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A Sharp Markov Brothers-Type Inequality in the Spaces L_∞ and L_1 on the Segment

I. E. Simonov*

Institute of Mathematics and Computer Science, Ural Federal University, Ekaterinburg

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Abstract—The inequality between the uniform norm of the derivative of order ℓ of an algebraic polynomial of degree n and the L_1-norm of the polynomial itself on a segment are studied. For all $\ell \geq (n-1)/3$, the exact constant and the extremal polynomial are written out.

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1. STATEMENT OF THE PROBLEM AND OBTAINED RESULTS

Let $\mathscr{P}_n$ be the set of algebraic polynomials of degree at most n with real coefficients. In this paper, we study the inequality

$$\|P^{(\ell)}\|_\infty \leq C(n,\ell)\|P\|_1, \qquad P \in \mathscr{P}_n, \tag{1.1}$$

with the exact constant $C(n,\ell)$. Here and elsewhere,

$$\|P\|_p = \left(\frac{1}{2}\int_{-1}^{1} |P(x)|^p\,dx\right)^{1/p}, \quad 0 < p < \infty, \qquad \|P\|_\infty = \max_{x\in[-1,1]} |P(x)|.$$

Inequality (1.1) is a particular result in the general problem of estimating the L_q-mean of the ℓth derivative of an algebraic polynomial via the L_p-mean of the polynomial itself:

$$\|P^{(\ell)}\|_q \leq C_{q,p}(n,\ell)\|P\|_p, \qquad P \in \mathscr{P}_n, \quad 0 < p,q \leq \infty, \quad 1 \leq \ell \leq n. \tag{1.2}$$

Inequality (1.2) was studied in a large number of papers; an extensive survey of the results is given in [1], [2]. Ivanov [3] and Konyagin [4] obtained the most general estimates of the best constants in inequalities of type (1.2). Results from [3] imply the order of growth of the quantity $C_{q,p}(n,\ell)$ in n as $n \to \infty$ for fixed ℓ, p, q; in particular,

$$C(n,\ell) = C_{\infty,1}(n,\ell) \asymp n^{2\ell+2}, \qquad n \to \infty.$$

The exact values of $C_{q,p}(n,\ell)$ are known only for certain values of p, q, and ℓ. The Markov brothers [5], [6] obtained the sharp inequality (1.2) for $p = q = \infty$, $1 \leq \ell \leq n-1$; in this case, the Chebyshev polynomial of the first kind

$$T_n(x) = \cos(n \arccos x), \qquad x \in [-1,1],$$

is an extremal polynomial. Bojanov [7] (see also [8]) proved that the polynomial T_n is extremal also for all $q \in [1,\infty)$, $p = \infty$, $\ell = 1$. The case $p = q = 2$ was studied by Schmidt, Hille, Szegö, Tamarkin, Milovanović, Dörfler, and Kroó (see [9], [1], [10], [11]). Labelle [12] found the exact constant for $q = \infty$, $p = 2$, $1 \leq \ell \leq n-1$. If $\ell = n$, then the problem can be reduced to the determination of the polynomial of least deviation from zero in the metric of $\|\cdot\|_p$, with a fixed leading coefficient. For $p = \infty$, such a

*E-mail: isimonov@k66.ru

polynomial is the Chebyshev polynomial of the first kind [13], for $p = 2$, the Legendre polynomial, and, for $p = 1$, the Chebyshev polynomial of the second kind

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}, \qquad x \in [-1,1].$$

In the last case, the solution was obtained by Korkin and Zolotarev [14].

The main result of the present paper is the following theorem.

