Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 162 (2010) 1204–1210
Journal of Approximation Theory
www.elsevier.com/locate/jat

Necessary conditions for metrics in integral Bernstein-type inequalities ☆

Polina Yu. Glazyrina

Department of Mathematics and Mechanics, Ural State University, Lenin Avenue, 51, Ekaterinburg, 620083, Russia

Received 23 February 2009; accepted 28 December 2009
Available online 6 January 2010

Communicated by Paul Nevai

Abstract

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n. Denote by Φ^+ the class of all functions φ: $(0,\infty) \to \mathbb{R}$ of the form $\varphi(u) = \psi(\ln u)$, where ψ is nondecreasing and convex on $(-\infty,\infty)$. In 1979, Arestov extended the classical Bernstein inequality $\|T_n'\|_C \le n\|T_n\|_C$, $T_n \in \mathcal{T}_n$, to metrics defined by $\varphi \in \Phi^+$:

$$\int_0^{2\pi} \varphi(|T_n'(t)|)\mathrm{d}t \le \int_0^{2\pi} \varphi(n|T_n(t)|)\mathrm{d}t, \quad T_n \in \mathcal{T}_n.$$

We study the question whether it is possible to extend the class Φ^+, and prove that under certain assumptions Φ^+ is the largest possible class.

Keywords: Algebraic polynomials; Trigonometric polynomials; Bernstein-type inequalities; Integral inequalities

1. Introduction

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n with complex coefficients. The inequality

$$\|T_n'\|_C \le n\|T_n\|_C, \quad T_n \in \mathcal{T}_n, \tag{1}$$

☆ This work was supported by the Russian Foundation for Basic Research (Project No. 08-01-00213) and by the Program for State Support of Leading Scientific Schools of the Russian Federation (Project No. NSh-1071.2008.1).
E-mail address: polina.glazyrina@usu.ru.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2009.12.009

is well known in approximation theory and is called the Bernstein inequality. Inequality (1) turns into equality iff $T_n(t) = a\cos nt + b\sin nt$, where $a, b \in \mathbb{C}$. The inequality was stated by Bernstein and Landau for polynomials with real coefficients (for details, see [5, Section 10, pp. 25–26; Section 3.4, p. 527], [8, Ch. 6, Theorems 1.2.4, 1.2.5]) in 1912–1914 and by Riesz for polynomials with complex coefficients ([10], [11, Vol. 2, Ch. 10]) in 1914.

We say that a function φ is increasing on an interval I if $\varphi(u_1) \le \varphi(u_2)$ for all $u_1 \le u_2$, $u_1, u_2 \in I$; φ is convex on I if $\varphi\big(\alpha u_1 + (1-\alpha)u_2\big) \le \alpha\varphi(u_1) + (1-\alpha)\varphi(u_2)$ for all $u_1, u_2 \in I$ and $\alpha \in [0, 1]$; φ is concave on I if $-\varphi$ is convex on I.

In 1933, Zygmund [11, Vol. 2, Ch. 10, (3.25)] proved the following statement. If φ is an increasing and convex function on $[0, \infty)$, then

$$\int_0^{2\pi} \varphi\big(|T_n'(t)|\big)\,\mathrm{d}t \le \int_0^{2\pi} \varphi\big(n|T_n(t)|\big)\,\mathrm{d}t, \qquad T_n \in \mathcal{T}_n. \tag{2}$$

For $\varphi(u) = u^p$, $p \ge 1$, inequality (2) implies the Bernstein inequality in the space L_p:

$$\|T_n'\|_p \le n\|T_n\|_p, \qquad T_n \in \mathcal{T}_n,$$

where $\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,\mathrm{d}t\right)^{1/p}$.

In 1979, Arestov [1–3] found weaker conditions on functions φ which provide the validity of inequality (2). Before we give Arestov's result, we introduce some notation [2,4].

We denote by Φ^+ the class of functions φ defined on $(0, \infty)$ with the following properties:

(i) φ is locally absolutely continuous;
(ii) φ increases on $(0, \infty)$;
(iii) $u\varphi'(u)$ increases on $(0, \infty)$.

Put $\psi(v) = \varphi(\mathrm{e}^v)$; that is, $\varphi(u) = \psi(\ln u)$. Clearly, φ belongs to Φ^+ iff the function ψ is increasing and convex on $(-\infty, \infty)$. For example, all increasing convex functions, the functions $\ln u$, $\ln^+ u = \max\{0, \ln u\}$, $\ln(1 + u^p)$, and u^p, $p > 0$, belong to Φ^+.

