Semigroup Forum (2013) 86:155–161
DOI 10.1007/s00233-012-9416-0

RESEARCH ARTICLE

Epigroups whose subepigroup lattice is lower semimodular

A.J. Ovsyannikov

Received: 16 February 2012 / Accepted: 25 June 2012 / Published online: 3 July 2012

Abstract We characterize epigroups mentioned in the title.

Keywords Epigroup · Lower semimodular lattice · Subepigroup lattice of an epigroup

A semigroup S is called an *epigroup* if some power of each element in S lies in a subgroup of S. An epigroup can be considered as a unary semigroup, i.e. as a semigroup with an additional unary operation of taking pseudo-inverse (see [6, 7]). Investigations of connections between epigroups *per se* (i.e. those which are neither periodic semigroups nor groups) and their subepigroup lattices have started in [10]. First results obtained in this direction have been surveyed in [8]. In the latter paper the problem of studying epigroups with lower semimodular subepigroup lattice has been posed as well.

If x, y are elements of a lattice L, we write $x \succ y$ to denote that $x > y$ and there is no $z \in L$ such that $x > z > y$. Recall that L is called *lower semimodular* if for all $x, y \in L$ from $x \vee y \succ x$ it follows that $y \succ x \wedge y$. An upper semimodular lattice is defined in a dual way. The structure of epigroups with upper semimodular subepigroup lattice is determined in [10]. The problem of investigation of semigroups with lower semimodular lattice of (usual) subsemigroups posed in [9], Problem 5.14, is deeply studied in [4]. Certain ideas and some of the auxiliary results of this paper are used to obtain the main result of the present paper. The lower semimodularity condition was

Communicated by M.V. Volkov.

The author acknowledges support from the Russian Foundation for Basic Research, grant 10-01-00524.

A.J. Ovsyannikov (✉)
Institute of Mathematics and Computer Science, Ural Federal University, Lenina 51, 620051 Ekaterinburg, Russia
e-mail: Alexander.Ovsyannikov@usu.ru

considered for the lattice of all full inverse subsemigroups of an inverse semigroup in [1–3].

We treat as well-known basic notions of semigroup theory (such as Green's relations, null semigroup, Brandt semigroup, principal factor, singular semigroup and so on). The reader can find corresponding information for example in [5]. We treat as known certain simple properties of epigroups; corresponding information can be found in [6, 7, 10]. We recall the definition of a pseudo-inverse element only. Let S be an epigroup and $x \in S$. A unique maximal subgroup of S that contains some power of x is denoted by G_x; we denote the identity of G_x by e_x. It is known that $xe_x = e_x x$ and this element is in G_x so that we can consider the inverse element $(xe_x)^{-1}$ in G_x. This element is called the *pseudo-inverse* for x and is denoted by $\overline{x}$.

For an element $a \in S$ we denote by J_a the $\mathcal{J}$-class of the epigroup S which contains the element a, by $J(a)$ the principal ideal generated by a, and by $I(a)$ the ideal $J(a)\backslash J_a$. If $I(a) \neq \emptyset$, then $J(a)/I(a) = J_a \cup \{0\}$ is the *principal factor of* S *associated with* a. We denote the elements of the Brandt semigroup B_2 by 0 and e_{ij}, where $i, j = 1, 2$; then $e_{ij}e_{kl} = 0$ when $j \neq k$ and $e_{ij}e_{kl} = e_{il}$ when $j = k$.

Let $\langle\langle X\rangle\rangle$ denote the subepigroup generated by a subset X of an epigroup S. Each element in $\langle\langle X\rangle\rangle$ can be represented by an epigroup term over X, where the operations are multiplication and taking pseudo-inverse. The *depth* of an epigroup term is defined as follows: the depth of each element in X is 0; if u, v are terms of depth m and respectively n, then uv has depth $m + n + 1$ and $\overline{u}$ has depth $m + 1$.

