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State Estimation for Linear Stochastic Differential Equations with Uncertain Disturbances via BSDE Approach

B. I. Ananyev

IMM UB of RAS, Ural Federal University, 16 Kovalevskaya str., 620990, Yekaterinburg, Russia

Abstract. A backward stochastic differential equation (BSDE) is an Ito stochastic differential equation (SDE) for which a random terminal condition on the state has been specified. The paper deals with estimation problems for partly observed stochastic processes described by linear SDEs with uncertain disturbances. The disturbances and unknown initial states are supposed to be constrained by the inequality including mathematical expectation of the integral quadratic cost. We consider our equations as BSDEs, and construct at given instant the random information set of all possible states which are compatible with the measurements and the constraints. The center of this set represents the best estimation of the process' state. The evolutionary equations for the random information set and for the best estimation are given. Some examples and applications are considered.

Keywords: BSDE approach, random information sets, optimal estimate
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INTRODUCTION

In the traditional stochastic filter model, state dynamics are governed by white-noise driven stochastic differential equations, and observation errors are also modeled as white noises. The objective is to minimize an expected mean squared estimation error criterion. For linear-gaussian filter models, the Kalman-Bucy filter provides a highly successful solution. This filter involves only the solution off-line of a matrix Riccati differential equation and the solution on-line of a linear stochastic differential equation for the estimate, Fleming and Rishel [1], Liptser and Shiryayev [2]. For nonlinear stochastic filter models, the expected mean squared estimation error depends on the conditional distribution of the state at time t given past observations for $0 < s < t$.

Nonlinear filtering theory reduces the problem to the solution of the Zakai stochastic PDE for an unnormalized version of the conditional density, followed by integrations, Kunita [3]. Except in the lowest state dimensions this is a hard computational task. In engineering practice, extended Kalman filters are frequently used instead. The extended Kalman filter approximation has been rigorously justified in some special cases, in which observation noise is of low intensity, Picard [4].

Instead of the traditional stochastic filter model, we consider in this paper a combined model. Errors in the state dynamics are modeled by unknown random functions and Brownian motion whereas observation errors are modeled only as unknown random functions (called disturbances) rather than as white noises. The unnormalized conditional density is replaced by a certain "information state" function $V(t,x)$, where $V(t,x)$ is defined by minimizing a least squared disturbance error criterion. The function V evolves either via Pontryagin's maximum principle, or according to the first-order PDE of Hamilton-Jacobi-Bellman type in the determinate case, and is interpreted as a viscosity sense solution. In this way, we interpret the system as BSDE and solve a stochastic backward linear-quadratic (BLQ) observation problem of special type with the help of stochastic maximum principle. BSDEs have received considerable research attention in recent years due to their nice structure and wide applicability in a number of different areas, especially in mathematical finance. A deterministic analog of our problem was considered in Bertsecas and Rhodes [5], Mortensen [6], Kurzhanski [7], Kurzhanski and Varaiya [8, 9], Schmitendorf [10], Schweppe [11]. An approach used in this paper may be applied to a more general nonlinear problem.

PROBLEM FORMULATION

We assume throughout that $(\Omega,\mathscr{F},\mathscr{F}_t,P)$ is a given and fixed complete filtered probability space and that $W(\cdot)$ is a scalar-valued Brownian motion on this space, $W(0)=0$, $\mathscr{F}_t=\sigma\{W(s):0\le s\le t\}$. Suppose that $\mathscr{F}_t$ contains all the

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P-null sets of $\mathscr{F}$.