Theorem 1. *Let n and ℓ be natural numbers such that $(n-1)/3 \le \ell \le n-1$. Denote $u_k = U_k^{(\ell)}(1)$, $k = \ell, \dots, n$,*

$$\mathbf{U} = \begin{pmatrix} u_n & u_{n-1} & \cdots & u_\ell \\ u_{n-1} & u_{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ u_\ell & 0 & \cdots & 0 \end{pmatrix},$$

and let λ^* be the greatest eigenvalue of the matrix $\mathbf{U}$. Then

1) *λ^* exceeds the absolute values of all the other eigenvalues of $\mathbf{U}$, has algebraic multiplicity 1, and the eigenvector $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-\ell})^\top$ corresponding to it has positive coordinates*;

2) *the exact constant in inequality* (1.1) *is $C(n,\ell) = \lambda^*$*;

3) *the extremal polynomials in inequality* (1.1) *are $P^*(x)$ and $P^*(-x)$, where*

$$P^*(x) = \frac{aG^*(\arccos x)}{\sqrt{1-x^2}}, \qquad a \in \mathbb{R} \setminus \{0\},$$

$$G^*(\theta) = \sum_{j=0}^{n-\ell} \sum_{k=0}^{n-\ell} \alpha_j \alpha_k \sin((n+1-j-k)\theta),$$

and here $\|P^{*(\ell)}\|_\infty = |P^{*(\ell)}(1)|$.

2. AUXILIARY STATEMENTS

We shall need the following definitions (see [15, Chap. 13]).

Let

$$A = \|a_{j,k}\|_1^n \qquad \text{and} \qquad C = \|c_{j,k}\|_1^n$$

be two real matrices of size $n \times n$. We shall write

$$A \ge C \qquad \text{if} \quad a_{j,k} \ge c_{j,k} \quad \text{for all } j = 1, \dots, n,\ k = 1, \dots, n.$$

Let $A = \|a_{j,k}\|_1^n$ be a complex matrix. Denote by A^+ the matrix obtained by replacing all the entries of the matrix A by their absolute values: $A^+ = \|\,|a_{j,k}|\,\|_1^n$.

A matrix $A = \|a_{j,k}\|_1^n$ is said to be *decomposable* if, for some partition of all indices $1, 2, \dots, n$ into two systems (without common indices)

$$j_1, j_2, \dots, j_\mu, \quad k_1, k_2, \dots, k_\nu, \qquad \mu + \nu = n,$$

we have

$$a_{j_\alpha, k_\beta} = 0, \qquad \alpha = 1, 2, \dots, \mu, \quad \beta = 1, 2, \dots, \nu.$$

Otherwise, the matrix A is said to be *indecomposable*.

By E_n we denote the unit matrix of size $n \times n$.

Lemma 1. *Let*

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & 0 & \cdots & 0 \end{pmatrix},$$

where $b_{jk} \neq 0$, $j + k \leq n + 1$. *Then the matrix* $\mathbf{B}$ *is indecomposable.*

Proof. For $n = 1$, the statement is obvious. For $n \geq 2$, let us use the criterion for the indecomposability of a matrix [16, Chap. VIII, Sec. 1, Theorem 8.11]: the matrix $B = \|b_{j,k}\|_1^n$ is indecomposable if and only if

$$(E_n + B^+)^{n-1} > 0.$$

If $n = 2$, then $E_n + \mathbf{B}^+ > 0$. Let $n \geq 3$. It suffices to prove that, for some $1 \leq m \leq n - 1$,

$$(E_n + \mathbf{B}^+)^m > 0.$$

It is readily verified that $\mathbf{B}^{+2} > 0$; therefore,

$$(E_n + \mathbf{B}^+)^2 \geq \mathbf{B}^{+2} > 0,$$

which proves the assertion. □

Let $\mathscr{T}_m$ be the set of trigonometric polynomials $G(\theta)$ of order m with real coefficients:

$$\begin{aligned} G(\theta) &= \alpha_0 + \sum_{k=1}^{m} (\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)) \\ &= \operatorname{Re}\biggl(\sum_{k=0}^{m} \overline{\gamma}_k e^{ik\theta} \biggr), \qquad \gamma_k = \alpha_k + i\beta_k, \quad \alpha_k, \beta_k \in \mathbb{R}, \quad \beta_0 = 0. \end{aligned}$$