We denote by $\mathcal{P}_n$ the set of all algebraic polynomials of degree at most n with complex coefficients. Let polynomials Λ_n and P_n from $\mathcal{P}_n$ be given by $\Lambda_n(z) = \sum_{k=0}^n \binom{n}{k}\lambda_k z^k$ and $P_n(z) = \sum_{k=0}^n \binom{n}{k} c_k z^k$. The polynomial

$$\Lambda_n P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k c_k z^k \tag{3}$$

is called the composition of Λ_n and P_n (for details, see [9, Vol. 2, Section 5]). Suppose that Λ_n is fixed, then Eq. (3) defines a linear operator on $\mathcal{P}_n$, which we denote by the same symbol Λ_n. For example, if $\Lambda_n(z) = (1 + \mathrm{e}^{\mathrm{i}\theta}z)^n$, $\theta \in \mathbb{R}$, then $(\Lambda_n P_n)(z) = P_n(\mathrm{e}^{\mathrm{i}\theta}z)$ is the operator of rotation by angle θ; in particular, $\Lambda_n(z) = (1+z)^n$ defines the identity operator. The polynomial $\Delta_n(z) = \frac{n}{2}(1+z)^{n-1}(z-1)$ defines the differential operator

$$(\Delta_n P_n)(z) = zP_n'(z) - \frac{n}{2}P_n(z).$$

In the sequel, if $P_n \in \mathcal{P}_n$ has degree $m < n$, then we say that $z = \infty$ is a zero of P_n with multiplicity $n - m$. Let $\mathcal{P}_n^0$ be the set of all polynomials $P_n \in \mathcal{P}_n$ such that all n zeros of P_n lie in the unit disk $|z| \le 1$, and let $\mathcal{P}_n^\infty$ be the set of all polynomials $P_n \in \mathcal{P}_n$ such that all zeros of P_n lie in the domain $|z| \ge 1$. Furthermore, we say that an operator Λ_n belongs to the class

Ω_n^0 if $\Lambda_n \mathcal{P}_n^0 \subset \mathcal{P}_n^0$, and that Λ_n belongs to the class Ω_n^∞ if $\Lambda_n \mathcal{P}_n^\infty \subset \mathcal{P}_n^\infty$. Using Theorems 151 and 152 from [9, Section 5] (see also [2]), one can easily prove that $\Lambda_n \in \Omega_n^0$ iff the polynomial $\Lambda_n \in \mathcal{P}_n^0$, and that $\Lambda_n \in \Omega_n^\infty$ iff the polynomial $\Lambda_n \in \mathcal{P}_n^\infty$. Finally, let $\Omega_n = \Omega_n^0 \cup \Omega_n^\infty$.

Theorem A (*Arestov [2]*)**.** *If $\varphi \in \Phi^+$ and $\Lambda_n \in \Omega_n$, then, for all $P_n \in \mathcal{P}_n$,*

$$\int_0^{2\pi} \varphi\big(|\Lambda_n P_n(\mathrm{e}^{\mathrm{i}t})|\big)\,\mathrm{d}t \le \int_0^{2\pi} \varphi\big(C(\Lambda_n)|P_n(\mathrm{e}^{\mathrm{i}t})|\big)\,\mathrm{d}t, \tag{4}$$

where $C(\Lambda_n) = \max\big\{|\lambda_0|, |\lambda_n|\big\}$. Equality holds in (4) if and only if P_n has the form

$$P_n(z) = az^n, \qquad P_n(z) \equiv a, \quad \text{or} \quad P_n(z) = az^n + b \quad (a, b \in \mathbb{C}),$$

depending on whether

$$\Lambda_n \in \Omega_n^0, \qquad \Lambda_n \in \Omega_n^\infty, \quad \text{or} \quad \Lambda_n \in \Omega_n^0 \cap \Omega_n^\infty.$$

The space $\mathcal{T}_n$ can be identified with the space $\mathcal{P}_{2n}$ by the mapping $T_n(t) = \mathrm{e}^{-\mathrm{i}nt} P_{2n}(\mathrm{e}^{\mathrm{i}t})$, $P_{2n} \in \mathcal{P}_{2n}$; moreover,

$$|T_n(t)| = |P_{2n}(\mathrm{e}^{\mathrm{i}t})|, \qquad |T_n'(t)| = |(\Delta_{2n} P_{2n})(\mathrm{e}^{\mathrm{i}t})|.$$

Note that $\Delta_{2n} \in \Omega_{2n}^0 \cap \Omega_{2n}^\infty$ and $C(\Delta_{2n}) = n$. Hence, inequality (2) is a consequence of Theorem A.

