessary and sufficient to have $2n\theta = 2\pi k, k \in \mathbb{Z}$, or

$$\theta = \theta_k = \frac{k\pi}{n}, \quad k \in \mathbb{Z}.$$

For this value θ, we have

$$f_n(t) = f_{n,k}(t) = \frac{y}{2} e^{-int} S_n(e^{i(t-\theta_k)}) = \frac{y}{2} e^{in\theta_k} e^{-in(t-\theta_k)} S_n(e^{i(t-\theta_k)}) = (-1)^k f_{n,0}(t - \theta_k)$$

or, by (4.4),

$$f_n(t) = f_{n,k}(t) = (-1)^k T_n \left(y^{\frac{1}{n}} \cos(t - \theta_k) - y^{\frac{1}{n}} + 1 \right). \tag{4.6}$$

Thus, we have shown that extremal polynomials of problem (1.4) are described by formula (4.6); this is precisely the assertion of Theorem 1. Theorem 1 is proved. $\quad\square$

4.2. *On trigonometric polynomials that deviate the least from zero on compact sets of a given measure*

For $0 < \alpha < \pi$, we denote by $\mathcal{T}(2\alpha)$ the set of all compact subsets Q of the torus $\mathbb{T}$ whose measure $|Q|$ is equal to the number $2\alpha : |Q| = 2\alpha$. For $n \geq 1$ and a compact subset $Q \subset \mathbb{T}$, we define the value

$$\mathcal{E}_n(Q) = \inf\{\| \cos nt - f_{n-1} \|_{C(Q)} : f_{n-1} \in \mathcal{F}_{n-1}\} \tag{4.7}$$

of the best uniform approximation of the function $\cos nt$ by the family $\mathcal{F}_{n-1}$ of trigonometric polynomials of order $n - 1$ on the set Q. We are interested in the least value

$$\mathcal{E}_n(2\alpha) = \mathcal{E}_n(\mathcal{T}(2\alpha)) = \inf\{\mathcal{E}_n(Q) : |Q| = 2\alpha\} \tag{4.8}$$

of (4.7) over all compact sets $Q \in \mathcal{T}(2\alpha)$.

Problems (4.7) and (4.8) can also be considered as analogs of problems (3.6) and (3.8) studied by Chebyshev and Polya. However, for the study of (4.7) and (4.8), other methods are applied.

Lukashov [6] gave a solution of problem (4.7) for a finite set of segments; however, terms that he used to obtain these results do not allow one to conclude anything about problem (4.8).

Problem (4.7) for the segments

$$I_k(2\alpha) = I(2\alpha) + \frac{k\pi}{n} = \left[-\alpha + \frac{k\pi}{n}, \alpha + \frac{k\pi}{n} \right], \quad k \in \mathbb{Z}, \tag{4.9}$$

which are shifts of the segment $I(2\alpha) = [-\alpha, \alpha]$, plays an important role. For these segments, a solution of problem (4.7) can be easily given.

Lemma 9. *Let $n \geq 1$ and $0 < \alpha < \pi$. Then,*

$$\mathcal{E}_n(I_k(2\alpha)) = \sin^{2n} \frac{\alpha}{2}, \quad k \in \mathbb{Z}, \tag{4.10}$$

and the polynomials

$$\widetilde{f}_{n,k}(t) = (-1)^k \left(\sin^{2n} \frac{\alpha}{2} \right) \cdot T_n \left(y^{\frac{1}{n}} \cos\left(t + \frac{\pi k}{n} \right) - y^{\frac{1}{n}} + 1 \right), \quad y = \sin^{-2n} \frac{\alpha}{2}, \tag{4.11}$$

are extremal; they differ from polynomials (1.6) only by the proper normalization.

Proof. We restrict our attention to the case $k = 0$. The polynomial

$$\widetilde{f}_n(t) = \widetilde{f}_{n,0}(t) = \left(\sin^{2n} \frac{\alpha}{2} \right) \cdot T_n \left(y^{\frac{1}{n}} \cos t - y^{\frac{1}{n}} + 1 \right), \quad y = \sin^{-2n} \frac{\alpha}{2}, \tag{4.12}$$

has the form $\widetilde{f}_n(t) = \cos nt + f_{n-1}(t)$, $f_{n-1} \in \mathcal{F}_{n-1}$; it has a $(2n + 1)$-point alternance on the segment $[-\alpha, \alpha]$. Therefore, $\mathcal{E}_n(I_k(2\alpha)) = \| \widetilde{f}_n \|_{C[-\alpha,\alpha]} = \sin^{2n} \alpha/2$. The lemma is proved. □

The following assertion contains a solution of problem (4.8).