The lattice of all subepigroup of an epigroup S is denoted by Subepi S.

The main result of the paper is the following statement.

Theorem *Let S be an epigroup. The lattice* Subepi S *is lower semimodular if and only if the following conditions hold*:

(1) *each principal factor of S is a semigroup of some of the following types*:
 (1a) *a null semigroup*;
 (1b) *a group with lower semimodular subgroup lattice* (*with zero adjoined*);
 (1c) *a singular semigroup* (*with zero adjoined*);
 (1d) *the* 5*-element Brandt semigroup* B_2;
(2) *for all $e \in E_S$, $a \in S$, from $ea \in H_e\backslash\{e\}$ it follows that $e \in \langle\langle a\rangle\rangle$*;
(3) *for all $a, b, x \in S$ such that x is not in a non-trivial group*,
 (3a) *if $ab \notin J_x$ and $x = xab$ $[x = abx]$, then either $x \in \langle\langle a, b\rangle\rangle$ or $x = xa$ $[x = bx]$*;
 (3b) *if the associated principal factor of x is null and $x = bax$, then either $x \in \langle\langle a, b\rangle\rangle$ or $x = xa = bx$.*

Observe that, as was shown in [4], for periodic semigroups, condition (3b) follows from condition (3a). We do not know whether condition (3a) implies condition (3b) for epigroups which are not periodic semigroups; the proof for the periodic case presented in [4] does not extend to arbitrary epigroups.

Our first auxiliary result is parallel to statement 1 of Lemma 1.3 in [4].

Lemma 1 *Let S be an epigroup. If $U \succ V$ in the lattice* Subepi S, *then the set $U\backslash V$ is contained in a $\mathcal{J}$-class of S.*

Proof Take $x, y \in U \backslash V$. Since $x \in \langle\langle V, y\rangle\rangle$ and $x \notin V$, we can represent x by a term which involves some elements of V and necessarily involves y. If this term is a product that involves y or its pseudo-inverse $\overline{y}$, then x is divided by y or by $\overline{y}$, so in this case $x \in J(y)$, since $\overline{y}$ is divided by y. If this term is the pseudo-inverse element for such a product, then x is divided by the product and therefore is divided by y. We have $x \in J(y)$. By symmetry it follows that $y \in J(x)$. Thus, $J_x = J_y$, as required. □

Now we are ready to prove that the conditions of Theorem are sufficient. Let S be an epigroup which satisfies all these conditions. It is easy to see that a lattice L is lower semimodular if and only if for all $u, v, w \in L$ from $u \succ v \ngeq u \wedge w$ it follows that $u \wedge w \succ v \wedge w$. Thus, we ought to show that, for all $U, V, W \in \operatorname{Subepi} S$, from $U \succ V \not\supseteq U \cap W$ it follows that $U \cap W \succ V \cap W$. First we prove several auxiliary statements.

Let us verify that if the difference $U \backslash V$ is not contained in a subgroup of S, then this difference consists of a unique element, and if $U \backslash V \subseteq H$ for a subgroup H of S, then $U \cap H \succ V \cap H$. By Lemma 1, $U \backslash V \subseteq J_a$ for an element $a \in U \backslash V$. Arguing by contradiction, assume that J_a is not a group and the difference $U \backslash V$ consists of more than 1 element. By condition 1 the associated principal factor P of a is either a null semigroup or a combinatorial completely [0-]simple semigroup. Assume that P is a null semigroup. Let x, y be distinct elements in $(U \backslash V) \cap P$. Since $U \succ V$, we have $x \in \langle\langle y, V\rangle\rangle$ and $y \in \langle\langle x, V\rangle\rangle$, so x can be represented by a term t of y and elements of V and y can be represented by a term t' of x and elements of V. The pseudo-inverse $\overline{y}$ does not occur in t because $\overline{y} \in I(y)$. In a similar way, the pseudo-inverse $\overline{x}$ does not occur in t'. Since $axb \in P \cup V$ for all $a, b \in V$ and P is null, we conclude that $t = a_0 y b_0$ for some $a_0, b_0 \in V^1$, so $x = a_0 y b_0$. In a similar way we have $y = a_1 x b_1$ for some $a_1, b_1 \in V^1$. Thus $x = a_0 a_1 x b_1 b_0$. By condition (3b), it follows that $x = a_0 a_1 x = x b_1 b_0$. Now we come to a contradiction repeating arguments in the last paragraph of part (1) of the proof of Lemma 4.1 in [4].

