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On Hamilton Paths in Grid Graphs

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Abstract

In this paper we consider an approach to solve the Hamilton path problem for grid graphs. This approach is based on an explicit reduction from the problem to the satisfiability problem.

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Keywords: Hamilton path, grid graph, **NP**-complete, satisfiability, vacuum cleaning robot

Gridworlds are extensively used for design of experiments as models of real world. In particular, grid graphs are popular testbeds for different hard problems. In particular, grid graphs used for investigation of robotic planning problems (see e.g. [1]). For instance, grid graph can be considered as a map of some indoor environment. Vacuum cleaning robot needs cover the environment. The robot must visit each vertex only once to prevent a damage of the floor. It is clear that we can consider the Hamilton path problem (HP) for grid graphs as a model for this robotic problem. In this paper we consider an approach to solve HP. Note that HP is **NP**-complete [2]. Encoding different hard problems as instances of SAT has caused considerable interest (see e.g. [3] – [15]). We consider an explicit reduction from HP to the satisfiability problem.

Let $G = (V, E)$ be a grid graph. Clearly, we can consider G as a part of two-dimensional integer grid. We assume that $V = \{G(i_1, j_1), \ldots, G(i_n, j_n)\}$, $(G(i,j), G(s,t)) \in E \Leftrightarrow |i-s| + |j-t| = 1 \wedge \{G(i,j), G(s,t)\} \subseteq V$. Also, we can assume that $1 \leq i_k \leq p$, $1 \leq j_k \leq q$, for any $1 \leq k \leq n$. Let

$$
\begin{aligned}
\varphi[1] &= \wedge_{1\leq k\leq n} \vee_{1\leq i\leq p} \vee_{1\leq j\leq q} g[k,i,j], \\
\varphi[2] &= \wedge_{\substack{1\leq k\leq n,\\ 1\leq i[1]\leq p,\\ 1\leq i[2]\leq p,\\ 1\leq j[1]\leq q,\\ 1\leq j[2]\leq q,\\ |i[1]-i[2]|+|j[1]-j[2]|\neq 0}} (\neg g[k,i[1],j[1]] \vee \neg g[k,i[2],j[2]]), \\
\varphi[3] &= \wedge_{\substack{1\leq k\leq n,\\ 1\leq i\leq p,\\ 1\leq j\leq q,\\ G(i,j)\notin E}} \neg g[k,i,j], \\
\varphi[4] &= \wedge_{\substack{1\leq k\leq n-1,\\ 1\leq i[1]\leq p,\\ 1\leq i[2]\leq p,\\ 1\leq j[1]\leq q,\\ 1\leq j[2]\leq q,\\ |i[1]-i[2]|+|j[1]-j[2]|\neq 1}} (\neg g[k,i[1],j[1]] \vee \neg g[k+1,i[2],j[2]]), \\
\xi &= \wedge_{i=1}^{4} \varphi[i].
\end{aligned}
$$

Theorem. *There is a Hamilton path in G if and only if ξ is satisfiable.*

Proof. Let $\xi = 1$. By definition, in this case, $\varphi[i] = 1$, for all $1 \leq i \leq 4$. Since $\varphi[1] = 1$, it is clear that for any $1 \leq k \leq n$, there are i and j such that $1 \leq i \leq p$, $1 \leq j \leq q$, and $g[k,i,j] = 1$. In view of $\varphi[2] = 1$, it is easy to check that for any $1 \leq k \leq n$, there is only one pair $(i,j) \in \{1,2,\ldots,p\} \times \{1,2,\ldots,q\}$ such that $g[k,i,j] = 1$. Therefore, we can assume that for any $1 \leq k \leq n$, $g[k,i,j] = 1$ if and only if $(i,j) = (a[k], b[k])$ where $\{(a[1], b[1]), (a[2], b[2]), \ldots, (a[n], b[n])\}$ is some fixed set. Since $\varphi[3] = 1$, it is clear that for any $1 \leq k \leq n$, if $g[k,i,j] = 1$, then $G(i,j) \in V$. In view of $\varphi[4] = 1$, it is easy to see that for all $1 \leq k \leq n-1$, $1 \leq i[1] \leq p$, $1 \leq i[2] \leq p$, $1 \leq j[1] \leq q$, $1 \leq j[2] \leq q$, if $g[k,i[1],j[1]] = g[k+1,i[2],j[2]] = 1$, then $|i[1]-i[2]| + |j[1]-j[2]| = 1$. Therefore, if $g[k,i[1],j[1]] = g[k+1,i[2],j[2]] = 1$, then $(G(a[k], b[k]), G(a[k+1], b[k+1])) \in E$. So, $G(a[1], b[1]), G(a[2], b[2]), \ldots, G(a[n], b[n])$ is a Hamilton path in G. Now, let $G(a[1], b[1]), G(a[2], b[2]), \ldots, G(a[n], b[n])$ be a Hamilton path in G. Let $g[k,i,j] = 1$ if and only if $i = a[k]$, $j = b[k]$. It is easy to check that $\xi = 1$. □

It is clear that ξ is a CNF. So, ξ gives us an explicit reduction from HP to SAT. Now, using standard transformations (see e.g. [16]) we can obtain an

time	N1	N2	N3	R1	R2	R3
average	3.1 min	15.2 min	1.62 h	19 sec	9.4 min	57 min
maximum	41 min	2.7 h	11.6 h	22 min	2.1 h	9.36 h
best	12 sec	43 sec	32.3 min	7 sec	28 sec	19.8 min

Table 1: Experimental results for different reductions to 3SAT where N1 is ζ for natural instances, N2 is $\gamma[1]$ from [18] for natural instances, N3 is the reduction from [19] for natural instances, R1 is ζ for robotic instances, R2 is $\gamma[1]$ from [18] for robotic instances, R3 is the reduction from [19] for robotic instances.

explicit transformation ξ into ζ such that $\xi \Leftrightarrow \zeta$ and ζ is a 3-CNF. Clearly, ζ gives us an explicit reduction from HP to 3SAT.

We consider our genetic algorithm which was proposed in [17]. We have used heterogeneous cluster for our computational experiments. Each test was runned on a cluster of at least 100 nodes. We have created a generator of natural instances for HP. Also, we have created a generator of instances for HP which allow us to construct a plan for a vacuum cleaning robot. Selected experimental results are given in Table 1.

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