Professor Arestov asked the author whether it is possible to extend the class Φ^+ in Theorem A. In this paper we prove that, under certain assumptions, Φ^+ is the largest possible class.

2. Main result

We study inequality (4) for the class $\Phi = \Phi_n$ of functions φ defined on $(0, \infty)$ with the following properties:

(i) φ is continuous on $(0, \infty)$;
(ii) φ increases on $(0, \infty)$;
(iii) for all $P_n \in \mathcal{P}_n$, $\int_0^{2\pi} \varphi\big(|P_n(\mathrm{e}^{\mathrm{i}t})|\big)\,\mathrm{d}t < \infty$.

An example of the function $\varphi(u) = -\exp(1/u)$ shows that the third condition cannot be removed.

Now we will introduce a class $\Phi^- \subset \Phi$ with the property that, for every $\varphi \in \Phi^-$, inequality (4) is not satisfied (as will be stated in Theorem 1).

Definition. Denote by Φ^- the set of all functions $\varphi(u) = \psi(\ln u)$, where $\varphi \in \Phi$, and there exist points $v_1 < v_* < v_2$ and a real number k such that the function

$$\psi(v) - k \cdot v$$

(i) increases on $[v_1, v_*]$ and decreases on $[v_*, v_2]$,
(ii) does not coincide with a constant in any neighborhood of the point v_*.

Remark 1. Let us clarify this definition. Suppose that $\varphi \notin \Phi^+$ and the corresponding function ψ has a locally absolutely continuous derivative ψ' everywhere. Then $\psi''(v_*) < 0$ for some v_*. Hence, there exist points $v_1 < v_* < v_2$ such that

$$\begin{aligned} &\psi'(v) > \psi'(v_*), \quad v \in [v_1, v_*], \\ &\psi'(v) < \psi'(v_*), \quad v \in [v_*, v_2]. \end{aligned} \tag{5}$$

Furthermore, for the function ψ we have the representation

$$\psi(v) - \psi'(v_*)(v - v_*) = \psi(v_*) + \int_{v_*}^{v} \big(\psi'(\eta) - \psi'(v_*)\big)\, \mathrm{d}\eta.$$

It follows from (5) that the function $\psi(v) - \psi'(v_*)(v - v_*)$ increases on $[v_1, v_*]$ and decreases on $[v_*, v_2]$. Therefore, φ belongs to Φ^-.

Thus, if $\varphi \in \Phi$ has a locally absolutely continuous derivative on $(0, \infty)$, then either $\varphi \in \Phi^+$ or $\varphi \in \Phi^-$.

Remark 2. If ψ is strictly concave on some interval $[v_1, v_2]$, then $\varphi \in \Phi^-$.

Remark 3. Let us give two examples of functions from Φ^-. For the function $\varphi(u) = u/(1+u)$, by means of which convergence in measure can be defined [6, Ch. 4, Ex. 4.7.60°], the corresponding function $\psi(v) = \mathrm{e}^v/(1 + \mathrm{e}^v)$ is concave on $[0, \infty)$ and, therefore, $\varphi \in \Phi^-$.

Let $C_0(v)$, $v \in [0, 1]$, be the Cantor function [6, Ch. 3, Prop. 3.6.5], and let $[v]$ denote the integer part of v. The singular function φ defined by $\varphi(\mathrm{e}^v) = C_0(v - [v]) + [v]$ also belongs to Φ^-.