Let P be a combinatorial completely [0-]simple semigroup. Then the class J_a is either a singular semigroup or a Brandt semigroup B_2. Consider each of the two possibilities.

1. Let J_a be a singular semigroup. For definiteness, suppose that J_a is left singular, i.e. $xy = x$ for all $x, y \in J_a$ (the right singular case is treated in a symmetric way). Fix an element $b \in J_a \cap (U \setminus V)$ which is not equal to a. Then $U = \langle\langle a, V\rangle\rangle = \langle\langle b, V\rangle\rangle$, whence $a \in \langle\langle b, V\rangle\rangle$ and $b \in \langle\langle a, V\rangle\rangle$. Let us prove that

$$a = v_1 b, \qquad b = v_2 a \quad \text{for some } v_1, v_2 \in V \backslash J(a). \tag{1}$$

We establish the first equality, the second is verified in a similar way. Since $a \in \langle\langle b, V\rangle\rangle$, take a representation of the element a by a term t of the least depth from the elements of $\{b\} \cup V$. Since $a^2 = a$, we have $\overline{a} = a$. Then $t = t_1 t_2$, where t_1, t_2 are terms whose depths are less than the depth of t. Denote the elements in S represented by these terms by the same letters. So we conclude that $t_i \in J_a$ or $t_i \in V \backslash J(a)$, $i = 1, 2$. Clearly, $t_2 a \in J_a$. If $t_1 \in J_a$, then $a = a^2 = t_1 t_2 a = t_1$, because J_a is left singular. We have a contradiction with the choice of t. If $t_1, t_2 \notin J_a$, then $t_1, t_2 \in V$ and so $a \in V$, which is impossible. Therefore, $t_1 \in V \backslash J(a)$, $t_2 \in J_a \backslash V$. Now we are to show that $t_2 = vb$ for some $v \in V$. Let us consider the leftmost occurrence of b in t_2. Since

$vb, bv \in J_a$ for all subwords of the kind vb or bv occurring in the term t_2 and J_a is an idempotent semigroup, we conclude that the operation of taking pseudo-inverse does not occur in t_2. Since J_a is a singular semigroup, by the choice of the term t, we have $t_2 = vbw = vb \cdot bw = vb$ for all $w \in \langle\langle b, v\rangle\rangle$. Thus $a = t_1 vb$, whence putting $v_1 = t_1 v$, we obtain the first equality in (1).

The equalities (1) give $a = v_1 b = v_1 v_2 a$. Since $v_1, v_2 \in V \backslash J(a)$, the condition $v_1 v_2 \in J_a$ implies that $a = v_1 v_2 a = v_1 v_2$, i.e. $a \in V$, which is impossible. So, $v_1 v_2 \notin J_a$. Since $a \notin \langle\langle v_1, v_2\rangle\rangle$, condition (3a) gives $a = v_2 a$, whence $a = b$ in view of the second equality in (1). This contradicts the assumption $a \neq b$. Case 1 is completely treated.

2. Let $J_a \cup \{0\}$ be a Brandt semigroup B_2. We denote the elements of J_a by $e_{11}, e_{12}, e_{21}, e_{22}$ in accordance with the agreement in the beginning of the paper. Consider all the possibilities which can arise here.