Theorem 8. *For any $\alpha, 0 < \alpha < \pi$, for any compact subset $Q \subset \mathbb{T}$ of the torus of measure $|Q| = 2\alpha$, the following inequality is valid:*

$$\mathcal{E}_n(Q) \geq \sin^{2n} \frac{\alpha}{2}; \tag{4.13}$$

it turns into an equality only on segments (4.9). As a consequence, the following equality holds for value (4.8):

$$\mathcal{E}_n(2\alpha) = \sin^{2n} \frac{\alpha}{2}. \tag{4.14}$$

Proof. The proof of this assertion is carried out with the help of Theorem 1 by the same scheme as the proof of Theorem 7, starting from Theorem 6. Indeed, let us assume that, for a compact subset $Q \subset \mathbb{T}$ of measure $|Q| = 2\alpha$, $0 < \alpha < \pi$, the inequality $\mathcal{E}_m(Q) \leq \sin^{2n} \alpha/2$ is valid. Let $f_n(t) = \cos nt - f_{n-1}$, $f_{n-1} \in \mathcal{F}_{n-1}$, be a polynomial on which an infimum in (4.7) is reached for this set Q. We have $d' = \| f_n \|_{C(Q)} \leq d = \sin^{2n} \alpha/2$. Let us estimate the measure of the set

$$\{ t \in \mathbb{T} : |y' f_n| \geq 1 \}, \quad y' = 1/d',$$

from above. This set can be represented in the form

$$\{ t \in \mathbb{T} : |y' f_n| \geq 1 \} = \{ t \in \mathbb{T} : |y' f_n| > 1 \} \cup \{ t \in \mathbb{T} : |y' f_n| = 1 \}.$$

The embedding $\{ t \in \mathbb{T} : |y' f_n| > 1 \} \subset \mathbb{T} \setminus Q$ is valid; consequently, $|\{ t \in \mathbb{T} : |y' f_n| > 1 \}| \leq 2\pi - 2\alpha$. However, the set $\{ t \in \mathbb{T} : |y' f_n| = 1 \}$ is finite; so, $|\{ t \in \mathbb{T} : |y' f_n| = 1 \}| = 0$.

Therefore, the following estimate is valid:

$$|\{t \in \mathbb{T} : |y' f_n(t)| \geq 1\}| \leq 2\pi - 2\alpha.$$

The function $y' f_n$ belongs to the set $\mathcal{F}_n(y')$ and satisfies the following inequalities:

$$\sigma_n(y') \leq |\{t \in \mathbb{T} : |y' f_n| \geq 1\}| \leq 2\pi - 2\alpha = \sigma_n(y), \quad y = 1/d = \sin^{-2n} \frac{\alpha}{2}. \tag{4.15}$$

By Theorem 1, value (1.4) increases with respect to its argument. Since $y' \geq y$, (4.15) implies that $y' = y$ and

$$\sigma_n(y) = |\{t \in \mathbb{T} : |y f_n(t)| \geq 1\}| = 2\pi - 2\alpha. \tag{4.16}$$

This fact means that the polynomial $y f_n$ is extremal in problem (1.4). By Theorem 1, the polynomial $y f_n$ coincides with one of polynomials (1.6). Consequently, the set

$$\{t \in \mathbb{T} : |y f_n(t)| \geq 1\} = \{t \in \mathbb{T} : |f_n(t)| \geq d\}, \quad d = \sin^{2n} \frac{\alpha}{2}, \tag{4.17}$$

is one of the segments (4.9) of length 2α. The set Q is compact; its measure is also equal to 2α; this set belongs to segment (4.17). Consequently, Q coincides with segment (4.17); i.e., it coincides with one of the segments (4.9). In this case, inequality (4.13) turns into an equality. Theorem 7 is proved. $\quad\square$

As a consequence of Theorems 1 and 8, the following analog of Theorem 3 is valid.

Corollary 3. *Problems* (1.4) *and* (4.8) *are related as follows:*

$$\mathcal{E}_n(2\alpha) = y^{-1}, \quad 2\alpha = 2\pi - \sigma_n(y), \ y > 1. \tag{4.18}$$

4.3. The sharp constant in inequality (1.7)

Theorem 1 allows us to find a value of the best constant β_n in inequality (1.7).

Theorem 9. *For any* $n \geq 1$*, the following formula is valid for the best constant* β_n *in inequality* (1.7)*:*

$$\beta_n = \sqrt{2n}. \tag{4.19}$$

Proof. The leading harmonic of trigonometric polynomial (1.1) can be written in the form $a_n \cos nt + b_n \sin nt = y \cos(nt + t_n)$, where $y = y(f_n) = \sqrt{a_n^2 + b_n^2}$, and t_n is the respective shift of the argument. Functional (1.2) is invariant with respect to a shift of argument of the polynomial; hence, we can assume that $y \cos nt$ is the leading harmonic of the polynomial f_n; i.e., $f_n \in \mathcal{F}_n(y)$. Studying inequality (1.7), we have to restrict our attention to only polynomials with $y = y(f_n) > 1$; in addition, it is reasonable to choose lower harmonics of the polynomial so that functional (1.2) will have the least value. As a result, we arrive at the following representation of the best constant β_n in inequality (1.7):

$$\beta_n = \sup_{y > 1} \frac{\mu(y \cos(nt))}{\sigma_n(y)}.$$

We have

$$\mu(y \cos(nt)) = 4 \arccos \frac{1}{y}.$$

Now, applying Theorem 1, we obtain

$$\beta_n = \sup_{y>1} \frac{4 \arccos \frac{1}{y}}{4 \arccos \frac{1}{y^{\frac{1}{2n}}}} = \sup_{0 \le t < 1} \frac{\arccos t^{2n}}{\arccos t} = \lim_{t \to 1-0} \frac{\arccos t^{2n}}{\arccos t} = \sqrt{2n}.$$

Assertion (4.19) is proved. $\square$

Remark. Problems (1.3), (4.7) and (4.8) for the leading harmonic of the general form $A \cos nt + B \sin nt$ are reduced to the case $y \cos nt$ considered in this paper by an appropriate change of variable.

Acknowledgments

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