Remark 4. It is sufficient to consider only one of the following two cases: $\Lambda_n \in \Omega_n^0$ or $\Lambda_n \in \Omega_n^\infty$. Indeed, applying the methods of de Bruijn and Springer [7] and Arestov [3], consider the map $I = I_n$ on $\mathcal{P}_n$ defined by

$$(IP_n)(z) = z^n P_n(1/z), \quad P_n \in \mathcal{P}_n.$$

It is clear that $|P_n(\mathrm{e}^{\mathrm{i}t})| = |(IP_n)(\mathrm{e}^{-\mathrm{i}t})|$, $t \in [0, 2\pi]$, $P_n \in \mathcal{P}_n$, and

$$\left|(\Lambda_n P_n)(\mathrm{e}^{\mathrm{i}t})\right| = \left|(I(\Lambda_n P_n))(\mathrm{e}^{-\mathrm{i}t})\right| = \left|\big((I\Lambda_n)(IP_n)\big)(\mathrm{e}^{-\mathrm{i}t})\right|, \quad \Lambda_n \in \Omega_n.$$

Moreover, the map I is a bijection of $\mathcal{P}_n^\infty$ onto $\mathcal{P}_n^0$. Therefore, if, say, $\Lambda_n \in \Omega_n^\infty$, then $I\Lambda_n \in \Omega_n^0$. Thus, inequality (7) is valid for an operator Λ_n and a polynomial P_n iff it is valid for $I\Lambda_n$ and IP_n.

The polynomial $\Lambda_n(z) = c(1 + \mathrm{e}^{\mathrm{i}\theta} z)^n$ defines on $\mathcal{P}_n$ the operator

$$(\Lambda_n P_n)(z) = c P_n(\mathrm{e}^{\mathrm{i}\theta} z), \quad c \in \mathbb{C}, \theta \in \mathbb{R}. \tag{6}$$

For this operator, inequality (4) turns into equality for every $P_n \in \mathcal{P}_n$, and so operators (6) are excluded from the further consideration.

Theorem 1. *If $\varphi \in \Phi^-$, $\Lambda_n \in \Omega_n$, and Λ_n is not of the form (6), then there exists a polynomial $P_n \in \mathcal{P}_n$ such that*

$$\int_0^{2\pi} \varphi\Big(|\,(\Lambda_n P_n)\,(\mathrm{e}^{\mathrm{i}t})|\Big)\, \mathrm{d}t > \int_0^{2\pi} \varphi\Big(C(\Lambda_n)|P_n(\mathrm{e}^{\mathrm{i}t})|\Big)\, \mathrm{d}t, \tag{7}$$

where $C(\Lambda_n) = \max\big\{|\lambda_0|, |\lambda_n|\big\}$.

Proof. In view of Remark 4, it is sufficient to prove the theorem for

$$\Lambda_n \in \Omega_n^0, \quad \Lambda_n(z) \neq c(1 + e^{i\theta} z)^n, \quad c \in \mathbb{C},\ \theta \in \mathbb{R}. \tag{8}$$

Without loss of generality, we can assume that $\lambda_n = 1$. We claim that $|\lambda_0| \leq 1$ and $|\lambda_{n-1}| < 1$. Indeed, by conditions (8), Λ_n has n zeros according to multiplicity $z_1, \ldots, z_n$ and all the zeros lie on the unit circle. Consequently,

$$|\lambda_0| = |z_1 \cdots z_n| \leq 1, \qquad |\lambda_{n-1}| = \left| \frac{1}{n}(z_1 + \cdots + z_n) \right| \leq 1.$$

The last inequality turns into equality only if $z_1 = \cdots = z_n = e^{i\theta}$ for some $\theta \in \mathbb{R}$, but then Λ_n is an operator of the form (6) and we do not consider such operators. Consequently, under our assumptions, $C(\Lambda_n) = \max\{|\lambda_0|, |\lambda_n|\} = 1$, and we must prove that there exists a polynomial $P \in \mathcal{P}_n$ such that

$$\int_0^{2\pi} \varphi\left(|\Lambda_n P(e^{it})|\right) dt - \int_0^{2\pi} \varphi\left(|P(e^{it})|\right) dt > 0. \tag{9}$$

Suppose that $\varphi(u) = \psi(\ln u)$, points $v_1 < v_* < v_2$ and a constant k satisfy conditions (i) and (ii) of the definition of the class Φ^-. Consider the function

$$\widetilde{\varphi}(u) = \varphi(u) - k \cdot \ln u = \psi(\ln u) - k \cdot \ln u,$$

and set $u_1 = e^{v_1}$, $u_2 = e^{v_2}$, $u_* = e^{v_*}$. Clearly, $\widetilde{\varphi}$ increases on $[u_1, u_*]$, decreases on $[u_*, u_2]$, and does not coincide with a constant in any neighborhood of the point u_*.