2.1. $e_{11}, e_{12} \in U \backslash V$. Then $e_{11} \in \langle\langle e_{12}, V\rangle\rangle$ and $e_{12} \in \langle\langle e_{11}, V\rangle\rangle$. Consider a representation of the element e_{11} by a term t of the least depth which involves elements of V and necessarily involves e_{12}. Since either $\overline{x} = x$ or $\overline{x} \in I(a)$ for all $x \in J_a$ and the term t is of the least depth, the operation of taking pseudo-inverse does not occur in t. The equality $e_{11} = e_{11}^2$ gives $e_{11} = e_{11} t$. Since $e_{11} v e_{12} = e_{12}$ and $e_{12} v e_{12} = e_{12}$ for all $v \in V \backslash I(a)$, by the minimality of depth of the term t we obtain $t = e_{12} v_1$ for some $v_1 \in V \backslash I(a)$. Therefore, $e_{11} = e_{12} v_1$. In a similar way we prove that $e_{12} = e_{11} v_2$ for some $v_2 \in V \backslash I(a)$. So, $e_{11} = e_{11} v_2 v_1$. Observe that $v_2 v_1 \notin J_a$ since otherwise $v_2 v_1 = e_{11}$, which is impossible. Since $e_{11} \neq e_{11} v_2$, by condition (3a) we obtain that $e_{11} \in \langle\langle v_1, v_2\rangle\rangle$, which is a contradiction showing that case 2.1 is impossible.

2.2. $e_{11}, e_{21} \in U \backslash V$ or $e_{21}, e_{22} \in U \backslash V$ or $e_{12}, e_{22} \in U \backslash V$. These cases are treated in a way similar to case 2.1.

2.3. $e_{11}, e_{22} \in U \backslash V$. Since $J_a \subseteq \langle\langle e_{12}, e_{21}\rangle\rangle$, we conclude that either $e_{12} \in U \backslash V$ or $e_{21} \in U \backslash V$, and we come to the conditions which were considered in case 2.1 and case 2.2.

2.4. $e_{12}, e_{21} \in U \backslash V$. In view of cases 2.1 and 2.2 we may assume that $e_{11}, e_{22} \in V$. We have $e_{12} \in \langle\langle e_{21}, V\rangle\rangle$. Consider a representation of the element e_{12} by a term t of the least depth which involves elements of V and necessarily involves e_{21}. In a similar way to the case 2.1 we note that the operation of taking pseudo-inverse does not occur in t. Since $e_{21} v e_{21} = e_{21}$ for all $v \in V \backslash I(a)$, the term t has a unique occurrence of e_{21}, i.e. $e_{12} = v_1 e_{21} v_2$. By multiplying the last equality through by e_{22} on the right and taking into account that $e_{21} v e_{22} = e_{22}$ for all $v \in V \backslash I(a)$, we obtain $e_{12} = e_{12} e_{22} = v_1 e_{21} v_2 e_{22} = v_1 e_{22} \in V$. Therefore, $e_{12} \in V$, which is impossible.

So, we have proved that if $U \backslash V$ is not contained in a subgroup of S, then this difference has a unique element. Now we are to prove that if $U \backslash V \subseteq H$ for some subgroup H, then $U \cap H \succ V \cap H$. Let $U \backslash V \subseteq H$ and H is a non-trivial group. If $|U \backslash V| = 1$, then the required statement is obvious.

Suppose that $|U \backslash V| > 1$. Let e denote the identity of the subgroup H. Pick an element $a \in U \backslash V$ distinct from e. Let us prove that $e \in V$. Arguing by contradiction, assume that $e \notin V$. Then $U = \langle\langle e, V\rangle\rangle$, in particular, $a \in \langle\langle e, V\rangle\rangle$. Let us fix a representation of the element a by a term t of the least depth. Since $\overline{e} = e$ and $\overline{x} \in H$ for all $x \in H$, the term t does not contain the operation of taking pseudo-inverse which is applied to the elements e or V. Then t contains at least one of the products ev, ve

for $v \in V$. Clearly, $ev, ve \in H$ and therefore $ev = eve = ve$, so these products can not be equal to e. By condition 2 we have $e \in \langle\langle v \rangle\rangle$. So, $e \in V$, which contradicts our assumption.