Let us introduce the notation. The set of symmetric $n \times n$ matrices with real elements is denoted by S^n. We write $M > (\geq) 0$, if M is positive (semi)definite and $M \in S^n$. Let X be a given Hilbert space. By $C(0,T;X)$ we denote the set of X-valued continuous functions. We write $\eta \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$ if $E|\eta|^2 < \infty$ and random element η is $\mathscr{F}_T|\mathscr{B}(\mathbb{R}^n)$ measurable. Consider now a $(\mathscr{B}([0,T]) \times \mathscr{F})|\mathscr{B}(\mathbb{R}^n)$ measurable function $f(t,\omega)$, which is $\mathscr{F}_t$ adapted for every $t \in [0,T]$. If besides $E\int_0^T |f(t)|^2 dt < \infty$, we shall write $f \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$. If f is uniformly bounded, we write $f \in L^\infty_{\mathscr{F}}(0,T;\mathbb{R}^n)$. If f has P-a.s. continuous sample paths and $E\sup_{t\in[0,T]}|f(t)|^2 < \infty$, we write $f \in L^2_{\mathscr{F}}(0,T;C(0,T;\mathbb{R}^n))$. It is clear what the inclusion $f \in L^\infty_{\mathscr{F}}(0,T;C(0,T;\mathbb{R}^n))$ means.

Consider the BSDE

$$\begin{gathered} dx_t = (A(t)x_t + B(t)v(t) + C(t)z(t))dt + z(t)dW(t), \\ y_t = G(t)x_t + w(t), \quad x_T = \xi, \end{gathered} \tag{1}$$

where $y_t \in \mathbb{R}^m$ is an observation process and $\xi \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$ is a terminal state. In (1), the functions $v \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^k)$, $z \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$, $w \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$, and the initial state x_0 are supposed to be uncertain and constrained by the inequality

$$J(T,\xi,v) = E\Big[x_0' H x_0 + \int_0^T (w'(t)Q(t)w(t) + z'(t)S(t)z(t) + v'(t)R(t)v(t))dt\Big] \leq 1, \tag{2}$$

where "$'$" means the transposition. We shall assume the following:

Assumption (A)**.**

$$\begin{gathered} A, C \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B \in L^\infty(0,T;\mathbb{R}^{n\times k}), \quad Q \in L^\infty(0,T;S^m),\ Q \geq 0, \\ S \in L^\infty(0,T;S^n), S \geq 0, \quad R \in L^\infty(0,T;S^k), R > 0, \quad H \in S^n, H \geq 0. \end{gathered} \tag{3}$$

Assumption (A) will guarantee the existence of a unique solution pair $(x(\cdot),z(\cdot)) \in L^2_{\mathscr{F}}(0,T;C(0,T;\mathbb{R}^n)) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ of (1) for every admissible disturbance $v(\cdot) \in \mathscr{V} = L^2_{\mathscr{F}}(0,T;\mathbb{R}^k)$. This result is proved, for example, in Yong and Zhou [12, Chapter 7]. The presence of the additional term $z(t)dW(t)$ in (1) makes the backward problem correct.

The *BLQ observation problem* can be stated as follows:

$$\begin{gathered} \text{Find the function } V(T,\xi,y(\cdot)) = \min_v J(T,\xi,v) \\ \text{subject to } v \in \mathscr{V},\ \text{where } (x(\cdot),z(\cdot),v(\cdot)) \text{ satisfies (1).} \end{gathered} \tag{4}$$

Knowing the function $V(T,\xi,y(\cdot))$ we can introduce the *random information set* $\mathscr{X}(T,y(\cdot)) = \{\xi \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n) : V(T,\xi,y(\cdot)) \leq 1\}$, which contains all the terminate states compatible with measurements, and investigate its dynamics with varying T. For multistage systems it was done in Ananyev [13]. The approximations for information sets via ellipsoid calculous in the deterministic case were suggested in Kurzhanski and Vályi [14], Chernousko [15].

