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Defining Relations of a Free Modular Lattice of Rank 3

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Abstract—For a 3-generated free modular lattice we obtain a set of 11 defining relations and prove that this set is minimal.

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Recall that the *rank* of free algebra from some manifold is the cardinal number of the set of its free generators. We concentrate our attention on a free lattice of the rank 3 in the manifold of modular lattices; we denote it by A. Let F be a free lattice of the rank 3 in the manifold of all lattices; let f, g, and h be its free generators, and let φ be a homomorphism from F to A. By standard considerations of a universal algebra, elements $a = \varphi(f)$, $b = \varphi(g)$, and $c = \varphi(h)$ are free generators of the lattice A. Relations defining this lattice in the manifold of all lattices were considered in [1] and [2]. In the paper [1] one has particularly shown that A can be defined by 21 relations. In [2] one has proved that this set of defining relations is not minimal; namely, it was shown there that 15 relations among those mentioned above define the lattice A, moreover, they form the minimal set of defining relations for A. Note that in [1] one has described a set of seven defining relations for a free distributive lattice of the rank 3, and in [3] this set was proved to be minimal.

The following assertion is the main result of this paper: There exists a set of 11 defining relations for the lattice A. Note that this set is not a subset of the set of defining relations indicated in [1]. Let us enumerate these relations:

$$(a \vee (b \wedge c)) \wedge (b \vee c) = (a \wedge (b \vee c)) \vee (b \wedge c), \tag{1}$$
$$(b \vee (c \wedge a)) \wedge (c \vee a) = (b \wedge (c \vee a)) \vee (c \wedge a), \tag{2}$$
$$(c \vee (a \wedge b)) \wedge (a \vee b) = (c \wedge (a \vee b)) \vee (a \wedge b), \tag{3}$$

$$\begin{aligned}(a \vee b) \wedge (a \vee c) \wedge (b \vee c) = {} & ((a \wedge (b \vee c)) \vee ((a \vee b) \wedge c)) \\ & \wedge ((b \wedge (a \vee c)) \vee ((b \vee a) \wedge c)) \wedge ((a \wedge (c \vee b)) \vee ((a \vee c) \wedge b)),\end{aligned} \tag{4}$$

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = {} & ((a \vee (b \wedge c)) \wedge ((a \wedge b) \vee c)) \\ & \vee ((b \vee (a \wedge c)) \wedge ((b \wedge a) \vee c)) \vee ((a \vee (c \wedge b)) \wedge ((a \wedge c) \vee b)),\end{aligned} \tag{5}$$

$$(a \vee b) \wedge (a \vee c) = a \vee ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)), \tag{6}$$
$$(b \vee a) \wedge (b \vee c) = b \vee ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)), \tag{7}$$
$$(c \vee a) \wedge (c \vee b) = c \vee ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)), \tag{8}$$
$$(a \wedge b) \vee (a \wedge c) = a \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c)), \tag{9}$$
$$(b \wedge a) \vee (b \wedge c) = b \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c)), \tag{10}$$

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$$(c \wedge a) \vee (c \wedge b) = c \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c)). \tag{11}$$

For short we denote this set of relations by ρ.

Theorem. *The lattice with generating elements a, b, and c and with the set ρ of defining relations is isomorphic to a free modular lattice of the rank 3. If a subset of relations is strictly contained in ρ, then the lattice with generating elements a, b, and c and with this set of defining relations is not modular.*

The proof of the first statement of the theorem is based on the following assertions. Let L be the lattice with generating elements a, b, and c defined by relations of the set ρ. Since the lattice A satisfies the indicated relations, it is a homomorphic image of the lattice L. It is known that A contains 28 elements, therefore, the lattice L contains at least 28 elements. But direct calculations show that L cannot contain more than 28 elements. This means that two mentioned lattices are isomorphic.

Let us prove the second assertion. For each set of defining relations obtained from ρ by eliminating one arbitrary relation we construct an example of a 3-generated non-modular lattice satisfying all relations from the mentioned set. For instance, the lattice presented in Fig. 1 is non-modular; it satisfies relations (2)–(11) but does not satisfy relation (1).

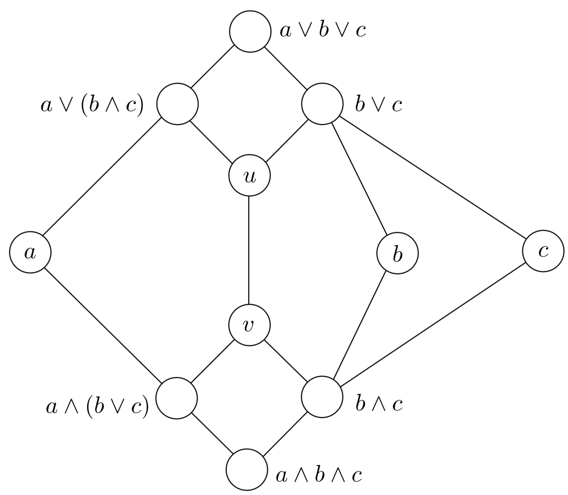

Fig. 1.

In Fig. 1 we use the denotations $u = (a \vee (b \wedge c)) \wedge (b \vee c)$ and $v = (a \wedge (b \vee c)) \vee (b \wedge c)$.

The lattice presented in Fig. 2 satisfies relations (1)–(3) and (5)–(11) but does not satisfy relation (4).

In Fig. 2 we use the denotations $t = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$, $a_1 = a \wedge (b \vee c)$, and $b_1 = b \wedge (a \vee c)$.

The lattice presented in Fig. 3 satisfies relations (1)–(5) and (7)–(11) but does not satisfy relation (6).

The rest examples are obtained from those described above by permutations of elements a, b, and c and by passing to the dual lattices.

In this connection it is interesting to clear out whether there exists a set of defining relations for a free modular lattice of the rank 3, where the number of relations is less than 11.

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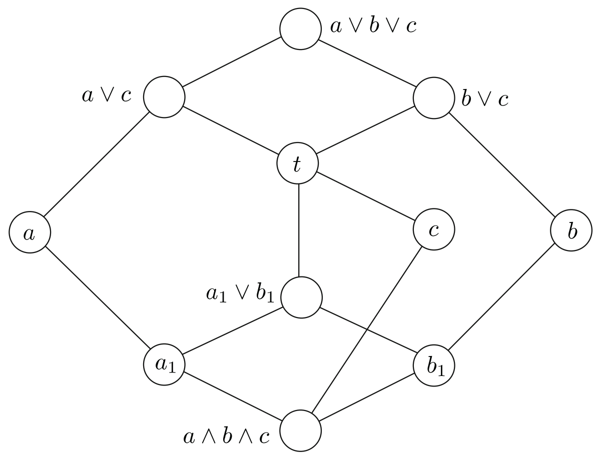

Fig. 2.

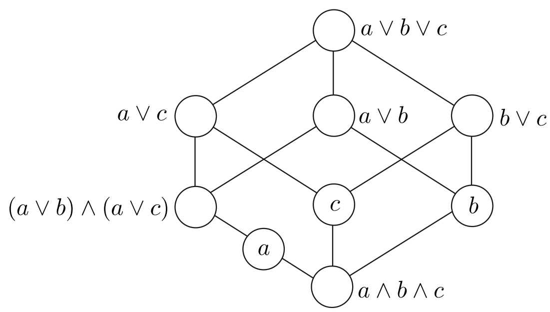

Fig. 3.

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