Now we are to show that $\langle\langle a, V \cap H \rangle\rangle = U \cap H$ for all $a \in U \setminus V$, this will prove the required statement. Obviously, $\langle\langle a, V \cap H \rangle\rangle \subseteq U \cap H$. Let us establish the converse inclusion. Fix $x \in U \cap H$. Since $U = \langle\langle a, V \rangle\rangle$, the element x can be represented by a term t which contains a and elements of V. Let us take such a term of the least depth. Since $a, x \in H$ and $x \notin V$, we see that a occurs in the term t. By the choice of t, this term does not contain the operation of taking pseudo-inverse which is applied to the elements of V. In addition, all the products of kind $a^n v$, va^n for $v \in V$ and integers n, which occur in t, are in H. Since $e \in V$, we have $a^n v = a^n eve$ and $va^n = evea^n$, so $eve \in V \cap H$. Therefore, x can be represented by a term t' which can be obtained from t by replacing $v \in V$ by eve; observe that $eve \in V \cap H$. Thus, $x \in \langle\langle a, V \cap H \rangle\rangle$, as required.

Finally we prove that the lattice $\mathrm{Subepi}\, S$ is lower semimodular. Take subepigroups $U, V, W \in \mathrm{Subepi}\, S$ such that $U \succ V \not\supseteq U \cap W$. Using arguments similar to those of the last paragraph of Sect. 4 of [4], we prove that $U \cap W \succ V \cap W$. If $|U \setminus V| = 1$, then obviously $U \cap W \succ V \cap W$. Let $U \setminus V \subseteq H$, where H is a group with lower semimodular subgroup lattice. Let us pick distinct elements x, y in $(U \cap W) \setminus (V \cap W)$. Then $x, y \in H$. Above we have proved that $U \cap H \succ V \cap H$. Since H is lower semimodular, it follows that $U \cap W \cap H \succ V \cap W \cap H$. Thus $y \in \langle\langle x, V \cap W \cap H \rangle\rangle$ whence $y \in \langle\langle x, V \cap W \rangle\rangle$ and in a similar way $x \in \langle\langle y, V \cap W \rangle\rangle$. Therefore $U \cap W \succ V \cap W$. Sufficiency is proved.

Now we prove necessity of the conditions of Theorem. Let S be an epigroup whose lattice $\mathrm{Subepi}\, S$ is lower semimodular. Pick $a \in S$ and let $F = J(a)/I(a)$ be a principal factor of the epigroup S. Then, analogously to Lemma 3.1 of [4], we conclude that either $\mathrm{Subepi}\, F = L$ or $\mathrm{Subepi}\, F = L^0$, where $L = \{T \in \mathrm{Subepi}\, S | I(a) \subseteq T \subseteq J(a)\}$ is an interval in the lattice $\mathrm{Subepi}\, S$. Thus L and therefore L^0 are lower semimodular. We conclude that every principal factor of the epigroup S has lower semimodular subepigroup lattice. As is known (see [6], Corollary of Proposition 1), a principal factor of an epigroup is a null semigroup or a completely 0-simple semigroup, or (if it is the kernel of the epigroup) a completely simple semigroup. By repeating the proof of Proposition 3.6 in [4], we obtain that a completely 0-simple semigroup with lower semimodular subepigroup lattice either is combinatorial or is a group with zero adjoined. In particular, every non-trivial subgroup of S is isolated. A completely simple semigroup with lower semimodular subepigroup lattice is either a group or a singular semigroup. Since a combinatorial epigroup is periodic, by Theorem 5.3 of [4] it follows that a combinatorial completely 0-simple epigroup with lower semimodular subepigroup lattice is isomorphic to the Brandt semigroup B_2 or is a singular semigroup with zero adjoined. Thus, condition 1 holds for S.