Let us formulate the stochastic maximum principle for BSDEs (Dokuchaev and Zhou [16]), which we use further. Consider the following control problem for BSDE:

$$\begin{gathered} \text{Minimize } J(u(\cdot)) = E\Big[g(x(0)) + \int_0^T \varphi(t,x_t,z(t),u(t))dt\Big], \\ \text{subject to } dx_t = f(t,x_t,z(t),u(t))dt + z(t)dW(t),\ x_T = \xi. \end{gathered}$$

It is supposed that functions g, f and φ are smooth enough. Consider the forward adjoint Itô equation

$$\begin{gathered} d\psi_t = -\Big\{\Big(\frac{\partial f}{\partial x}\Big)'_0(t)\psi_t + \Big(\frac{\partial \varphi}{\partial x}\Big)'_0(t)\Big\}dt - \Big\{\Big(\frac{\partial f}{\partial z}\Big)'_0(t)\psi_t + \Big(\frac{\partial \varphi}{\partial z}\Big)'_0(t)\Big\}dW(t), \\ \psi_0 = -\frac{\partial g}{\partial x}(x_0^0), \end{gathered} \tag{5}$$

and introduce the Hamiltonian

$$H(t,u) = -E[\psi'(t)f(t,x_t^0,z^0(t),u) + \varphi(t,x_t^0,z^0(t),u)|\mathscr{F}_t].$$

Theorem 1 (Dokuchaev, Zhou)**.** *If u^0 is optimal, the following equality holds:*

$$H(t,u^0(t,\omega),\omega) = \max_u H(t,u,\omega),\ a.e.\ t \in [0,T],\ P\text{-}a.s. \tag{6}$$

GENERAL SOLUTION

Let us first present a solution of the *deterministic* BLQ observation problem. This corresponds to $\xi \in \mathbb{R}^n$ being deterministic, $C = 0$, $S = 0$, and $v(\cdot) \in \mathscr{V} = L^2(0,T;\mathbb{R}^k)$. The other parameters satisfy assumption (A), while the inequality and the dynamics are given by

$$\begin{gathered} J(T,\xi,v) = x_0'Hx_0 + \int_0^T \big(w'(t)Q(t)w(t) + v'(t)R(t)v(t)\big)dt \le 1, \\ \dot{x}_t = A(t)x_t + B(t)v(t), \quad y_t = G(t)x_t + w(t), \quad x_T = \xi, \end{gathered} \tag{7}$$

respectively. The deterministic problem can be easily solved by the dynamic programming method. Let $V(t,x)$ be the Bellman function for the deterministic BLQ observation problem. The HJB-equation for $V(t,x)$ is of the form

$$\begin{gathered} V_t = \min_v \left\{ -(A(t)x + B(t)v)'V_x + f_0(t,v,y_t - G(t)x) \right\}, \\ f_0(t,v,w) = w'Q(t)w + v'R(t)v, \quad V(0,x) = x'Hx. \end{gathered}$$

If we seek a solution to HJB-equation in the form $V(t,x) = x'Z(t)x - 2x'd(t) + g(t)$, we obtain the following equations

$$\begin{gathered} \dot{Z}(t) + Z(t)A(t) + A'(t)Z(t) + Z(t)B(t)R^{-1}(t)B'(t)Z(t) \\ -G'(t)Q(t)G(t) = 0, \quad Z(0) = H; \\ \dot{d}(t) = G'(t)Q(t)y_t - (Z(t)B(t)R^{-1}(t)B'(t) + A'(t))d, \quad d(0) = 0; \\ \dot{g}(t) = y_t'Q(t)y_t - d'(t)B(t)R^{-1}(t)B'(t)d(t), \quad g(0) = 0. \end{gathered} \tag{8}$$

Let the following assumption be valid.

Assumption (D)**.** Either $H > 0$, or system (1) under $v = 0$, $w = 0$ is completely observable Bryson and Ho [17] on every subinterval $[\tau,\theta] \subset [0,T]$.

Under assumption (D), it is known that the matrix $Z(t) > 0$ for every $t > 0$. Then the information set $\mathscr{X}(t,y(\cdot)) = \{x : (x - \hat{x}(t))'Z(t)(x - \hat{x}(t)) + h(t) \le 1\}$ is a bounded ellipsoid for every $t > 0$, where parameters $\hat{x}(t) = Z^{-1}(t)d(t)$, $h(t) = g(t) - d(t)'Z^{-1}(t)d(t)$ satisfy the equations

$$\begin{gathered} \dot{\hat{x}}(t) = Z^{-1}(t)G'(t)Q(t)(y_t - G(t)\hat{x}(t)) + A(t)\hat{x}(t), \quad \hat{x}(0) = 0; \\ \dot{h}(t) = (y_t - G(t)\hat{x}(t))'Q(t)(y_t - G(t)\hat{x}(t)), \quad h(0) = 0. \end{gathered} \tag{9}$$

Thus, we obtain the following result.