Following Geronimus' paper [17], let $L[G]$ denote the integral

$$L[G] = \frac{1}{2\pi} \int_0^{2\pi} |G(\theta)| \, d\theta. \tag{2.1}$$

Suppose that we are given the collection c of $s + 1$ complex numbers

$$c = (c_{m-s}, \dots, c_m), \qquad c_k = a_k + ib_k.$$

Let $\omega[G, c]$ denote the linear functional of $s + 1$ leading coefficients of the polynomial $G(\theta)$:

$$\omega[G, c] = \sum_{k=m-s}^{m} \alpha_k a_k + \beta_k b_k = \operatorname{Re} \sum_{k=m-s}^{m} \overline{\gamma}_k c_k.$$

In what follows, the essential role is played by the following result due to Geronimus; in it the norm of $\omega[\,\cdot\,, c]$ is calculated as that of a linear functional on the space $\mathscr{T}_m$ with norm (2.1).

Theorem A (Geronimus [17, Theorem II], [18, Theorem I]). *For any* $G \in \mathscr{T}_m$, *the following inequality holds*:

$$\frac{|\omega[G, c]|}{L[G]} \leq \frac{|\omega[G_0, c]|}{L[G_0]} = \frac{\pi \delta_0}{2}, \qquad s \leq \left[\frac{2m - 1}{3} \right].$$

Here δ_0 is the greatest root of the secular equation

$$\mathbf{C}(\delta,c)=\begin{vmatrix} \delta & 0 & \cdots & 0 & \overline{c}_{m-s} & \overline{c}_{m-s+1} & \cdots & \overline{c}_m \\ 0 & \delta & \cdots & 0 & 0 & \overline{c}_{m-s} & \cdots & \overline{c}_{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \delta & 0 & 0 & \cdots & \overline{c}_{m-s} \\ c_{m-s} & 0 & \cdots & 0 & \delta & 0 & \cdots & 0 \\ c_{m-s+1} & c_{m-s} & \cdots & 0 & 0 & \delta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_m & c_{m-1} & \cdots & c_{m-s} & 0 & 0 & \cdots & \delta \end{vmatrix}=0. \tag{2.2}$$

The equality sign holds only for polynomials of the form

$$G_0(\theta)=\mathrm{Re}(q^2(z)z^{m-2s+\nu})|\tau(z)|^2, \qquad z=e^{i\theta},$$

where $\nu+1$ is the multiplicity of the root δ_0, $\tau(z)$ is an arbitrary polynomial of degree at most ν, $q(z)$ is a polynomial of degree $s-\nu$ determined from the expansion

$$\delta_0\frac{q(z)}{z^{s-\nu}\overline{q(1/\overline{z})}}=\overline{c}_{m-s}+\overline{c}_{m-s+1}z+\dots+\overline{c}_m z^s+\cdots, \qquad |z|\le 1.$$

The following lemma is the main component of the proof of of Theorem 1.

Lemma 2. *Suppose that we are given natural numbers m and s such that $s\le(2m-1)/3$ and a collection $b^*=(b^*_{m-s},\dots,b^*_m)$, $b^*_k>0$. Then, for any collection $b=(b_{m-s},\dots,b_m)$, $b_k\in\mathbb{R}$, with the property*

$$|b_k|\le b^*_k, \qquad k=m-s,\dots,m, \tag{2.3}$$

and any odd polynomial $G(\theta)$ of degree m,

$$G(\theta)=\sum_{k=1}^m \beta_k\sin(k\theta)=\mathrm{Re}\sum_{k=0}^m\overline{\gamma}_k e^{ik\theta}, \qquad \gamma_k=0+i\beta_k,$$

the following inequality holds:

$$\frac{|\omega[G,ib]|}{L[G]}\le\frac{\pi\lambda^*}{2}, \tag{2.4}$$

where λ^* is the greatest eigenvalue of the matrix

$$\mathbf{B}^*=\begin{pmatrix} b^*_m & b^*_{m-1} & \cdots & b^*_{m-s} \\ b^*_{m-1} & b^*_{m-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b^*_{m-s} & 0 & \cdots & 0 \end{pmatrix}. \tag{2.5}$$