Let us verify condition 2. This statement is parallel to Lemma 3.7 of [4]. Let $e \in E_S$, $a \in S$, and H_e is a non-trivial group. Assume that $ea \in H_e$ and $ea \neq e$. We are to prove that $e \in \langle\langle a \rangle\rangle$. Let $b = ea$. Since $a \notin I(e)$ and $H_e = J_e$, we conclude that $J(e) \subseteq J(a)$ and $b \in H_e$. Moreover, $ae, \overline{a}e, e\overline{a} \in H_e$. From $b \in H_e$ it follows that $e \in \langle\langle b \rangle\rangle$. We are to prove that, for all $h \in H_e$, $c \in \langle\langle a \rangle\rangle$, we have $hc, ch \in H_e$. Let $c = a^n$, where n is a positive integer. Then $hc = hea^n = h(ea)a^{n-1} = heaea^{n-1} = h(ea)^2 a^{n-2} =$

$\dots = h(ea)^n = hb^n$. Since $h, b \in H_e$, we have $hc \in H_e$. In a similar way, using the element ae instead b, we obtain that $ch \in H_e$. Using the same arguments to $e\overline{a}$ and $\overline{a}e$, we conclude that $hc, ch \in H_e$ for all $c = \overline{a}^n$, n is a positive integer. Let f be the identity of the group $\langle\langle a\rangle\rangle$. Since $f = a\overline{a}$, clearly $hf, fh \in H_e$.

We are now to prove that $\langle\langle a, b\rangle\rangle = \langle\langle a\rangle\rangle \cup \langle\langle b\rangle\rangle$. Using the statement proved in the previous paragraph, we have $ba^n = (ea)a^n = (ea)^{n+1} = b^{n+1}$ for any positive integer n. In a similar way we have $a^n b = b^{n+1}$. We are to show that $\overline{b} = e\overline{a}$. We have $b(e\overline{a}) = eae\overline{a} = ea\overline{a} = ef$. Since $ef \in H_e$ and $(ef)^2 = efef = eff = ef$, we conclude that $ef = e$, so $b(e\overline{a}) = e$. Since $e\overline{a} \in H_e$, we have $\overline{b} = e\overline{a}$. Thus $\langle\langle a, b\rangle\rangle = \langle\langle a\rangle\rangle \cup \langle\langle b\rangle\rangle$.

Let us prove that $\langle\langle a, b\rangle\rangle \succ \langle\langle a\rangle\rangle$. For all $x \in \langle\langle a, b\rangle\rangle \setminus \langle\langle a\rangle\rangle$ we have $x \in \langle\langle b\rangle\rangle$. Since $e \in \langle\langle b\rangle\rangle$, it follows that $e \in \langle\langle x\rangle\rangle$. Since $b = ea$, we conclude that $b \in \langle\langle a, x\rangle\rangle$, as required. From the condition $\langle\langle a, b\rangle\rangle \succ \langle\langle a\rangle\rangle$, by lower semimodularity of the lattice Subepi S, it follows that $\langle\langle b\rangle\rangle \succ \langle\langle a\rangle\rangle \cap \langle\langle b\rangle\rangle$, so $\langle\langle a\rangle\rangle \cap \langle\langle b\rangle\rangle \neq \emptyset$ and $e \in \langle\langle a\rangle\rangle$. Therefore, condition 2 is proved.

The following statement can be proved in a similar way to Lemma 3.8 of [4].