Proposition 2 (deterministic BLQ observation problem)**.** *The function $V(T,\xi,y(\cdot)) = \xi'Z(T)\xi - 2\xi'd(T) + g(T)$ (or $V(T,\xi,y(\cdot)) = (\xi - \hat{x}(T))'Z(T)(\xi - \hat{x}(T)) + h(T)$ under assumption* (D)*), where parameters satisfy equations* (8) *or* (9)*. Under assumption* (D) *the information set $\mathscr{X}(t,y(\cdot)) = \{x : V(T,\xi,y(\cdot)) \le 1\}$ is a bounded ellipsoid for every $t > 0$. The disturbance minimizer to problem* (4) *is equal to $\bar{v}(t) = R^{-1}(t)B'(t)(Z(t)x(t) - d(t))$.*

Let's notice that under $B = 0$ we can formally assume $R = 0$. All the formulas (7) – (9) will be valid.

It is important to understand that the above time reversal technique cannot be extended to the stochastic BLQ observation problem, (4), as it would destroy the adaptiveness which is essential in the model. In particular, a disturbance obtained in this way will not, in general, be $\mathscr{F}_t$-adapted and hence is not admissible.

The solution to (4) is more complicated. But using the results of Dokuchaev and Zhou [16], Lim and Zhou [18] we can use the stochastic maximum principle (Theorem 1) and reveal the solution.

Let us write the adjoint equation (5) and optimal control from maximum equality (6) for our linear case:

$$\begin{gathered} d\psi_t = \big\{ -A'(t)\psi_t - G'(t)Q(t)G(t)x_t + G'(t)Q(t)y_t \big\} dt - \big\{ C'(t)\psi_t \\ + S(t)z(t) \big\} dW(t), \quad \psi_0 = -Hx_0, \quad v(t) = -R^{-1}(t)B'(t)\psi_t. \end{gathered} \tag{10}$$

Now we join this equation to BSDE (1):

$$dx_t = (A(t)x_t - B(t)R^{-1}(t)B'(t)\psi_t + C(t)z(t))dt + z(t)dW(t), \quad x_T = \xi. \tag{11}$$

Let's try to solve system (10), (11) with equality $x_t = \Sigma(t)\psi_t - h_t$, where

$$\begin{gathered} \dot{\Sigma}(t) - A(t)\Sigma(t) - \Sigma(t)A'(t) - \Sigma(t)G'(t)Q(t)G(t)\Sigma(t) + B(t)R^{-1}(t)B'(t) \\ + C(t)\Sigma(t)(I + S(t)\Sigma(t))^{-1}C'(t) = 0, \quad \Sigma(T) = 0; \end{gathered} \tag{12}$$

$$dh_t = \{A(t)h_t + \Sigma(t)G'(t)Q(t)(G(t)h_t + y_t) + C(t)(I + \Sigma(t)S(t))^{-1}\eta(t)\}dt + \eta(t)dW(t), \quad h_T = -\xi. \tag{13}$$

In Lim and Zhou [18] under assumptions (A), it is proved that Riccati-type equation (12) has unique solution $\Sigma \in C(0,T;S^n)$ and $\Sigma(\cdot) \geq 0$. Analogously to Lim and Zhou [18] we can prove the following assertion.