Let us construct a polynomial $P \in \mathcal{P}_n$ that satisfies (9) in the form

$$P(z) = mz^{n-1}(z - a), \quad a \in (0, 1), \quad m > 0.$$

We have $\Lambda_n P(z) = m(z^n - \lambda_{n-1} a z^{n-1}) = mz^{n-1}(z - \lambda_{n-1} a)$,

$$\int_0^{2\pi} \left|\Lambda_n P(e^{it})\right| dt = \int_0^{2\pi} m \left|e^{it} - \lambda_{n-1} a\right| dt = \int_0^{2\pi} m \left|e^{it} - |\lambda_{n-1}| a\right| dt.$$

Let $Q(e^{it}) = m(e^{it} - |\lambda_{n-1}| a)$; then inequality (9) is equivalent to the inequality

$$\int_0^{2\pi} \left[\varphi\left(|Q(e^{it})|\right) - \varphi\left(|P(e^{it})|\right)\right] dt > 0. \tag{10}$$

Let us compare $\left|P(e^{it})\right|^2$ and $\left|Q(e^{it})\right|^2$ on the interval $[0, 2\pi]$. We have

$$\begin{aligned} \left|P(e^{it})\right|^2 &= m^2(1 + a^2 - 2a\cos t), \\ \left|Q(e^{it})\right|^2 &= m^2\left(1 + |\lambda_{n-1}|^2 a^2 - 2|\lambda_{n-1}| a \cos t\right), \end{aligned} \tag{11}$$

and, consequently,

$$\begin{aligned} \frac{1}{m^2}\left(\left|P(e^{it})\right|^2 - \left|Q(e^{it})\right|^2\right) &= 1 + a^2 - 2a\cos t - 1 - |\lambda_{n-1}|^2 a^2 + 2|\lambda_{n-1}| a \cos t \\ &= a\big(1 - |\lambda_{n-1}|\big)\big(a + |\lambda_{n-1}| a - 2\cos t\big). \end{aligned} \tag{12}$$

Let $t_* = \arccos\big((a + |\lambda_{n-1}|a)/2\big)$. Evidently, $t_* \in (0, \pi)$, and it can be verified easily that

$$|P(e^{it_*})| = |Q(e^{it_*})| = m\sqrt{1 - |\lambda_{n-1}|a^2}. \tag{13}$$

It follows from (11) that the absolute values $|P(e^{it})|$ and $|Q(e^{it})|$ are even functions of t that are increasing on $[0, \pi]$; by (12),

$$|Q(e^{it})| > |P(e^{it})|, \quad t \in [0, t_*), \quad \text{and} \quad |Q(e^{it})| < |P(e^{it})|, \quad t \in (t_*, \pi]. \tag{14}$$

Thus, we conclude that the values $\left|Q(e^{it})\right|$ and $\left|P(e^{it})\right|$ belong to the interval $[|P(1)|, |P(-1)|]$ for all $t \in [0, 2\pi]$ and

$$|P(1)| = m(1 - a), \quad |P(-1)| = m(1 + a). \tag{15}$$

Now, we choose parameters m and a such that

$$\left|P(e^{it_*})\right| = \left|Q(e^{it_*})\right| = u_* \quad \text{and} \quad \big[|P(1)|, |P(-1)|\big] \subset [u_1, u_2]. \tag{16}$$

This can be done the following way. Let a_k be a sequence such that $a_k \to +0$, $k \to \infty$. Define m_k by

$$m_k\sqrt{1 - |\lambda_{n-1}|a_k^2} = u_*.$$

Then $m_k \to u_*$, $k \to \infty$. Therefore,

$$m_k(1 - a_k) \to u_* > u_1, \quad \text{and} \quad m_k(1 + a_k) \to u_* < u_2.$$

Thus we can take $a = a_k$ and $m = m_k$ for a sufficiently large value of k.