Lemma 2 *Let S be an epigroup. If $S = \langle\langle x, y\rangle\rangle$, $x \neq y$, and neither of the elements x, y is contained in a non-trivial subgroup, then at least one of the subsets $S \setminus \{x\}$ or $S \setminus \{y\}$ is a subepigroup of S.*

We now are to prove that for the epigroup S condition (3a) holds. Arguing by contradiction, assume that $a, b, x \in S$, the element x is not in a non-trivial group and $a, b \notin J_x$, $x = xab$, $x \neq xa$, $x \notin \langle\langle a, b\rangle\rangle$. Set $T = \langle\langle a, b, x\rangle\rangle$. Since $x \notin \langle\langle a, b\rangle\rangle$, by Zorn's Lemma, in T there exists a subepigroup M which is maximal with the properties $a, b \in M$ and $x \notin M$. Clearly $T \succ M$. Since $x = xa \cdot b$ and $b \in M$, we conclude that $xa \notin M$. Let us prove that $xa \notin \langle\langle x\rangle\rangle$. Since $\langle\langle M, x\rangle\rangle = T$ and $T \succ M$, lower semimodularity of the lattice Subepi S implies $\langle\langle x\rangle\rangle \succ M \cap \langle\langle x\rangle\rangle$. Since the element x is non-group, we have $\langle\langle x\rangle\rangle \setminus \{x\} \subseteq M$. Since $x \neq xa$, the inclusion $xa \in \langle\langle x\rangle\rangle$ implies $xa \in M$, which is impossible. Observe that $J_x = J_{xa}$. Since x does not belong to a non-trivial group, J_x is not an isolated subgroup, and therefore xa does not belong to a non-trivial group. Changing the roles of x and xa, we obtain $x \notin \langle\langle xa\rangle\rangle$. Put $U = \langle\langle x, xa\rangle\rangle$. Since $T \succ M$, lower semimodularity of the lattice Subepi S implies $U \succ M \cap U$, which contradicts Lemma 2. This contradiction proves condition (3a) of Theorem.

To prove condition (3b), assume that $x = bxa$, $x \notin \langle\langle a, b\rangle\rangle$ and $x \neq xa$ or $x \neq bx$. The end of the proof is almost identical to that in the previous case. The theorem is completely proved.

Acknowledgements The author thanks L.N. Shevrin and M.V. Volkov for helpful discussions. The author is also grateful to the anonymous referee for pointing out a gap in a proof in the first version of the paper and for other useful remarks.

References

1. Cheong, K.H., Jones, P.R.: Lower semimodular inverse semigroups. Semigroup Forum **75**, 393–416 (2007)

2. Cheong, K.H., Jones, P.R.: Lower semimodular inverse semigroups II. Commun. Algebra **39**, 955–971 (2011)
3. Jones, P.R.: Semimodular inverse semigroups. J. Lond. Math. Soc. **17**, 457–466 (1978)
4. Jones, P.R.: On semigroups with lower semimodular lattice of subsemigroups. J. Algebra **324**, 2089–2111 (2010)
5. Mikhalev, A.V., Pilz, G.F. (eds.): The Concise Handbook of Algebra. Kluwer Academic, Dordrecht (2002). Semigroups, Chap. A
6. Shevrin, L.N.: On the theory of epigroups. I, II. Mat. Sb. **185**, 129–160, 153–176 (1994) (Russian; English translation: Russ. Acad. Sci. Sb. Math. **82**, 485–512 (1995), **83**, 133–154 (1995))
7. Shevrin, L.N.: Epigroups. In: Kudryavtsev, V.B., Rosenberg, I.G. (eds.) Structural Theory of Automata, Semigroups and Universal Algebra. NATO Science Series, II. Mathematics, Physics and Chemistry, vol. 207, pp. 331–380. Springer, Berlin (2005)
8. Shevrin, L.N.: Lattice properties of epigroups. Fundam. Appl. Math. **14**, 219–229 (2008) (Russian)
9. Shevrin, L.N., Ovsyannikov, A.J.: Semigroups and Their Subsemigroup Lattices. Kluwer Academic, Dordrecht (1996)
10. Shevrin, L.N., Ovsyannikov, A.J.: On lattice properties of epigroups. Algebra Univers. **59**, 209–235 (2008)