Proposition 3. *For every $\xi \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$ and the admissible y_t BSDE system* (13) *has a unique solution $(h_\cdot, \eta(\cdot)) \in L^2_{\mathscr{F}}(0,T;C(0,T;\mathbb{R}^n)) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$. Besides, the Hamiltonian system* (10)*,* (11) *also has a unique solution $(x_\cdot, z(\cdot), \psi_\cdot)$. Moreover, the following relations are satisfied:*

$$\begin{gathered} x_t = \Sigma(t)\psi_t - h_t, \\ z(t) = -\Sigma(t)(S(t)\Sigma(t) + I)^{-1}C'(t)\psi_t - (\Sigma(t)S(t) + I)^{-1}\eta(t), \\ x_0 = -(\Sigma(0)H + I)^{-1}h_0. \end{gathered} \tag{14}$$

Now we are able to substitute $(x_\cdot, z(\cdot))$ in (10) and differentiate with the help of Itô formula the expression $\psi'(t)\Sigma(t)\psi(t)$. Substituting (14) into functional $J(T,\xi,v)$, see (1), we come to the conclusion.

Proposition 4. *Let $(x_\cdot, z(\cdot), \psi_\cdot)$ be the solution of the Hamiltonian system* (10)*,* (11)*, and let $\overline{v}(t) = -R^{-1}(t)B'(t)\psi_t$. Then the optimal value of the functional $J(T,\xi,v)$ is equal to*

$$\begin{gathered} \bar{J} = h_0'H(I + \Sigma(0)H)^{-1}h_0 + E\int_0^T \big[(y_t + G(t)h_t)'Q(t)(y_t + G(t)h_t) \\ + \eta'(t)S(t)(\Sigma(t)S(t) + I)^{-1}\eta(t)\big]dt. \end{gathered} \tag{15}$$

The terminal state ξ is present in this representation implicitly. In order to include ξ explicitly, introduce the following equations

$$\begin{gathered} \dot{N}(t) + N(t)(A(t) + \Sigma(t)G'(t)Q(t)G(t)) + (A(t) + \Sigma(t)G'(t)Q(t)G(t))'N(t) - G'(t)Q(t)G(t)) = 0, \\ N(0) = (H(I + \Sigma(0)H)^{-1} + (I + H\Sigma(0))^{-1}H)/2; \\ \dot{Z}(t) + Z(t)A(t) + A'(t)Z(t) + Z(t)[B(t)R^{-1}(t)B'(t) + C(t)\Sigma(t)(I + S(t)\Sigma(t))^{-1}C'(t)]Z(t) \\ -G'(t)Q(t)G(t) = 0, \quad Z(0) = H; \end{gathered} \tag{16}$$

In Lim and Zhou [18] it is shown that $N(t) = (Z(t)(I + \Sigma(t)Z(t))^{-1} + (I + Z(t)\Sigma(t))^{-1}Z(t))/2$.

MAIN RESULTS

If we make Itô differentiation of the expression $h'(t)N(t)h(t)$ and compare the result with formula (15), then we obtain the following theorem.

Theorem 5. *The optimal cost in problem* (4) *is given by*

$$\begin{aligned}\bar{J} = E\Big[\xi' N(T)\xi + \int_0^T \big(y_t' Q(t) y_t + 2y_t' Q(t) G(t)(I - \Sigma(t)N(t))h_t \\ +\eta'(t)(S(t)(\Sigma(t)S(t)+I)^{-1} - N(t))\eta(t) - 2\eta'(t)(I+S(t)\Sigma(t))^{-1}C'(t)N(t)h_t\big)dt\Big],\end{aligned} \tag{17}$$

where $(h_\cdot, \eta(\cdot)) \in L^2_{\mathscr{F}}(0,T;C(0,T;\mathbb{R}^n)) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ *is a unique solution of* (13).

Remark 6. Since the pair $(h_\cdot, \eta(\cdot))$ represents a continuous linear functional of the terminal state ξ under given $y_\cdot$, the random information set $\mathscr{X}(T, y(\cdot)) = \{\xi \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n) : V(T,\xi,y(\cdot)) \le 1\}$, where $V(T,\xi,y(\cdot)) = \min_v J(T,\xi,v)$, represents an ellipsoid in Hilbert space $L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$.