The eigenvalue λ^* exceeds the absolute values of all the other eigenvalues, has algebraic multiplicity 1, and to it corresponds the eigenvector $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_s)^\top$ with positive coordinates $\alpha_k>0$, $k=0,\dots,s$.

Inequality (2.4) becomes an equality if and only if, for some $\varepsilon\in\{-1,1\}$ and $\nu\in\{0,1\}$, the representation

$$b_k=\varepsilon(-1)^{\nu k}\,b^*_k, \qquad k=m-s,\dots,m, \tag{2.6}$$

is valid and the polynomial G has the form

$$G(\theta) = aG^*(\theta + \nu\pi), \qquad a \neq 0;$$

here

$$G^*(\theta) = \sum_{j=0}^{s} \sum_{k=0}^{s} \alpha_j \alpha_k \sin((m-j-k)\theta) = \sum_{k=1}^{m} \beta_k^* \sin(k\theta). \tag{2.7}$$

Besides, the first $s+1$ coefficients of the polynomial $G^*(\theta)$ are positive:

$$\beta_k^* > 0, \qquad k = m-s, \dots, m. \tag{2.8}$$

Proof. The proof is carried out in several steps.

1. Let us establish inequality (2.4). By Theorem A, we have

$$\frac{|\omega[G, ib]|}{L[G]} \le \frac{\pi\delta_0}{2},$$

where δ_0 is the greatest root of Eq. (2.2) $\mathbf{C}(\delta, ib) = 0$.

Denote

$$\mathbf{B} = \begin{pmatrix} b_m & b_{m-1} & \cdots & b_{m-s} \\ b_{m-1} & b_{m-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-s} & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} b_{m-s} & b_{m-s+1} & \cdots & b_m \\ 0 & b_{m-s} & \cdots & b_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{m-s} \end{pmatrix}.$$

For any δ, in view of the properties of block matrices [15, Chap. II, Sec. 5, item 3], the following relation holds:

$$\mathbf{C}(\delta, ib) = \begin{vmatrix} \delta E & -iB \\ iB^\top & \delta E \end{vmatrix} = |\delta^2 E - BB^\top|, \qquad E = E_{s+1}.$$

By direct multiplication, we verify the equality $BB^\top = \mathbf{B}^2$; therefore,

$$\mathbf{C}(\delta, ib) = |\delta^2 E - \mathbf{B}^2| = |\delta E - \mathbf{B}||\delta E + \mathbf{B}|. \tag{2.9}$$

Similarly, we obtain the equalities

$$\mathbf{C}(\delta, ib^*) = |\delta^2 E - \mathbf{B}^{*2}| = |\delta E - \mathbf{B}^*||\delta E + \mathbf{B}^*|. \tag{2.10}$$

It is easy to verify that the matrix $(\mathbf{B}^*)^2$ is positive; therefore, by Perron's theorem [15, Chap. XIII, Sec. 2, Theorem 1], it has an eigenvalue $(\lambda^*)^2$, $\lambda^* > 0$, that exceeds the absolute values of all the other eigenvalues and has multiplicity of 1. The matrix $\mathbf{B}^*$ is nonnegative and, by Lemma 1, is indecomposable. Therefore, it follows from relation (2.10) and the Frobenius theorem [15, Chap. XIII, Sec. 2, Theorem 2] that λ^* is an eigenvalue of $\mathbf{B}^*$ of multiplicity of 1, is greater that the absolute values of all the other eigenvalues of $\mathbf{B}^*$, and to it corresponds the eigenvector $\alpha = (\alpha_0, \dots, \alpha_s)^\top$ with positive coordinates $\alpha_k > 0$, $k = 0, \dots, s$.