Combining (14) and (16), we conclude that

$$\begin{aligned} &u_1 \le |P(1)| < |P(e^{it})| < |Q(e^{it})| < u_*, \quad t \in (0, t_*), \\ &u_* < |Q(e^{it})| < |P(e^{it})| < |P(-1)| \le u_2, \quad t \in (t_*, \pi). \end{aligned} \tag{17}$$

It remains to verify inequality (10) for the constructed polynomial P. By the well-known Jensen formula (see, for example, [9, Section 3, Problem 175]),

$$\int_0^{2\pi} \ln|P(e^{it})|\,dt = \int_0^{2\pi} \ln|m(e^{it} - a)|\,dt = 2\pi \ln m,$$
$$\int_0^{2\pi} \ln|Q(e^{it})|\,dt = \int_0^{2\pi} \ln|m(e^{it} - |\lambda_{n-1}|a)| = 2\pi \ln m.$$

Thus,

$$\begin{aligned} &\int_0^{2\pi} \left[\varphi\left(|Q(e^{it})|\right) - \varphi\left(|P(e^{it})|\right)\right] dt \\ &= \int_0^{2\pi} \left[\varphi\left(|Q(e^{it})|\right) - \varphi\left(|P(e^{it})|\right) - k\ln|Q(e^{it})| + k\ln|P(e^{it})|\right] dt \\ &= 2\int_0^{\pi} \left[\widetilde{\varphi}\left(|Q(e^{it})|\right) - \widetilde{\varphi}\left(|P(e^{it})|\right)\right] dt \\ &= 2\int_0^{t_*} \left[\widetilde{\varphi}\left(|Q(e^{it})|\right) - \widetilde{\varphi}\left(|P(e^{it})|\right)\right] dt + 2\int_{t_*}^{\pi} \left[\widetilde{\varphi}\left(|Q(e^{it})|\right) - \widetilde{\varphi}\left(|P(e^{it})|\right)\right] dt. \end{aligned}$$

Relations (17) yield that the last expression is greater than 0. This completes the proof of the theorem. □

Corollary 1. *For any $\varphi \in \Phi^-$, there exists $T_n \in \mathcal{T}_n$ such that*

$$\int_0^{2\pi} \varphi\left(|T_n'(t)|\right) \, \mathrm{d}t > \int_0^{2\pi} \varphi\left(n|T_n(t)|\right) \, \mathrm{d}t.$$

For smooth functions $\varphi \in \Phi$, Arestov's theorem and Theorem 1 give the necessary and sufficient conditions on φ for validity of inequality (4).

Corollary 2. *Suppose that an operator $\Lambda_n \in \Omega_n$ is not of the form (6) and a function $\varphi \in \Phi$ has a locally absolutely continuous derivative. Then inequality (4) is valid if and only if $\varphi \in \Phi^+$.*

The proof immediately follows from Theorem A, Remark 1, and Theorem 1.

Acknowledgments

The author thanks Professor V.V. Arestov and Professor E.E. Berdysheva for useful discussions and the referees for many valuable suggestions.

References

[1] V.V. Arestov, S.N. Bernšteĭn inequalities for algebraic and trigonometric polynomials, Dokl. Akad. Nauk SSSR 246 (6) (1979) 1289–1292 (in Russian). Translation in Soviet Math. Dokl. 20 (3) (1979) 600–603.
[2] V.V. Arestov, Integral inequalities for trigonometric polynomials and their derivatives, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1) (1981) 3–22 (in Russian). Translation in Math. USSR-Izv. 18 (1982) 1–18.
[3] V.V. Arestov, Integral inequalities for algebraic polynomials on the unit circle, Mat. Zametki 48 (4) (1990) 7–18 (in Russian). Translation in Math. Notes 48 (4) (1990) 977–984.
[4] V.V. Arestov, Inequalities of Bernstein and Szegö for trigonometric polynomials, Izv. Ural. Gos. Univ. Ser. Mat. Mekh. Inform. 58 (2008) 43–58 (in Russian).
[5] S.N. Bernstein, Collected Works. Vol. I. The Constructive Theory of Functions [1905–1930], Izdat. Akad. Nauk SSSR, Moscow, 1952 (in Russian).
[6] V.I. Bogachev, Measure Theory, Vol. I, Springer-Verlag, Berlin, 2007.
[7] N.G. de Bruijn, T.A. Springer, On the zeros of composition-polynomials, Indag. Math. 9 (1947) 406–414.
[8] G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
[9] G. Pólya, G. Szegö, Problems and Theorems in Analysis, Vol. II, Springer-Verlag, New York–Heidelberg, 1976.
[10] M. Riesz, Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome, Deutsche Math. Ver. 23 (1914) 354–368.
[11] A. Zygmund, Trigonometric Series, Vol. II, Cambridge University Press, New York, 1959.