Remark 7. The same way as it was in the deterministic case we can obtain with the help of Itô differentiation the equality $\psi_t = -Z(t)x_t + q_t$, where ψ, x are the solutions of the Hamiltonian system (10), (11) and q is a solution of the system

$$\begin{aligned}dq_t = \big\{-[A+BR^{-1}B'Z + C(I+\Sigma S)^{-1}\Sigma C'Z]' q_t + G'Qy_t \\ +ZC(I+\Sigma S)^{-1}\eta(t)\big\}dt + \big\{(S-Z)(I+\Sigma S)^{-1}\eta(t) - (Z\Sigma + I)(I+S\Sigma)^{-1}C'\psi_t\big\}dW(t), \quad q_0 = 0.\end{aligned} \tag{18}$$

Remark 8. The second equation in (16) is a generalization of the Riccati equation (8) associated with the deterministic problem, where $C=0$, $S=0$, and ξ, v, w are nonrandom. Equation (18) in the deterministic case coincides with (8) and $q_t = d(t)$, since $\eta(t) \equiv 0$. Therefore optimal disturbance functions are the same, and values of the costs (15), (17) also coincides with the deterministic analog.

Optimal estimate and its properties

Let us begin from the definition.

Definition. We call the element $\hat{\xi}(T)$ of the random information set $\mathscr{X}(T, y(\cdot))$ *the optimal estimate* if it coincides with the geometric center of $\mathscr{X}(T, y(\cdot))$.

The optimality of $\hat{\xi}(T)$ means the following

$$\sup_{y\in\mathscr{X}(T,y(\cdot))} \|\hat{\xi}(T) - y\| = \min_x \sup_y \|x-y\|, \quad x,y \in \mathscr{X}(T,y(\cdot)). \tag{19}$$

Equality (19) easily follows from the fact that the informational set is convex and symmetric (Kurzhanski [7]).

In order to find $\hat{\xi}(T)$ let us introduce a linear bounded operator $\mathscr{A} : L^2_{\mathscr{F}}(0,T;\mathbb{R}^{2n}) \to L^2_{\mathscr{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$ according to the formula

$$\mathscr{A}(\phi(\cdot),\psi(\cdot)) = (-p_\cdot, p_T),$$

where $p(\cdot)$ is the solution of the following linear SDE:

$$dp_t = \{-(A(t)+\Sigma(t)G'(t)Q(t)G(t))'p_t + \phi(t)\}dt + \{-(I+S(t)\Sigma(t))^{-1}C'(t)p_t + \psi(t)\}dW(t), \quad p(0)=0.$$

It is known, see Yong and Zhou [12, p. 353], that the linear adjoint operator $\mathscr{A}^*(f(\cdot),\xi) = (\bar{h}_\cdot, \bar{\eta}(\cdot))$ gives the solution of the equation

$$d\bar{h}_t = \{(A(t)+\Sigma(t)G'(t)Q(t)G(t))\bar{h}_t + C(t)(I+\Sigma(t)S(t))^{-1}\bar{\eta}(t) + f(t)\}dt + \bar{\eta}(t)dW(t), \quad \bar{h}_T = \xi. \tag{20}$$

Comparing (13) and (20) we get $(h_\cdot,\eta(\cdot)) = \mathscr{A}^*(\Sigma G'Qy_\cdot,-\xi)$. Let $\pi_1(\phi(\cdot),\psi(\cdot)) = \phi(\cdot)$, $\pi_2(\phi(\cdot),\psi(\cdot)) = \psi(\cdot)$ be the projectors. Then we can rewrite the (17) in the form

$$\begin{aligned}\bar{J} = E\Big[\xi' N(T)\xi + \int_0^T y_t' Q(t) y_t dt\Big] - 2 < \mathscr{A}\pi_1^*(I-N\Sigma)G'Qy_\cdot,(\Sigma G'Qy_\cdot,-\xi) > \\ -2 < \mathscr{A}\pi_2^*(I+S\Sigma)^{-1}C'N\pi_1\mathscr{A}^*(\Sigma G'Qy_\cdot,-\xi),(\Sigma G'Qy_\cdot,-\xi) > \\ + < \mathscr{A}\pi_2^*(S(\Sigma S+I)^{-1}-N)\pi_2\mathscr{A}^*(\Sigma G'Qy_\cdot,-\xi),(\Sigma G'Qy_\cdot,-\xi) >,\end{aligned} \tag{21}$$