Let us prove that $\delta_0 \le \lambda^*$, and study the cases of equality. Let δ_1 be the greatest (in absolute value) eigenvalue of the matrix $\mathbf{B}$. In view of relation (2.9), we have $|\delta_1| = \delta_0$. Condition (2.3) implies the inequality $\mathbf{B}^+ \le \mathbf{B}^*$. Thus, all the assumptions of Lemma [15, Chap. XIII, Sec. 2, Lemma 2] hold; hence

$$|\delta_1| = \delta_0 \le \lambda^*;$$

here the equality is attained only in the case where

$$\mathbf{B} = e^{i\varphi} D \mathbf{B}^* D^{-1}; \tag{2.11}$$

here $e^{i\varphi} = \delta_1/\lambda^*$ and $D = \|d_{j,k}\|_0^s$ is a complex diagonal matrix all of whose diagonal elements are equal (in absolute value) to 1: $d_{jj} = d_j$, $|d_j| = 1$.

Let us prove that, in this case, condition (2.11) is equivalent to (2.6). First, it is easy to see that if (2.6) holds, then so does (2.11) with $e^{i\varphi} = \varepsilon(-1)^{\nu m}$ and $d_j = (-1)^{\nu j}$.

Now assume that condition (2.11) holds. The matrix $\mathbf{B}$ is symmetric; therefore, all of its characteristic numbers are real. Hence the ratio δ_1/λ^* can only be equal to either 1 or -1. To be definite, let $\delta_1/\lambda^* = 1$. Equating (2.11) elementwise, we obtain the relation

$$b_{m-k} = \frac{d_j}{d_{k-j}}\, b^*_{m-k}, \qquad j = 0, \dots, s, \quad j, \dots, s. \tag{2.12}$$

For $j = 0$, we divide relation (2.12) for k by the same relation for $k-1$, obtaining

$$\frac{b_{m-k}}{b^*_{m-k}} = \frac{d_{k-1}}{d_k}\frac{b_{m-(k-1)}}{b^*_{m-(k-1)}}, \qquad k = 1, \dots, s.$$

Dividing relations (2.12) for $j = 1$ and $j = 0$ by each other, we obtain

$$\frac{d_1}{d_0} = \frac{d_{k-1}}{d_k}, \qquad k = 1, \dots, s.$$

Comparing the last two expressions, we find that

$$\frac{b_{m-k}}{b^*_{m-k}} = \frac{d_1}{d_0}\frac{b_{m-(k-1)}}{b^*_{m-(k-1)}}, \qquad k = 1, \dots, s.$$

All the b_j and b_j^* are real; therefore, $d_1/d_0 \in \{-1, 1\}$. Hence we easily obtain (2.6) with $\varepsilon = d_0 e^{i\varphi}$ and

$$\nu = \begin{cases} 0 & \text{for } d_1/d_0 = 1, \\ 1 & \text{for } d_1/d_0 = -1. \end{cases}$$

2. Let us prove relation (2.7). As already noted, the multiplicity of λ^*, and hence that of the greatest root of the equation $\mathbf{C}(\lambda, ib^*) = 0$, is equal to 1. Therefore, by Theorem A, the extremal polynomial in inequality (2.4) for $b = b^*$ is unique up to a numerical multiplier and is

$$G^*(\theta) = \operatorname{Re}(q^2(z) z^{m-2s}), \qquad z = e^{i\theta}, \tag{2.13}$$

where $q(z)$ is the polynomial of degree s determined from the expansion

$$i\lambda^* \frac{q(z)}{z^s \overline{q(1/\overline{z})}} = b^*_{m-s} + b^*_{m-s+1} z + \dots + b^*_m z^s + \cdots, \qquad |z| \le 1. \tag{2.14}$$