where $< (\phi(\cdot),\xi),(\phi_1(\cdot),\xi_1) = E\{\int_0^T \phi'(s)\phi(s)ds + \xi'\xi_1\}$ is the inner product in the space $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$. Introducing the element $d \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$, for which

$$\begin{aligned}Ed'\xi =< \mathscr{A}\pi_1^*(I-N\Sigma)G'Qy_\cdot,(0,-\xi) > + < \mathscr{A}\pi_2^*(I+S\Sigma)^{-1}C'N\pi_1\mathscr{A}^*(0,-\xi), \\ (\Sigma G'Qy_\cdot,0) > + < \mathscr{A}\pi_2^*(I+S\Sigma)^{-1}C'N\pi_1\mathscr{A}^*(\Sigma G'Qy_\cdot,0),(0,-\xi) > \\ - < \mathscr{A}\pi_2^*(S(\Sigma S+I)^{-1}-N)\pi_2\mathscr{A}^*(\Sigma G'Qy_\cdot,0),(0,-\xi) >, \quad \forall\xi,\end{aligned} \tag{22}$$

and the self-adjoint linear operator $\mathscr{Z}(\cdot)$ in the space $L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$ satisfying the relation

$$\begin{aligned}E\xi'\mathscr{Z}(\xi) = E\xi'N(T)\xi - 2 < \mathscr{A}\pi_2^*(I+S\Sigma)^{-1}C'N\pi_1\mathscr{A}^*(0,\xi),(0,\xi) > \\ + < \mathscr{A}\pi_2^*(S(\Sigma S+I)^{-1}-N)\pi_2\mathscr{A}^*(0,\xi),(0,\xi) >,\end{aligned} \tag{23}$$

we can represent the informational set in the form

$$\mathscr{X}(T,y(\cdot)) = \{\xi \in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n) : E[\xi'\mathscr{Z}(\xi) - 2d'\xi + g] \le 1\}, \tag{24}$$

where g is an appropriate random variable defining by relation (21). If $H > 0$ and the matrices Q, S, R in (2), (3) are positive and continuous, the operator $\mathscr{Z}$ is coercive and we have

$$\hat{\xi}(T) = \mathscr{Z}^{-1}d. \tag{25}$$

Consider a special case of (1), where $B(t) \equiv 0$. We can formally assume that $R(t) \equiv 0$. Then $\Sigma(t) \equiv 0$, $x_t = -h_t$, $z(t) = -\eta(t)$, and $N(t) = Z(t)$. Equality (22) is converted in

$$Ed'\xi =< \mathscr{A}\pi_1^* G'Qy_\cdot,(0,-\xi) >, \quad \forall\xi. \tag{26}$$

Therefore, according to definition of the operator $\mathscr{A}$ we have $d = p_T$, where

$$dp_t = \{-A'(t)p_t - G'Qy_t\}dt - C'(t)p_t dW(t), \quad p(0) = 0. \tag{27}$$

EXAMPLES AND APPLICATIONS

Consider two-dimensional system $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$, $0 \le t \le T$, describing the uniform motion on the line, and the observation: $y = x_1 + w$. The constraints are of the form $\int_0^T w^2(t)dt \le 1$. Let $T = 10$ and the signal be equal to $y(t) = -11 + 2t + A\cos(\pi t)$, where $A = \sqrt{2/T}$. We suppose that $w(t) = A\cos(\pi t)$ during all the process. Solutions of the equations (8) are: $Z(t) = [t, -t^2/2; -t^2/2, t^3/3]$, $d(t) = [\int_0^t y_s ds; \int_0^t (s-t)y_s ds]$, $g(t) = \int_0^t y_s^2 ds$.

Here the diameter of the informational set tends to zero and, therefore, the error of estimation also does. The optimal estimate $\hat{x}(t)$ well approximates the state vector as is shown on Fig. 1.