Let us show that the polynomial

$$q(z) = (1-i)(\alpha_s + \alpha_{s-1} z + \dots + \alpha_0 z^s) \tag{2.15}$$

satisfies the expansion (2.14). Let us multiply both sides of (2.14) by the denominator of its left-hand side

$$z^s \overline{q(1/\overline{z})} = (1+i)(\alpha_0 + \alpha_1 z + \dots + \alpha_s z^s),$$

obtaining

$$\lambda^*(\alpha_s + \alpha_{s-1} z + \dots + \alpha_0 z^s) = (\alpha_0 + \alpha_1 z + \dots + \alpha_s z^s)(b^*_{m-s} + b^*_{m-s+1} z + \dots + b^*_m z^s + \cdots). \tag{2.16}$$

Since α is an eigenvector of $\mathbf{B}^*$, we see that the following relations hold:

$$\begin{aligned} \lambda^*\alpha_0 &= b^*_s\alpha_0 + b^*_{s-1}\alpha_1 + \dots + b^*_0\alpha_s, \\ \lambda^*\alpha_1 &= b^*_{s-1}\alpha_0 + b^*_{s-2}\alpha_1 + \dots + b^*_0\alpha_1, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \lambda^*\alpha_s &= b^*_0\alpha_0. \end{aligned}$$

The left- and right-hand sides of the kth equality coincide, respectively, with the coefficients of z^{s-k} on the left- and right-hand sides of (2.16). Thus, we have establisheded that the expansion (2.14) is valid.

Substituting (2.15) into (2.13), we obtain

$$G^*(\theta) = \operatorname{Re}\biggl(-2i\sum_{j=0}^{s}\sum_{k=0}^{s}\alpha_j\alpha_k e^{(m-j-k)i\theta}\biggr) = 2\sum_{j=0}^{s}\sum_{k=0}^{s}\alpha_j\alpha_k\sin((m-j-k)\theta) = \sum_{k=0}^{m}\beta_k^*\sin(k\theta),$$

where

$$\beta_{m-k}^* = 2\sum_{j=0}^{k}\alpha_j\alpha_{k-j}, \qquad k = 0,\dots,s. \tag{2.17}$$

In view of item 1 of the proof, the equality in (2.4) is attained for the collection $b = \widetilde{b}$ as well:

$$\widetilde{b} = (\widetilde{b}_{m-s},\dots,\widetilde{b}_m), \qquad \widetilde{b}_k = \varepsilon(-1)^k b_k^*, \quad k = m-s,\dots,m, \quad \varepsilon \in \{-1,1\},$$

and the extremal polynomial is also unique up to a numerical multiplier. Consider the polynomial

$$\widetilde{G}(\theta) = G^*(\theta+\pi) = \sum_{k=0}^{m}(-1)^k\beta_k^*\sin(k\theta). \tag{2.18}$$

For it, the following relations hold:

$$|\omega[\widetilde{G}, i\widetilde{b}\,]| = |\omega[G^*, ib^*]|, \qquad L[\widetilde{G}] = L[G^*].$$

Therefore,

$$\frac{|\omega[\widetilde{G}, i\widetilde{b}\,]|}{L[\widetilde{G}]} = \frac{\pi\lambda^*}{2}\,.$$

In view of item 1, there are no other extremal collections b and hence also no extremal polynomials. The lemma is proved. $\square$

3. PROOF OF THEOREM 1

Let us write the quantity $C(n,\ell)$ as

$$C(n,\ell) = \sup_{P\in\mathscr{P}_n}\frac{\|P^{(\ell)}\|_\infty}{\|P\|_1}\,. \tag{3.1}$$

Let us expand the polynomial $P \in \mathscr{P}_n$ in the Chebyshev polynomials of the second kind:

$$P(x) = \sum_{k=0}^{n}\tau_k U_k(x) = \sum_{k=1}^{n+1}\beta_k U_{k-1}(x),$$

and, to it, let us assign the trigonometric polynomial

$$G(\theta) = \sum_{k=1}^{m}\beta_k\sin(k\theta), \qquad m = n+1. \tag{3.2}$$

Replacing $\theta = \arccos x$, we express the denominator of (3.1) as

$$\|P\|_1 = \frac{1}{2}\int_{-1}^{1}\biggl|\sum_{k=1}^{m}\beta_k\frac{\sin(k\arccos x)}{\sqrt{1-x^2}}\biggr|\,dx = \frac{1}{4}\int_{0}^{2\pi}\biggl|\sum_{k=1}^{m}\beta_k\sin(k\theta)\biggr|\,d\theta = \frac{\pi}{2}\,L[G].$$

It is well known that $\|U_k^{(\ell)}\|_\infty = U_k^{(\ell)}(1)$ for all k and ℓ. Consider all possible collections of numbers $\{\varepsilon_k\}_{k=\ell+1}^m$, where ε_k assumes the values 1 or -1. Then the numerator (3.1) can be estimated as follows:

$$\|P^{(\ell)}\|_\infty \le \sum_{k=1}^m \|\beta_k U_{k-1}^{(\ell)}\|_\infty = \sum_{k=\ell+1}^m |\beta_k| U_{k-1}^{(\ell)}(1) = \max_{\{\varepsilon_k\}_{k=\ell+1}^m} \sum_{k=\ell+1}^m \varepsilon_k U_{k-1}^{(\ell)}(1)\beta_k. \tag{3.3}$$

Set $s = m - (\ell+1) = n - \ell$,

$$b_k^* = U_{k-1}^{(\ell)}(1), \quad b_k = \varepsilon_k U_{k-1}^{(\ell)}(1), \quad b^* = (b_{m-s}^*, \dots, b_m^*), \quad b = (b_{m-s}, \dots, b_m).$$

In this notation, inequality (3.3) takes the form

$$\|P^{(\ell)}\|_\infty \le \max_{\{\varepsilon_k\}_{k=\ell+1}^m} \sum_{k=\ell+1}^m \varepsilon_k U_{k-1}^{(\ell)}(1)\beta_k = \max_b |\omega[G, ib]|.$$

It follows from the inequality $(n-1)/3 \le \ell \le n-1$ that

$$1 \le s \le \frac{2n+1}{3} = \frac{2m-1}{3}.$$

Therefore, the assumptions of Lemma 2 hold and, applying it, we obtain the estimate

$$\frac{\|P^{(\ell)}\|_\infty}{\|P\|_1} \le \max_b \frac{2}{\pi} \frac{|\omega[G, ib]|}{L[G]} \le \lambda^*, \tag{3.4}$$

where λ^* is the greatest characteristic number of the matrix (2.5). Besides, the right-hand inequality in (3.4) becomes an equality only on the polynomials (2.7). By formula (3.2), to the polynomial $G^*(\theta)$ corresponds the polynomial

$$P^*(x) = \sum_{k=1}^{n+1} \beta_k^* U_{k-1}(x)$$

and to the polynomial $G^*(\theta+\pi)$ corresponds the polynomial $P^*(-x)$. In view of condition (2.8) on the coefficients β_k^*, the following relations hold:

$$\|P^{*(\ell)}(x)\|_\infty = \|P^{*(\ell)}(-x)\|_\infty = P^{*(\ell)}(1) = \sum_{k=\ell+1}^{n+1} \beta_k^* U_{k-1}(1) = |\omega[G^*, ib^*]|,$$

$$\|P^*(x)\|_1 = \|P^*(-x)\|_1 = \frac{\pi}{2} L[G^*].$$

Therefore, $C(n,\ell) = \lambda^*$, and $P^*(x)$, $P^*(-x)$ are extremal polynomials.

The proof of the theorem is complete.

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