Now let us consider a stochastic analog of above system:

$$dx_{1t} = x_{2t}dt + z_1(t)dW(t), \quad dx_{2t} = z_2(t) + z_2(t)dW(t), \tag{28}$$

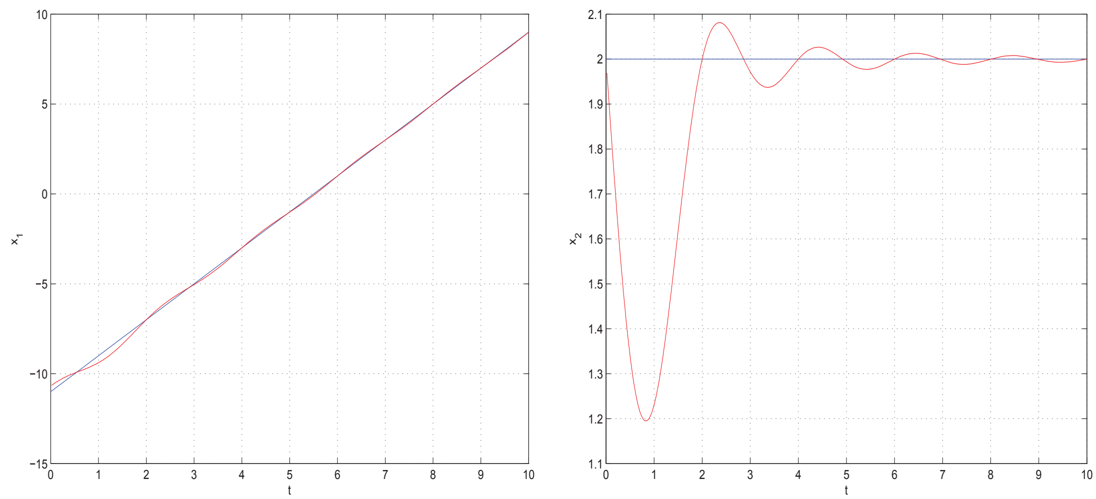

FIGURE 1. On the left: estimation of x_1; on the right: estimation of x_2

with the constraints

$$E\int_0^T \left[qw^2(t)+s_1z_1^2(t)+s_2z_2^2(t)\right]dt \le 1,$$

where positive constants q, s_1, s_2 regulate the influence of the corresponding disturbances.

According to formulas (22)-(27), in order to find the optimal estimate we must indicate such a vector $x(T)=\xi=\hat{\xi}(T)\in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^2)$ for system (28) that

$$dp_{1t}=qy_t dt,\quad dp_{2t}=-p_{1t}dt-p_{2t}dW(t),\quad p_0=0,$$
$$d\tilde{p}_{1t}=qx_{1t}dt+s_1z_1(t)dW(t),\quad d\tilde{p}_{2t}=-\tilde{p}_{1t}dt+(-\tilde{p}_{2t}+s_2z_2(t))dW(t),\quad \tilde{p}_0=0,$$

and $\tilde{p}_T=p_T$. This is equality (25). A numerical solution of such a problem will be considered in other work. This solution is founded on the four-step scheme for FBSDE and on the replacement of Hilbert space $L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^n)$ by its finite dimensional subspace.

Another example deals with option pricing problem. Consider a market with 2 assets (a bond and a stock). A stock price satisfies the SDE

$$ds(t)=(bdt+\sigma dW(t))s(t),\quad s(0)=s_0,$$

whereas the total wealth is subjected to the equation

$$dx_t=\{rx_t+(b-r)\pi(t)\}dt+\sigma\pi(t)dW(t),\quad x_T=(s(T)-q)^+,\tag{29}$$

where $\pi(t)$ is a portfolio. Suppose that we have the constraints

$$E\left[Hx_0^2+\int_0^T S\pi(t)^2dt\right]\le 1.$$

The problem is to describe the attainability domain for equation (29) at the final instant T (s_0 is unknown).

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